Introduction to Differential Topology

Special Course for students of 3-5 years, 2015–2016

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September 13, 2015

Form the book

- Lectures by John Milnor *Differential Topology*, Princeton University, Fall term (1958) Notes by James Munkres:

Differential topology may be defined as the study of those properties of differentiable manifolds which are invariant under diffeomorphism (differentiable homeomorphism) that is which do not depend of the choice of a sample from the class of diffeomorphic manifolds.

Typical problem falling under this heading are the following:

- Given two differentiable manifolds, under what conditions are they diffeomorphic?
- Given a differentiable manifold, is it the boundary of some differentiable manifold-with-boundary?
- Given a differentiable manifold, is it parallelisable?

All these problems concern more than the topology of the manifold, yet they do not belong to differential geometry, which usually assumes additional structure (e.g., a connection or a metric). The most powerful tools in this subject have been derived from the methods of algebraic topology. In particular, the theory of characteristic classes is crucial, where-by one passes from the manifold M to its tangent bundle, and thence to a cohomology class in M which depends on this bundle.

Outline:

- Smooth manifolds.
- Tangent bundles.
- Bundles. Vector bundles.
- Calculus on smooth manifolds. Differential Forms.
- Homology and Cohomology. De Rham Cohomology.
- Connections and Curvatures.
- Characteristic classes. Chern-Weil Theory.
- Immersions and embeddings. Bordisms
- Surgery. Smooth structures on homotopy type.

Let us consider an n -dimensional Euclidean space which is usually denoted by $\mathbf{R}^n$. We assume that this space is provided with an n – tuple of Cartesian coordinates $(x^1, x^2, \dots, x^n)$ which permits a unique determination of the position of any point $\vec{x} \in \mathbf{R}^n$ by associating with it a set of real numbers, the coordinates relative to a fixed orthogonal basis formed by mutually orthogonal unit vectors $(\mathbf{e}_1, \mathbf{e}_2, \dots \mathbf{e}_n)$:

$$\vec{x} = \sum_{i=1}^{n} x^i \mathbf{e}_i.$$

The idea of describing a point in an Euclidean space by a set of real numbers (which may also be considered as the coordinates of the radius vector coupling the origin with the point) underlies analytic geometry which solves various geometric problems by purely algebraic methods. This important approach was first, introduced (explicitly) into mathematics by des Cartes in whose honor we now say "Cartesian coordinates". Algebraization of geometry played a key role in the development not only of geometry as such but also of mathematics as a whole.

We shall not concentrate on the problems which are easily and elegantly solved by algebraic-analytic methods (for example, classification of second-order surfaces in a three-dimensional space) and refer the readers to numerous courses of algebra and analytic geometry. Let us only recall that Cartesian coordinates in $\mathbf{R}^n$ are closely related to the concept of the Euclidean scalar product, a bilinear form which associates with each pair of vectors $\vec{x}, \vec{y} \in \mathbf{R}^n$ a real number usually denoted by $\langle \vec{x}, \vec{y} \rangle$.

This operation is symmetric and linear in each argument, and the form itself is positive definite. In a Cartesian coordinate system we have

$$\langle \vec{x}, \vec{y} \rangle = x^1y^1 + x^2y^2 + \ldots + x^nyn = \sum_{i=1}^{n} x^iy^i,$$

where

$$\vec{x} = \left(x^1, x^2, \ldots, x^n\right),$$

$$\vec{y} = \left(y^1, y^2, \ldots, y^n\right).$$

Simple examples however show that Cartesian coordinates are not always the most convenient ones to solve analytically many particular problems. We shall demonstrate this by writing the equations of curves on a plane in Cartesian coordinates (x, y) .

Of course, for rather simple curves, for example, a circle or an ellipse, the analytic expressions in Cartesian coordinates are simple.

Indeed, the equation of a circle of radius R with the center at the point $\mathbf{A}$ is

$$\left(x - A^1\right)^2 + \left(y - A^2\right)^2 = R^2,$$

where $\mathbf{A} = \left(A^1, A^2\right)$. The equation of an ellipse is also simple:

$$\frac{\left(x - A^1\right)^2}{a^2} + \frac{\left(y - A^2\right)^2}{b^2} = R^2,$$

where a and b are the principal semi-axes.

However, in various applications and concrete mechanical and physical problems we often deal with smooth curves (say, trajectories of the motion of a particle in a force field) whose equations in Cartesian coordinates are rather cumbersome. For example, the equation

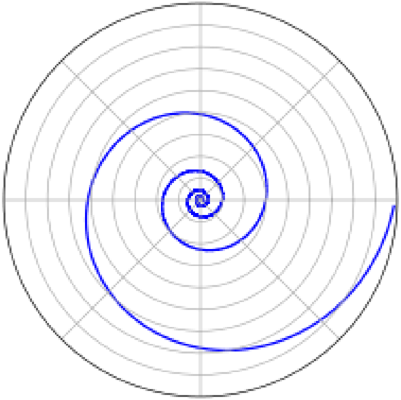
$$\sqrt{x^2+y^2} - e^{\lambda\left(\tan^{-1}\frac{y}{x}\right)} = 0$$

defines a spiral in Cartesian coordinates.

Although this equation is rather simple, it could be written in a simpler form in so called polar coordinates (r, φ) related to the Cartesian coordinates (x, y) by the formulas

$$\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi. \end{aligned} \tag{1}$$

In polar coordinates the equation of a spiral becomes $r = e^{\lambda\varphi}$, thereby clearly demonstrating the character of the motion along this trajectory.



Let us consider an arbitrary domain in a Euclidean space $\mathbf{R}^n$. We recall that, just as in mathematical analysis, by a domain we mean an arbitrary set U in an Euclidean space whose every point P is contained in U together with a ball of sufficiently small radius with center at P .

The system of coordinates of the point $P \in U$ is a set of numbers, called coordinates, that are associated with the point P , say,

$$x^1 = x^1(P), x^2 = x^2(P), \dots, x^n = x^n(P),$$

such that they satisfy the conditions:

- All coordinates are continuous functions of the variable P ,
- The common map
  $\vec{x}(P) = (x^1 = x^1(P), x^2 = x^2(P), \ldots, x^n = x^n(P))$,

$$\vec{x} : U \longrightarrow \mathbf{R}^n$$

  is the homeomorphism from U on the open set $V \subset \mathbf{R}^n$.

We say that the set of continuous functions

$$x^1 = x^1(P), x^2 = x^2(P), \dots, x^n = x^n(P),$$

forms a local system of coordinates if it satisfy the previous conditions locally that is for any point $P \in U$ there is a neighbourhood $P \in U' \subset U$, such that the common map

$$\vec{x}|_{U'} \longrightarrow \mathbf{R}^n$$

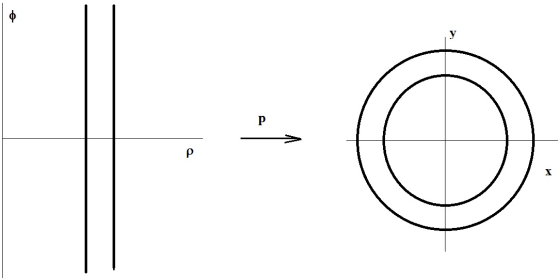
is a homeomorphism onto the open subset $V' = \vec{x}(U) \subset \mathbf{R}^n$.

The polar coordinates are relations

$$x = \rho \cos \varphi,$$
$$y = \rho \sin \varphi,$$

that we can consider as a map

$$p : R^{+}(\rho) \times R(\varphi) \longrightarrow \mathbf{R}^2(x, y).$$



ϕ
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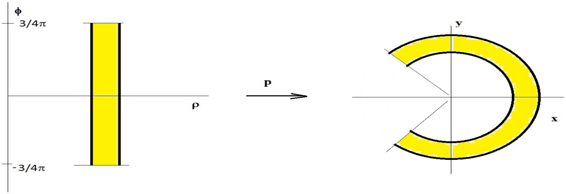
The map p is not homeomorphism but it is locally homeomorphism:



Among all continuous coordinate mappings of special interest are those that define a smooth mapping of a domain U onto V , i.e. when all functions $\left\{x^1\left(y^1,\ldots,y^n\right),\ldots,x^n\left(y^1,\ldots,y^n\right)\right\}$ are continuously smooth functions of their arguments $\left(y^1,\ldots,y^n\right)$. But the smoothness of the coordinate mapping f without the assumption of the smoothness of the inverse mapping f^{-1} does not lead to a meaningful coordinate system.

Let turn to defining coordinate systems in which f and f^{-1} are both smooth.

Definition (Diffeomorphism)

We say that the homeomorphism $f : U \longrightarrow V$ is *diffeomorphism* if both f and f^{-1} are smooth maps.

To this end, we shall need a new concept, the Jacobi matrix of a smooth mapping.

Let $f : U \to V$ be a smooth mapping defined by a family of functions

$$\begin{cases} x^1 = x^1\left(y^1, y^2, \dots, y^n\right), \\ x^2 = x^2\left(y^1, y^2, \dots, y^n\right), \\ \vdots \\ x^n = x^n\left(y^1, y^2, \dots, y^n\right). \end{cases}$$

Definition

The Jacobi matrix of a mapping f is a functional matrix

$$Df = \frac{\partial x}{\partial y} = \begin{pmatrix} \frac{\partial x^1}{\partial y^1} & \cdots & \frac{\partial x^1}{\partial y^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x^n}{\partial y^1} & \cdots & \frac{\partial x^n}{\partial y^n} \end{pmatrix},$$

composed of partial derivatives of the coordinates $\left(x^1(P), \ldots, x^n(P)\right)$.

The determinant of this matrix is denoted by $J(f) = \det Df$ and called the *Jacobian* of the mapping f . Sometimes the Jacobian is denoted by

$$J(f) = \frac{\partial(x^1, x^2, \dots, x^n)}{\partial(y^1, y^2, \dots, y^n)}.$$

The Jacobi matrix can be extended to maps which have different dimensions in the image and the inverse image:

$$\mathbf{R}^n(y^1, y^2, \ldots, y^n) \supset U \xrightarrow{f} V \subset \mathbf{R}^m(x^1, x^2, \ldots, x^m),$$

$$x = f(y) = \begin{cases} x^1 = x^1\left(y^1, y^2, \ldots, y^n\right), \\ x^2 = x^2\left(y^1, y^2, \ldots, y^n\right), \\ \vdots \\ x^m = x^n\left(y^1, y^2, \ldots, y^n\right). \end{cases}$$

Then the Jacobi matrix also is composed of all derivatives:

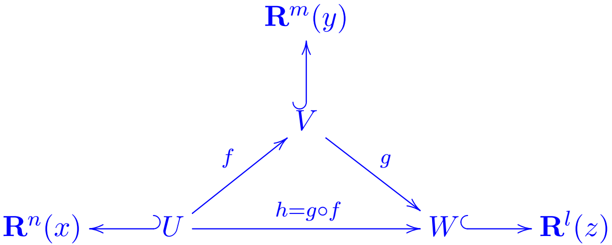
$$Df = \frac{\partial x}{\partial y} = \begin{pmatrix} \frac{\partial x^1}{\partial y^1} & \cdots & \frac{\partial x^1}{\partial y^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x^m}{\partial y^1} & \cdots & \frac{\partial x^m}{\partial y^n} \end{pmatrix}.$$

The Jacobi matrix in this case is not quadratic and has m rows and n columns

The fundamental property of the construction of the Jacobi matrices is known as so called chain rule. Notice that the Jacobi matrix Df is the matrix function that depends on the variables $y = (y^1, y^2, \dots, y^n)$ and in each point y the matrix $Df|_y$ induces a natural linear map

$$Df|_y : \mathbf{R}^n \longrightarrow \mathbf{R}^m$$

Consider two maps

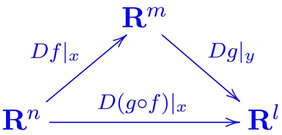


Theorem (The chain rule)

There is the chain rule for the Jacobi matrices of the composition:

$$D(g \circ f)|_x = Dg|_y \circ Df|_x,$$

where $y = f(x)$.



The second fundamental property is the following. Let $\mathbf{Id} = f : U \longrightarrow U$ be the identical map,

$$\begin{cases} f(x^1) = x^1, \\ f(x^2) = x^2, \\ \vdots \\ f(x^n) = x^n, \end{cases}$$

Theorem (Identity map)

The Jacobi matrix of the identity map is the identity matrix (in each point):

$$D(\mathbf{Id})|_x = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}.$$

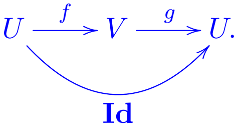
These two fundamental properties (chain rule and identity of the Jacobi matrix of the identical map) constitute so called functorial properties of the differential of a smooth maps. The simplest consequence from the functorial properties is

Theorem

Let $U, V \subset \mathbf{R}^n$ be open domains and $f : U \longrightarrow V$ be a diffeomorphism. Then for each point $x \in U$ one has

$$\det(Df|_x) \neq 0.$$

Proof. The map f is smooth. Let $g = f^{-1} : V \longrightarrow U$ be the inverse map (also smooth). We have the diagram



Using chain rule and identity of the differential of the identity one has the diagram that is formed by the Jacobi matrices

$$\mathbf{R}^n \xrightarrow{Df|_x} \mathbf{R}^n \xrightarrow{Dg|_y} \mathbf{R}^n, \qquad Dg|_y \circ Df|_x = \mathbf{Id}_n$$

where $y = f(x)$.

Hence

$$1 = \det \mathbf{Id}_n = \det(Dg|_y \circ Df|_x) = \det(Dg|_y)\det(Df|_x)$$
$$\Downarrow$$
$$\det(Df|_x) \neq 0.$$

There is an inverse theorem which states that nonvanishing of the Jacobian implies that the smooth map $f : U \longrightarrow V$ is a local coordinate system (or local diffeomorphism):

Theorem (Local diffeomorphism criterion)

Let $U, V \subset \mathbf{R}^n$ be open domains and $f : U \longrightarrow V$ be a smooth map. Consider a point $x \in U$, and put $y = f(x) \in V$. Assume that

$$\det Df|_x \neq 0.$$

Then there are neighborhoods $x \in U' \subset U$, $y \in V' \subset V$ such that the restriction

$$f|_{U'} : U' \longrightarrow V'$$

is a diffeomorphism.

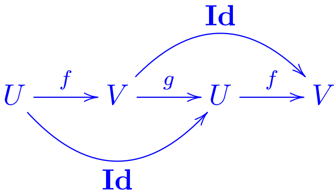
As an excellent example of using the categorical properties of the Jacobi matrices is the following theorem

Theorem (Invariance of dimension)

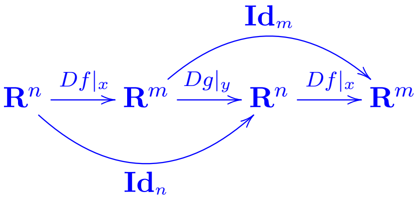
Let $f : U \longrightarrow V$ *be a diffeomorphism from an open domain* $U \subset \mathbf{R}^n$ *to an open domain* $V \subset \mathbf{R}^m$. *Then* $n = m$

We can say that the dimension of the Euclidean space is an invariant with respect to smooth homeomorphisms.
As a matter of fact one can prove that the dimension of the Euclidean space is an invariant with respect to all (non smooth) homeomorphisms.

Proof. The map f is smooth. Let $g = f^{-1} : V \longrightarrow U$ be the inverse map (also smooth). We have the diagram



The Jacobi matrices form the diagram



Hence

$$\mathbf{rank}\ (\mathbf{Id}_n) \leq \min\{\mathbf{rank}\ (Df|_x), \mathbf{rank}\ (Df|_y)\},$$
$$\mathbf{rank}\ (\mathbf{Id}_m) \leq \min\{\mathbf{rank}\ (Df|_x), \mathbf{rank}\ (Df|_y)\},$$

or

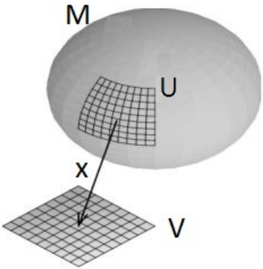
$$n \leq \min\{n, m\}, \quad m \leq \min\{n, m\}.$$

Thus

$$\max\{n, m\} \leq \min\{n, m\},$$
$$\Downarrow$$
$$n = m.$$

∎

A metric space M is called an n *-dimensional manifold* (or simply *manifold*) if any point P of the space M is contained in a neighbourhood $U \subset M$ that is homeomorphic to a domain V of an Euclidean space $\mathbf{R}^n$.



This condition can be formulated in brief as follows: an n -dimensional manifold M is locally homeomorphic to a domain in an Euclidean space $\mathbf{R}^n$. The dimension of M is said to be equal to n , $\dim M = n$.
Thus, if M is an n -dimensional manifold, we can find in M a system of open sets $\{U_\alpha\}$ numbered by finitely (or infinitely) many indices α and a system of homeomorphisms $\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbf{R}^n$.

The system $\{U_\alpha\}$ must cover the space M , i.e.

$$M = \bigcup_\alpha U_\alpha,$$

and the domains V_α may in general, intersects one another.

Fix a Cartesian coordinate system $(x^1, x^2, \ldots, x^n)$ in an Euclidean space $\mathbf{R}^n$.
The system of functions

$$x_\alpha^k = x_\alpha^k(P) = x^k\left(\varphi_\alpha(P)\right)$$

given on an open set U_α is called a *local coordinate system*, and the open set U_α together with a local coordinate system defined on it is called a *chart* of a manifold M .

The homeomorphism $\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbf{R}^n$ is called the *coordinate homeomorphism* and defined by the formula

$$\varphi_\alpha(P) = \left(x^1_\alpha(P), x^2_\alpha(P), \dots, x^n_\alpha(P)\right) \in V_\alpha \subset \mathbf{R}^n.$$

The inverse homeomorphism

$$\varphi^{-1}_\alpha : V_\alpha \longrightarrow U_\alpha \subset M, \quad P = \varphi^{-1}_\alpha\left(x^1_\alpha, x^2_\alpha, \dots, x^n_\alpha\right)$$

is called a *parametrization* of the region $U_\alpha \subset M$.

Thus, a chart is a pair $(U_\alpha, \{x^k_\alpha\})$, where the set U_α is called *the chart domain* and we shall denote the chart, for brevity, only by the first symbol, the chart domain, U_α . A set of charts $\{U_\alpha\}$ covering the entire manifold M is called an *atlas*. It is convenient to number local coordinates of a point $P \in M$ by an additional index α characterizing the chart U_α : $x^k_\alpha = x^k_\alpha(P)$. Since the point P can belong simultaneously to several charts, it has several sets of local coordinates.

The same manifold M can admit distinct atlases. Even though the chart domains, as open sets, remain unchanged, we can alter the local coordinate system in a chart by choosing another coordinate homeomorphism. The set of all chart domains of the atlas is covering of the manifold.
To compare different atlases of charts we consider following definitions

Definition (refinement of atlases)

Consider two atlases of charts $\mathfrak{U}=\{U_\alpha,\{x^k_\alpha\}\}$ and $\mathfrak{V}=\{V_\beta,\{y^k_\beta\}\}$. We say that the atlas $\mathfrak{V}$ refines the atlas $\mathfrak{U}$, $\mathfrak{V}\succ\mathfrak{U}$, if for any β there is $\alpha=\alpha(\beta)$ that

- $V_\beta\subset U_\alpha$.

If additionally

- $y^k_\beta=x^k_\alpha|_{V_\beta}, \quad 1\leq k\leq n$

we say that atlas $\mathfrak{V}$ strictly refines the atlas $\mathfrak{U}$ and write $\mathfrak{V}\gg\mathfrak{U}$, In particular if $\mathfrak{V} \subset \mathfrak{U}$ then $\mathfrak{V} \succ \mathfrak{U}$.

Theorem (Common refinement of atlases)

For any to atlases of charts $\mathfrak{U}'$, $\mathfrak{U}''$ there is an atlas $\mathfrak{V}$ such that

$$\mathfrak{V} \succ \mathfrak{U}', \quad \mathfrak{V} \succ \mathfrak{U}''.$$

Consider a continuous function $f : M \to \mathbf{R}^1$ defined on an n -dimensional manifold M . The restriction of $f|_{U_\alpha}$ to the chart domain U_α can be represent as the composition

$$f|_{U_\alpha}(P) = f_\alpha(x_\alpha^1(P), x_\alpha^2(P), \dots, x_\alpha^n(P))$$

for a proper usual function

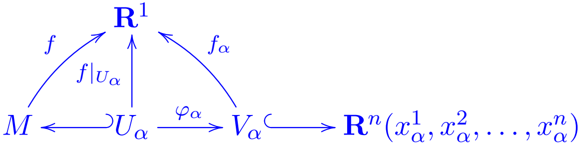
$$f_\alpha : V_\alpha \longrightarrow \mathbf{R}^1$$

of n independent variables $(x_\alpha^1, x_\alpha^2, \dots, x_\alpha^n)$.

The functions $f_\alpha : V_\alpha \longrightarrow \mathbf{R}^1$ is uniquely defined by the formula

$$f_\alpha(x_\alpha^1, x_\alpha^2, \ldots, x_\alpha^n) = f(\varphi_\alpha^{-1}(x_\alpha^1, x_\alpha^2, \ldots, x_\alpha^n))$$

for $(x_\alpha^1, x_\alpha^2, \ldots, x_\alpha^n) \in V_\alpha$ where $\varphi_\alpha : U_\alpha \longrightarrow V_\alpha$ is the coordinate homeomorphism:



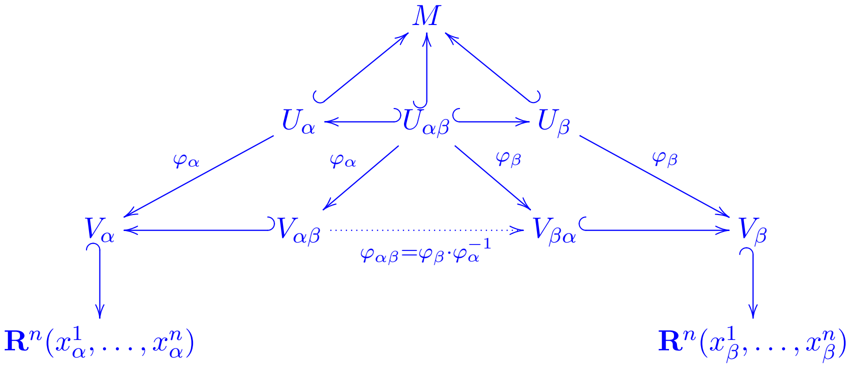
Therefore any function $f : M \to \mathbf{R}^1$ is uniquely defined by the system of functions $\{f_\alpha : V_\alpha \longrightarrow \mathbf{R}^1\}$, that satisfy the condition of compatibility: for any indices α and β and $P \in U_{\alpha\beta} = U_\alpha \cap U_\beta$

$$f_\alpha(x_\alpha^1(P), x_\alpha^2(P), \ldots, x_\alpha^n(P)) = f_\beta(x_\beta^1(P), x_\beta^2(P), \ldots, x_\beta^n(P)).$$

The condition of compatibility

$$f_\alpha(x_\alpha^1(P), x_\alpha^2(P), \dots, x_\alpha^n(P)) = f_\beta(x_\beta^1(P), x_\beta^2(P), \dots, x_\beta^n(P))$$

means that on the intersection $U_{\alpha\beta} = U_\alpha \cap U_\beta$ of two charts U_α and U_β there is a dependence of local coordinates of the point $P \in U_{\alpha\beta}$, namely $(x_\alpha^1(P), x_\alpha^2(P), \dots, x_\alpha^n(P))$ and $(x_\beta^1(P), x_\beta^2(P), \dots, x_\beta^n(P))$.


M
U_α
$U_{\alpha\beta}$
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V_α
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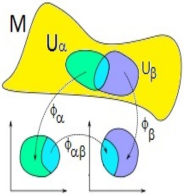
$V_{\beta\alpha}$
V_β
$\varphi_{\alpha\beta}=\varphi_\beta\cdot\varphi_\alpha^{-1}$
$\mathbf{R}^n(x_\alpha^1,\dots,x_\alpha^n)$
$\mathbf{R}^n(x_\beta^1,\dots,x_\beta^n)$

Changing of the coordinate system from a chart U_α to another chart U_β : $\varphi_{\alpha\beta} = \varphi_\beta \cdot \varphi_\alpha^{-1}$

$$(x_\beta^1, x_\beta^2, \dots, x_\beta^n) = \varphi_{\alpha\beta}(x_\alpha^1, x_\alpha^2, \dots, x_\alpha^n),$$

or

$$x_\beta^i = x_\beta^i(x_\alpha^1, x_\alpha^2, \dots, x_\alpha^n), \quad 1 \le i \le n.$$



Definition (Smooth structure)

The atlas of charts $\mathfrak{U} = \{U_\alpha, \varphi_\alpha\}$ represents a *smooth structure* on the manifold M if for any α, β the transition functions $\varphi_{\alpha\beta}$ are smooth. In this case the atlas $\mathfrak{U}$ is called *smooth atlas of charts*.

Two smooth atlases $\mathfrak{U}$ and $\mathfrak{V}$ are called *compatible* ($\mathfrak{U} \approx \mathfrak{V}$) if the union $\mathfrak{W} = \mathfrak{U} \cup \mathfrak{V}$ represents smooth structure.

If $\mathfrak{U}$ is a smooth atlas of charts and $\mathfrak{V} \ll \mathfrak{U}$ then atlas $\mathfrak{V}$ also is smooth atlas compatible with $\mathfrak{U}$.

Theorem (Compatibility relation)

The compatibility relation forms the equivalence relation.
PROOF.

- $\mathfrak{U} \approx \mathfrak{U}$ since $\mathfrak{U} \cup \mathfrak{U} = \mathfrak{U}$.
- $\mathfrak{U} \approx \mathfrak{V} \Leftrightarrow \mathfrak{V} \approx \mathfrak{U}$ since $\mathfrak{U} \cup \mathfrak{V} = \mathfrak{V} \cup \mathfrak{U}$.
- Let $\mathfrak{U} \approx \mathfrak{V}$ and $\mathfrak{V} \approx \mathfrak{W}$. Then $\mathfrak{U} \cup \mathfrak{V}$ and $\mathfrak{V} \cup \mathfrak{W}$ represent smooth structure. It is clear that $\mathfrak{U} \cup \mathfrak{V} \cup \mathfrak{W}$ also represent smooth structure.

∎

The *smooth structure* on the manifold M by definition is a collection of smooth atlases of charts which are pairwise compatible. Given smooth atlas $\mathfrak{U}$ there is maximal smooth atlas of carts $\mathfrak{U}_0$ which is compatible with $\mathfrak{U}$. The maximal smooth atlas of charts $\mathfrak{U}_0$ can be constructed as the union of all smooth atlases of charts compatible with $\mathfrak{U}$.

Example (Nonsmooth atlas)

There is an atlas of charts which is not smooth. As an example consider a real line $\mathbf{R}^1$ with parameter t . Consider two chart domains on $\mathbf{R}^1$, $U_\alpha = \mathbf{R}^1$ and $U_\beta = \mathbf{R}^1$. Define coordinate systems $x_\alpha \equiv t$ on U_α and $x_\beta \equiv t^3$ on U_β . Both maps $x_\alpha : \mathbf{R}^1 \longrightarrow \mathbf{R}^1$ and $x_\beta : \mathbf{R}^1 \longrightarrow \mathbf{R}^1$ are homeomorphisms. The transition function $x_\beta = x_\beta(x_\alpha) = x_\alpha^3$ clearly is smooth but the inverse transition function $x_\alpha = x_\alpha(x_\beta) = \sqrt[3]{x_\beta}$ is not smooth.

The definition of n -dimensional manifolds does not assume existence of smooth structure on manifold. So we say that the n -dimensional manifold is *topological manifold*. In the case of the presence of smooth atlas of charts we say that the topological manifold admits a smooth structure.

Form this point of view a natural question arises: given a topological manifold M if there exists an atlas of charts which represent a smooth structure? There exist topological manifolds which admit no smooth structure, a result proved by Kervaire (1960) [1].
The proof is based on the key invariant called the Kervaire invariant.

In dimensions smaller than 4, there is only one differential structure for each topological manifold. That was proved by Johann Radon for dimension 1 and 2, and by Edwin E. Moise in dimension 3 ([2]. By using obstruction theory, Robion Kirby and Laurent Siebenmann ([3]) were able to show that the number of PL structures for compact topological manifolds of dimension greater than 4 is finite.

John Milnor, Michel Kervaire, and Morris Hirsch proved that the number of smooth structures on a compact PL manifold is finite and agrees with the number of differential structures on the sphere for the same dimension (see the book Asselmeyer-Maluga, Brans, [1], chapter 7). By combining these results, the number of smooth structures on a compact topological manifold of dimension not equal to 4 is finite.

Dimension 4 is more complicated. For compact manifolds, results depend on the complexity of the manifold as measured by the second Betti number b_2 . For large Betti numbers $b_2 > 18$ in a simply connected 4-manifold, one can use a surgery along a knot or link to produce a new differential structure. With the help of this procedure one can produce countably infinite many differential structures.

But even for simple spaces like $\mathbf{S}^4$, $\mathbf{C}P^2, \ldots$ one doesn't know the construction of other differential structures. For non-compact 4-manifolds there are many examples like $\mathbf{R}^4, S^3 \times \mathbf{R}, M^4 \setminus \{*\}, \ldots$ having uncountably many differential structures.

Definition (smooth function)

Let M be a smooth manifold. The continuous function $f : M \longrightarrow \mathbf{R}^1$ is called smooth function if for any chart $(U_\alpha, \varphi_\alpha)$ the function

$$f_\alpha(x^1_\alpha, x^2_\alpha, \ldots, x^n_\alpha) = f(\varphi^{-1}_\alpha(x^1_\alpha, x^2_\alpha, \ldots, x^n_\alpha))$$

is smooth.

The smoothness of the function f_α is compatible with smooth structure of the manifold since

$$f_\alpha(x_\alpha^1, \dots, x_\alpha^n) =$$

$$= f_\beta(x_\beta^1(x_\alpha^1, \dots, x_\alpha^n), \dots, x_\beta^n(x_\alpha^1, \dots, x_\alpha^n))$$

that is a composition of smooth functions.

Similar if

$$f : M_1 \longrightarrow M_2$$

is a continuous map of smooth manifolds then one can define what does mean that map f is smooth. Let $\{U^1_\alpha, \varphi_\alpha\}$ and $\{U^2_\beta, \psi_\beta\}$ be smooth atlases of charts on manifolds M_1 and M_2 .

The restriction

$$f : U^1_\alpha \cap f^{-1}(U^2_\beta) \longrightarrow U^2_\beta$$

can be expressed in the term of local coordinate systems:

$$\begin{array}{ccccc}
U^1_\alpha & \hookleftarrow & (U^1_\alpha \cap f^{-1}(U^2_\beta)) & \xrightarrow{\ f\ } & U^2_\beta \\
\downarrow{\scriptstyle \varphi_\alpha} & & \downarrow & & {\scriptstyle \psi_\beta}\downarrow \\
V_\alpha & \hookleftarrow & \varphi_\alpha(U^1_\alpha \cap f^{-1}(U^2_\beta)) & \xrightarrow{\ f_{\alpha\beta}\ } & V_\beta
\end{array}$$

Where the function $f_{\alpha\beta}$ satisfies the condition

$$f_{\alpha\beta}(x^1_\alpha(P), x^2_\alpha(P), \dots, x^n_\alpha(P)) = \psi_\beta(f(P))$$

or

$$\begin{gathered} y^j_\beta = y^j_\beta(x^1_\alpha, x^2_\alpha, \dots, x^n_\alpha) = \\ = y^j_\beta\left(\psi_\beta \circ f \circ \varphi^{-1}_\alpha(x^1_\alpha, x^2_\alpha, \dots, x^n_\alpha)\right), \\ 1 \leq j \leq m. \end{gathered}$$

Definition (Smooth map of smooth manifolds)

We say that the map

$$f : M_1 \longrightarrow M_2$$

is smooth if for any point $P \in M_1$ and any chart $U_\alpha \ni P$ with local coordinates $(x_\alpha^1, x_\alpha^2, \ldots, x_\alpha^n)$ and any chart $V_\beta \ni f(P) = Q$ with local coordinates $(y_\beta^1, y_\beta^2, \ldots, y_\beta^m)$ the functions

$$y_\beta^j = y_\beta^j(x_\alpha^1, x_\alpha^2, \ldots, x_\alpha^n), \quad 1 \leq j \leq m$$

are smooth.

Definition (diffeomorphism of manifolds)

We say that the map

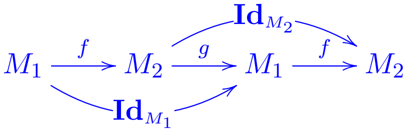
$$f : M_1 \longrightarrow M_2$$

is a diffeomorphism if f is a homeomorphism and both f and f^{-1} are smooth maps.

Another equivalent definition says that two smooth manifolds M_1 and M_2 are diffeomorphic if there are two smooth maps

$$f: M_1 \longrightarrow M_2, \quad g: M_2 \longrightarrow M_1$$

for which two possible compositions $f \circ g$ and $g \circ f$ are identities:



Proposition (Dimension of diffeomorphic manifolds)

If two manifolds are diffeomorphic,

$$M_1 \xrightarrow{\approx} M_2,$$

then

$$\dim M_1 = \dim M_2.$$

The function f has the smoothness of the class C^r , $r \geq 0$, if the function f and all its derivatives

$$\frac{\partial^{|\alpha|} f}{(\partial x)^{\alpha}}, \quad |\alpha| \leq r,$$

are continuous.

Definition (Support of the function)

For continuous function $f : M \longrightarrow \mathbf{R}$ the closed set $\mathbf{supp}\ (f) \subset M$ is called *support* of the function f if

$$\mathbf{supp}\ (f) = \overline{\{P \in M : f(P) \neq 0\}}.$$

Theorem (partition of unit)

Let M be a smooth compact manifold, $\mathfrak{U} = \{U_\alpha\}$ be an atlas of charts. There is a system of smooth functions $f_\alpha : M \longrightarrow [0, 1]$ such that

- $\mathbf{supp}\ f_\alpha \subset U_\alpha$,
- $\sum\limits_\alpha f_\alpha(P) \equiv 1, \quad P \in M$.

The system of functions $\{f_\alpha\}$ is called a *partition of unit* which is subordinated to the atlas of charts $\mathfrak{U}$.

Theorem (Urysohn lemma)

Let $F_1, F_2 \subset M$ be two closed subsets of a smooth manifold M , $F_1 \cap F_2 = \emptyset$. There is a smooth function $f : M \longrightarrow [0, 1]$ such that

- $f|_{F_1} \equiv 0$,
- $f|_{F_2} \equiv 1$.

Definition (Orientable manifold)

A smooth manifold M is called orientable if there is an atlas of charts $\mathcal{U} = \{U_\alpha\}$ such that for any indices α, β

$$\det \begin{pmatrix} \frac{\partial x_\alpha^1}{\partial x_\beta^1} & \cdots & \frac{\partial x_\alpha^1}{\partial x_\beta^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_\alpha^n}{\partial x_\beta^1} & \cdots & \frac{\partial x_\alpha^n}{\partial x_\beta^n} \end{pmatrix} > 0.$$

Under the condition the atlas $\mathcal{U}$ is called orientable atlas. The orientable atlas defines an orientation of the manifold M . Two orientable atlases $\mathcal{U}$ and $\mathcal{V}$ define the same orientation of the manifold M iff the union $\mathcal{U} \cup \mathcal{V}$ is orientable atlas.

Consider a smooth curve on a manifold M :

$$\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$$

Let $P_0 = \gamma(0) \in M$ be the point through which the curve passes. Let $\{x^i\}$ be a local coordinate system. Then

$$\gamma(t) = \left(x^1(t), x^2(t), \ldots, x^n(t)\right)$$

Definition (Tangent vector)

$$\frac{d\gamma}{dt}(0) = (\frac{dx^1}{dt}|_{t=0}, \frac{dx^2}{dt}|_{t=0}, \ldots, \frac{dx^n}{dt}|_{t=0})$$

is called the tangent vector to the curve γ in the point P_0 . $\{x^i(0)\}$ is called components of the tangent vector $\frac{d\gamma}{dt}(0)$ with respect to a local coordinate system $\left(x^1, x^2, \ldots, x^n\right)$.

For a pair of coordinate systems $\{x^i_\alpha\}$ and $\{x^j_\beta\}$ one has a tensor law of changing of components of the tangent vector

$$\frac{dx^i_\alpha}{dt}|_0 = \frac{\partial x^i_\alpha}{\partial x^j_\beta}(P_{t=0}) \cdot \frac{dx^j_\beta}{dt}|_{t=0}$$

Definition (abstract tangent vector to manifold)

Let M be a smooth manifold of the dimension n , $P \in M$ be a point. A tangent vector ξ to the manifold M in the point $P \in M$ is system of components $\xi = \{\xi^i_\alpha\}$ associated with the coordinate system $\{x^i_\alpha\}$ that satisfies the tensor law of changing components for two coordinate systems $\{x^i_\alpha\}$ and $\{x^j_\beta\}$

$$\xi^i_\alpha = \frac{\partial x^i_\alpha}{\partial x^j_\beta}(P) \cdot \xi^j_\beta.$$

For two abstract tangent vectors $\xi = \{\xi^i_\alpha\}$ and $\eta = \{\eta^i_\alpha\}$ in the same point $P \in M$ one can define the linear combination

$$\lambda\xi + \mu\eta = \{\lambda\xi^i_\alpha + \mu\eta^i_\alpha\}.$$

It is clear that the components $\{\lambda\xi^i_\alpha + \mu\eta^i_\alpha\}$ satisfy the tensor law.

The family of all tangent vector in the point $P \in M$ to the manifold M forms the vector space $T_P(M)$ with respect to the linear combination.

Definition (Tangent space)

The space $T_P(M)$ is called the tangent space to the manifold $T_P(M)$ in the point $P \in M$.

Three definitions of tangent vectors

- Tangent vector as a sheaf of osculating curves.
- Tangent vector as a tensor.
- Tangent vector as a differentiation operator.

Definition (osculating curves)

Two curves $\gamma' : (-\varepsilon, \varepsilon) \longrightarrow M$ and $\gamma'' : (-\varepsilon, \varepsilon) \longrightarrow M$ are osculating in the point $P_0 = \gamma'(0) = \gamma''(0)$ if for any coordinate system one has $\{x^i_\alpha\}$

$$\sum \left(x^i_\alpha(\gamma'(t)) - x^i_\alpha(\gamma''(t))\right)^2 = O(t^2) \quad (t \longrightarrow 0).$$

The condition does not depend of the choice of the coordinate system.

Theorem (Criteria of osculating curve)

Two curves $\gamma' : (-\varepsilon, \varepsilon) \longrightarrow M$ and $\gamma'' : (-\varepsilon, \varepsilon) \longrightarrow M$ are osculating in the point $P_0 = \gamma'(0) = \gamma''(0)$ if and only if

$$\frac{d\gamma'}{dt}|_{t=0} = \frac{d\gamma''}{dt}|_{t=0}$$

PROOF.

Definition (Differentiation operator)

Let $\mathcal{C}^\infty(M)$ be the linear space of all smooth functions on a smooth manifold M . A linear operator

$$D : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$$

is called a *differentiation operator* if it satisfies so called *Leibnitz law* with respect to the operation of pointwise multiplication:

$$D(fg) = D(f)g + fD(g)$$

or more detailed pointwise

$$D(f \cdot g)(P) = D(f)(P) \cdot g(P) + f(P) \cdot D(g)(P),$$

$$f, g \in \mathcal{C}^\infty(M), \quad P \in M.$$

Theorem (Constant function)

Let D be a differentiation operator, and $f(P) \equiv 1$. Then

$$D(f) \equiv 0.$$

Theorem (Preserving of support)

Let D be a differentiation operator. Then

$$\mathbf{supp}\ (D(f)) = \mathbf{supp}\ (f).$$

Tangent bundle
Tangent space

Let $i : U \hookrightarrow M$ be an open subset, $i^* : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(U)$ be the restriction map. Let $D : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$ be a differentiation operator.

Theorem (Restriction of differentiation)

There is a unique differentiation operator

$$D_U : \mathcal{C}^\infty(U) \longrightarrow \mathcal{C}^\infty(U)$$

such that the following diagram is commutative

$$\begin{array}{ccc}
\mathcal{C}^\infty(M) & \xrightarrow{D} & \mathcal{C}^\infty(M) \\
\downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} \\
\mathcal{C}^\infty(U) & \xrightarrow{D_U} & \mathcal{C}^\infty(U)
\end{array}$$

Theorem (3d definition of the tangent vector)

Each tangent vector $\xi \in T_P(M)$ can be uniquely described as a differentiation operator

$$\frac{\partial}{\partial \xi} = D_P : \mathcal{C}^\infty(M) \longrightarrow \mathbf{C}$$

which satisfies the Leibnitz law

$$D_P(fg) = D_P(f)g(P) + f(P)D_P(g) \in \mathbf{C}.$$

In local coordinate system $(x_\alpha^1, x_\alpha^2, \dots, x_\alpha^n)$ the operator $\frac{\partial}{\partial \xi}$ is described by

$$\frac{\partial}{\partial \xi}(f) = \sum_{i=1}^{n} \xi_\alpha^i \frac{\partial f}{\partial x_\alpha^i}(x_\alpha^1(P), x_\alpha^2(P), \dots, x_\alpha^n(P))$$

where

$$\xi_\alpha^i = D_P(x_\alpha^i).$$

Consider the space $TM = \coprod_{P \in M} T_P M$ with a proper topology locally generated by the Cartesian product:

$$\begin{array}{ccc} & TM & \\ & \uparrow & \\ \varphi_\alpha : & TU_\alpha & \xrightarrow{\approx} \mathbf{C}^n \times U_\alpha \end{array}$$

The tangent bundle has a natural smooth structure of a manifold of dimension $\dim TM = 2n$:

$$\begin{cases} x_\beta^j = x_\beta^j(x_\alpha^1, x_\alpha^2, \dots, x_\alpha^n), \\ \\ \xi_\beta^j = \xi_\alpha^i \cdot \frac{\partial x_\beta^j}{\partial x_\alpha^i}(x_\alpha^1, x_\alpha^2, \dots, x_\alpha^n). \end{cases}$$

Let $f: M_1 \longrightarrow M_2$ be a smooth map. Then

$$Df: TM_1 \longrightarrow TM_2$$

is called the differential of the map f and is defined using one of 3 definitions of the tangent vector:

- Tangent vector as a tensor:

$$\xi = \{\xi^i_\alpha\}, \eta = Df(\xi) = \{\eta^j_\beta\},$$
$$\eta^j_\beta = \xi^i_\alpha \frac{\partial y^j_\beta}{\partial x^i_\alpha}$$

- Tangent vector as the velocity vector of a curve:

$$\gamma = \gamma(t), \quad \gamma(0) = P \in M_1, \quad Q = f(P);$$
$$\xi = \frac{d\gamma}{dt}\Big|_{t=0},$$
$$\eta = Df(\xi) = \frac{d(f(\gamma))}{dt}\Big|_{t=0}$$

- Tangent vector as a differentiation operator:

$$\xi = D : \mathcal{C}^\infty(M_1) \longrightarrow \mathbf{C},$$
$$\eta = Df(\xi) = D' : \mathcal{C}^\infty(M_2) \longrightarrow \mathbf{C},$$
$$D'(g) = D(g \circ f) = D(f^*(g)), \quad g \in \mathcal{C}^\infty(M_2).$$

- Comparison with derivative:

$$\xi \in T_P M_1, \quad \eta = Df(\xi) \in T_Q(M_2),$$

$$\frac{\partial}{\partial\eta}(g) = \frac{\partial}{\partial(Df(\xi))}(g) = \frac{\partial}{\partial\xi}(f^*(g)),$$

$$f^*(g)(P) = g(f(P)), \quad P \in M_1.$$

Definition (Regular point)

Let $f:M_1 \longrightarrow M_2$ be a smooth map, $Q_0 \in M_2$ and $N=f^{-1}(Q_0)\subset M_1$. The point $Q_0 \in M_2$ is called *regular* value of the map f if for any $P \in N = f^{-1}(Q_0)$ the differential

$$Df : T_P M_1 \longrightarrow T_{Q_0} M_2$$

is surjective (or epimorphism). A point $P \in N = f^{-1}(Q_0)$ is called regular point. So regular value $Q_0 \in M_2$ is regular point if each inverse image $P \in N = f^{-1}(Q_0)$ is regular point. If the point $P \in N = f^{-1}(Q_0)$ is not regular then it is called *critical* point. Consequently Q_0 is called *critical* value.

If $f^{-1}(Q_0) \neq \emptyset$ then

$$\dim M_1 \geq \dim M_2.$$

So if $\dim M_1 < \dim M_2$ and $Q_0 \in M_2$ is regular point then

$$f^{-1}(Q_0) = \emptyset.$$

Theorem (Implicit function theorem)

Let $f: M_1 \longrightarrow M_2$ be a smooth map, $Q_0 \in M_2$ be a regular point of the map f and $N = f^{-1}(Q_0) \subset M_1$. Then $N \subset M_1$ is smooth manifold. More of that each local coordinate system on the manifold $N \subset M_1$ can be choose as a part of coordinate system on the manifold M_1 .
If $N \neq \emptyset$ then

$$\dim N = \dim f^{-1}(Q_0) = \dim M_1 - \dim M_2.$$

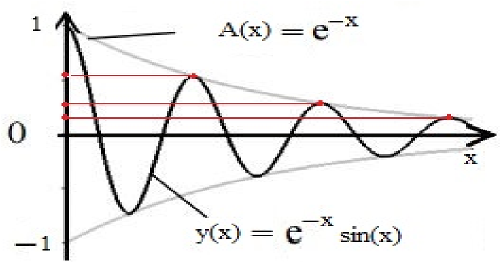
Theorem (Open set of regular points)

Let $f : M_1 \longrightarrow M_2$ be a smooth map of compact manifolds. The set $R \in M_2$ of all regular points of the map f is open. If the manifold M_1 is not compact then the set $R \in M_2$ of regular points may be non open.

Example (Non open set of regular points)

$$y = f(x) = e^{-x} \sin x, \quad x \in (-\infty, +\infty)$$

Example (Non open set of regular points)



Example (Non open set of regular points)

Singular points:

$$y = f(x) = e^{-x}\sin x,$$

$$f'(x) = 0 \Leftrightarrow e^{-x}(\cos x - \sin x) = 0 \Leftrightarrow (\cos x - \sin x) = 0,$$

$$x_k = \frac{\pi}{4} + k\pi;$$

$$y_k = f(x_k) = e^{-(\frac{\pi}{4}+k\pi)}\sin(\frac{\pi}{4} + k\pi)\longrightarrow 0.$$

Theorem (Implicit function theorem)

Let $f : M_1 \longrightarrow M_2$ be a smooth map of compact manifolds, $R \in M_2$ be the open set of all regular points of the map f . Then for each $Q_0 \in R \subset M_2$ there is a neighborhood $U \subset R$ such that $f^{-1}(U) \subset M_1$ is diffeomorphic to the cartesian product

$$f^{-1}(U) \approx U \times N = U \times f^{-1}(Q_0).$$

Definition (Zero measure subsets)

A subset $A \subset \mathbf{R}^n$ has measure zero if it may be covered by a countable collection of balls $B^n(x, r)$ having arbitrarily small total volume. In such a case, $\mathbf{R}^n \setminus A$ is everywhere dense (i.e., it intersects every non-empty open set).

Theorem (Image of Zero measure subsets)

Let $U \subset \mathbf{R}^n$ be an open subset; let $f : U \longrightarrow \mathbf{R}^n$ be differentiable. If $A \subset U$ has measure zero, so does $f(A)$.

Theorem (The Sard lemma)

The set of critical values of any differentiable map has measure zero.

Degree of map. Definition

Consider two orientable compact manifolds M and N , $\dim M = \dim N$, and a smooth map

$$f : M \longrightarrow N.$$

Let $y_0 \in N$ be a regular point of the map f . By definition the degree of the map f is the integer

$$\deg f = \sum_{x \in f^{-1}(y_0)} \mathbf{sign}\ \det df|_x .$$

Theorem (Homotopy invariance of the degree)

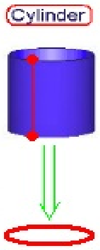
The degree of the map $f : M \longrightarrow N$ does not depend of regular point $y_0 \in N$ and of smooth homotopy of the map f .

Theorem (Fundamental theorem of algebra)

The fundamental theorem of algebra states that every non-constant single-variable polynomial with complex coefficients has at least one complex root.

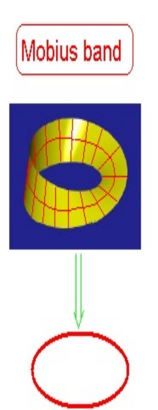
Cylinder

The surface of the cylinder can be seen as a disjoint union of a family of line segments continuously parametrized by points of a circle.



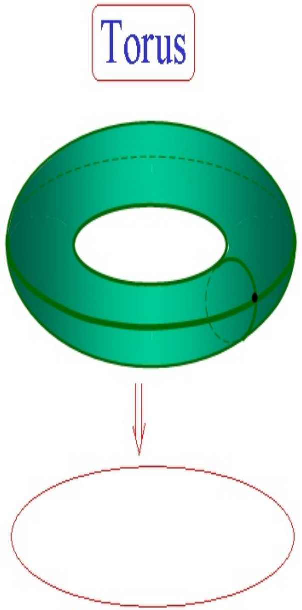
Móbius band

The Möbius band can be presented in similar way.



Torus

The two dimensional torus embedded in the three dimensional space can presented as a union of a family of circles (meridians) parametrized by points of another circle (a parallel).



The examples considered above share two important properties:

- any two fibers are homeomorphic,
- despite the fact that the whole space cannot be presented as a Cartesian product of a fiber with the base (the parameter space), if we restrict our consideration to some small region of the base the part of the fiber space over this region is such a Cartesian product.

The two properties above are the basis of the following definition.

Locally trivial bundles
Definition

Definition (Locally trivial bundle)

Let E and B be two topological spaces with a continuous map

$$\begin{array}{c} E \\ \downarrow p \\ B. \end{array}$$

The map p is said to define a *locally trivial bundle* if there is a topological space F such that for any point $x \in B$ there is a neighborhood $U \ni x$ for which the inverse image $p^{-1}(U)$ is homeomorphic to the Cartesian product $F \times U$.

Definition (Locally trivial bundle)

Moreover, it is required that the homeomorphism

$$\varphi : F \times U \longrightarrow p^{-1}(U)$$

preserves fibers, it is a 'fiberwise' map, that is, the following equality holds:

$$\varphi(F \times x) = p^{-1}(x) \subset p^{-1}(U) \subset E, \quad x \in U.$$

Definition (Locally trivial bundle)

In other words the following diagram is commutative

$$\begin{array}{ccccc} F\times U & \xrightarrow{\varphi} & p^{-1}(U) & \hookrightarrow & E \\ \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ U & = & U & \hookrightarrow & B. \end{array}$$

Definition (Locally trivial bundle)

The space E is called *total space of the bundle* or *the fiberspace*, the space B is called *the base of the bundle*, the space F is called *the fiber of the bundle* and the mapping p is called *the projection*.

Locally trivial bundles
Definition

A problem in the theory of fiber spaces is to classify the family of all locally trivial bundles with fixed base B and fiber F .

Definition (Isomorphic bundles)

Two locally trivial bundles $p : E \longrightarrow B$ and $p' : E' \longrightarrow B$ are considered to be isomorphic if there is a homeomorphism $\psi : E \longrightarrow E'$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\psi} & E' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ B & = & B \end{array}$$

is commutative.

Isomorphic fibers

It is clear that the homeomorphism ψ gives a homeomorphism of fibers $F \longrightarrow F'$.

Remark

To specify a locally trivial bundle it is not necessary to be given the total space E explicitly. It is sufficient to have a base B , a fiber F and a family of mappings such that the total space E is determined 'uniquely' (up to isomorphisms of bundles).

Atlas of charts

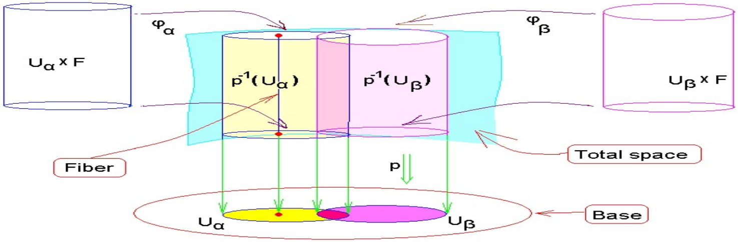
According to the definition of a locally trivial bundle, the base B can be covered by a family of open sets $\{U_\alpha\}$ such that each inverse image $p^{-1}(U_\alpha)$ is fiberwise homeomorphic to the Cartesian product $F \times U_\alpha$. This gives a system of fiberwise homeomorphisms

$$\begin{array}{ccc} F \times U_\alpha & \xrightarrow{\varphi_\alpha} & E \\ \downarrow & & \downarrow \\ U_\alpha & \hookrightarrow & B \end{array}$$

Intersection

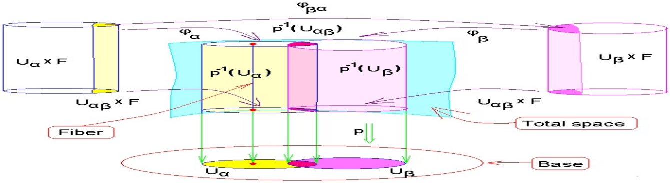
Since the homeomorphisms φ_α preserve fibers it is clear that for any open subset $V \subset U_\alpha$ the restriction of φ_α to $F \times V$ establishes the fiberwise homeomorphism of $F \times V$ onto $p^{-1}(V)$. Hence on the intersection of two charts $U_{\alpha\beta} = U_\alpha \cap U_\beta$ there are two fiberwise homeomorphisms

$$
\begin{array}{ccccccc}
 & & & E & & & \\
 & & & \uparrow & & & \\
 & F \times U_{\alpha\beta} & \xrightarrow{\varphi_\alpha} & p^{-1}(U_{\alpha\beta}) & \xleftarrow{\varphi_\beta} & F \times U_{\alpha\beta} & \\
 & \downarrow & & \downarrow & & \downarrow & \\
U_\alpha & \hookleftarrow U_{\alpha\beta} & = & U_{\alpha\beta} & = & U_{\alpha\beta} \hookrightarrow & U_\beta
\end{array}
$$



φ α
φ β
U α x F
p⁻¹(U α)
p⁻¹(U β)
U β x F
Fiber
Total space
p
Base
U α
U β

Let $\varphi_{\beta\alpha}$ denote the homeomorphism $\varphi_\beta^{-1}\varphi_\alpha$ which maps $(U_\alpha \cap U_\beta) \times F$ onto itself.



The locally trivial bundle is uniquely determined by the following collection: the base B , the fiber F , the covering $\{U_\alpha\}$ and the homeomorphisms

$$\varphi_{\beta\alpha} : F \times (U_\alpha \cap U_\beta) \longrightarrow F \times (U_\alpha \cap U_\beta).$$

The total space E should be thought of as a union of the Cartesian products $F \times U_\alpha$ with some identifications induced by the homeomorphisms $\varphi_{\beta\alpha}$.

By analogy with the terminology for smooth manifolds, the open sets U_α are called *charts*, the family $\{U_\alpha\}$ is called *the atlas of charts*, the homeomorphisms φ_α are called *the coordinate homeomorphisms, or trivializations* and the $\varphi_{\beta\alpha}$ are called *the transition functions* or *the sewing functions*.
Sometimes the collection $\{U_\alpha, \varphi_\alpha\}$ is called the atlas. Thus any atlas determines a locally trivial bundle. Different atlases may define isomorphic bundles but, beware, not any collection of homeomorphisms φ_α forms an atlas.

For the classification of locally trivial bundles, families of homeomorphisms $\varphi_{\beta\alpha}$,

$$\varphi_{\beta\alpha} : F \times (U_\alpha \cap U_\beta) \longrightarrow F \times (U_\alpha \cap U_\beta).$$

that actually determine bundles should be selected and then separated into classes which determine isomorphic bundles. In particular the homeomorphisms $\varphi_{\beta\alpha}$ should be selected to be transition functions for some locally trivial bundle:

$$\varphi_{\beta\alpha} = \varphi_\beta^{-1}\varphi_\alpha.$$

In the case for any three indices α, β, γ on the intersection $F \times (U_{\alpha\beta\gamma}) = F \times (U_\alpha \cap U_\beta \cap U_\gamma)$ the following relation holds:

$$\varphi_{\alpha\gamma}\varphi_{\gamma\beta}\varphi_{\beta\alpha} = \mathbf{Id},$$

where **Id** is the identity homeomorphism and for each α ,

$$\varphi_{\alpha\alpha} = \mathbf{Id}.$$

In particular

$$\varphi_{\alpha\beta}\varphi_{\beta\alpha} = \mathbf{Id},$$

thus

$$\varphi_{\alpha\beta} = \varphi_{\beta\alpha}^{-1}.$$

Hence for an atlas the homeomorphisms $\varphi_{\beta\alpha}$ should satisfy the condition of *cocyclicity*

$$\varphi_{\alpha\alpha} = \mathbf{Id}, \quad \varphi_{\alpha\gamma}\varphi_{\gamma\beta}\varphi_{\beta\alpha} = \mathbf{Id}.$$

These conditions are sufficient for a locally trivial bundle to be reconstructed from the base B , fiber F , atlas $\{U_\alpha\}$ and homeomorphisms $\{\varphi_{\beta\alpha}\}$.

To see this, let

$$E' = \coprod_{\alpha} (F \times U_\alpha)$$

be the disjoint union of the spaces $F \times U_\alpha$. Introduce the following relation: the point $(f, x) \in F \times U_\alpha$ is related to the point $(g, y) \in F \times U_\beta$,

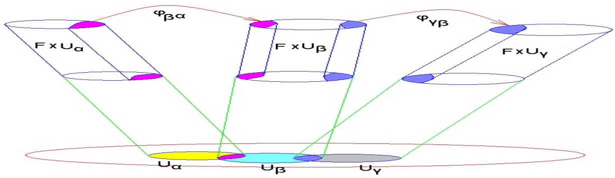
$$(f, x) \sim (g, y),$$

iff

$$x = y \in U_\alpha \cap U_\beta$$

and

$$(g, y) = \varphi_{\beta\alpha}(f, x).$$



φβα
φγβ
F x Uα
F x Uβ
F x Uγ
Uα
Uβ
Uγ

The conditions of cocyclicity

$$\varphi_{\alpha\alpha} = \mathbf{Id}, \quad \varphi_{\alpha\gamma}\varphi_{\gamma\beta}\varphi_{\beta\alpha} = \mathbf{Id}.$$

guarantee that this is an equivalence relation, that is, the space E' is partitioned into disjoint classes of equivalent points. Let $E = E'/\sim$ be the quotient space determined by this equivalence relation, that is, the set whose points are equivalence classes. Give E the quotient topology with respect to the projection

$$\pi : E' \longrightarrow E = E'/\sim$$

which associates to a (f, x) its the equivalence class. In other words, the subset $G \subset E$ is called open iff $\pi^{-1}(G)$ is open set.

There is the natural mapping p' from E' to B ,

$$\begin{array}{ccccc} E' & = & \coprod_\alpha (F\times U_\alpha) & \supset & F\times U_\alpha \\ \downarrow p' & & \downarrow p' & & \downarrow \mathbf{pr}_2 \\ B & = & B & \supset & U_\alpha \end{array}$$

Namely,

$$p'(f,x) = x.$$

Clearly the mapping p' is continuous and equivalent points maps to the same image. Hence the mapping p' induces a map

$$p : E \longrightarrow B$$

which associates to an equivalence class the point assigned to it by p':

$$\begin{array}{ccc} E' & \xrightarrow{\pi} & E = E'/\sim \\ \downarrow p' & & \downarrow p \\ B & = & B \end{array}$$

The mapping p is continuous.

It remains to construct the coordinate homeomorphisms. Put $\varphi_\alpha = \pi_{F\times U_\alpha}$

$$\begin{array}{ccc} E' = \coprod_\alpha (F\times U_\alpha) & \xrightarrow{\ \pi\ } & E \\ \uparrow & & \| \\ F\times U_\alpha & \xrightarrow[\ \varphi_\alpha\ ]{} & E \end{array}$$

Each class $z \in p^{-1}(U_\alpha)$ has a unique representative $(f,x) \in F \times U_\alpha$. Hence φ_α is a one to one mapping onto $p^{-1}(U_\alpha)$. By virtue of the quotient topology on E the mapping φ_α is homeomorphisms. It is easy to check that

$$\varphi_\beta^{-1}\varphi_\alpha = \varphi_{\beta\alpha}.$$

Theorem (Cocycle of transition functions)

So we have shown that locally trivial bundles may be defined by atlas of charts $\{U_\alpha\}$ and a family of homeomorphisms $\{\varphi_{\beta\alpha}\}$,

$$\varphi_{\beta\alpha} : F \times (U_\alpha \cap U_\beta) \longrightarrow F \times (U_\alpha \cap U_\beta).$$

satisfying the conditions of cocyclicity.

$$\varphi_{\alpha\alpha} = \mathbf{Id}, \quad \varphi_{\alpha\gamma}\varphi_{\gamma\beta}\varphi_{\beta\alpha} = \mathbf{Id}.$$

Let us now determine when two atlases define isomorphic bundles. First of all notice that if two bundles $p : E \longrightarrow B$ and $p' : E' \longrightarrow B$ with the same fiber F have the same transition functions $\{\varphi_{\beta\alpha}\}$ then these two bundles are isomorphic. Indeed, let

$$\varphi_\alpha : F \times U_\alpha \longrightarrow p^{-1}(U_\alpha).$$

$$\psi_\alpha : F \times U_\alpha \longrightarrow p'^{-1}(U_\alpha).$$

be the corresponding coordinate homeomorphisms and assume that

$$\varphi_{\beta\alpha} = \varphi_\beta^{-1}\varphi_\alpha = \psi_\beta^{-1}\psi_\alpha = \psi_{\beta\alpha}.$$

Then

$$\varphi_\alpha\psi_\alpha^{-1} = \varphi_\beta\psi_\beta^{-1}.$$

We construct a homeomorphism

$$h : E' \longrightarrow E.$$

Let $u \in E'$. The atlas $\{U_\alpha\}$ covers the base B and hence there is an index α such that $u \in p'^{-1}(U_\alpha)$. Set

$$h(u) = \varphi_\alpha\psi_\alpha^{-1}(u).$$

It is necessary to establish that the value of $h(u)$ is independent of the choice of index α . If $u \in p'^{-1}(U_\beta)$ also and since $\varphi_\alpha \psi_\alpha^{-1} = \varphi_\beta \psi_\beta^{-1}$ then

$$\varphi_\beta \psi_\beta^{-1}(u) = \varphi_\alpha \psi_\alpha^{-1}(x).$$

Hence the definition of $h(x)$ is independent of the choice of chart. Continuity and other necessary properties are evident.

Further, given an atlas $\{U_\alpha\}$ and coordinate homeomorphisms $\{\varphi_\alpha\}$, $F \times U_\alpha \xrightarrow{\varphi_\alpha} p^{-1}(U_\alpha) \hookrightarrow E$, if $\{V_\beta\}$ is a finer atlas (that is, $V_\beta \subset U_\alpha$ for some $\alpha = \alpha(\beta)$) then for the atlas $\{V_\beta\}$, the coordinate homeomorphisms are defined in a natural way

$$\varphi'_\beta = \varphi_{\alpha(\beta)}|_{F \times V_\beta} : F \times V_\beta \longrightarrow p^{-1}(V_\beta).$$

The transition functions $\varphi'_{\beta_1,\beta_2}$ for the new atlas $\{V_\beta\}$ are defined using restrictions

$$\begin{aligned}\varphi'_{\beta_1,\beta_2} &= \varphi_{\alpha(\beta_1),\alpha(\beta_2)}|_{F\times(V_{\beta_1}\cap V_{\beta_2})}: \\ &: F\times(V_{\beta_1}\cap V_{\beta_2})\longrightarrow F\times(V_{\beta_1}\cap V_{\beta_2}).\end{aligned}$$

Theorem (Common refinement)

For two atlases $\{U_\alpha\}$ and $\{V_\beta\}$ there is a common refinement $\{W_\gamma\}$,

$$W_\gamma \subset U_{\alpha(\gamma)} \cap V_{\beta(\gamma)}.$$

Proof.

$$W_\gamma = U_\alpha \cap V_\beta, \quad \gamma = (\alpha, \beta).$$

Thus if there are two atlases and transition functions for two bundles, with a common refinement, that is, a finer atlas with transition functions given by restrictions, it can be assumed that the two bundles have the same atlas. If $\varphi_{\beta\alpha}$, $\varphi'_{\beta\alpha}$ are two systems of the transition functions (for the same atlas), giving isomorphic bundles then the transition functions $\varphi_{\beta\alpha}$, $\varphi'_{\beta\alpha}$ must be related.

Theorem (Homology of transition function cocycle)

Two systems of the transition functions $\varphi_{\beta\alpha}$, and $\varphi'_{\beta\alpha}$ define isomorphic locally trivial bundles iff there exist fiber preserving homeomorphisms

$$h_\alpha : F \times U_\alpha \longrightarrow F \times U_\alpha$$

such that

$$\varphi_{\beta\alpha} = h_\beta^{-1} \varphi'_{\beta\alpha} h_\alpha.$$

Proof.

Suppose that two bundles $p : E \longrightarrow B$ and $p' : E' \longrightarrow B$ with the coordinate homeomorphisms φ_α and φ'_α are isomorphic. Then there is a homeomorphism $\psi : E' \longrightarrow E$. Let

$$h_\alpha = \varphi'^{-1}_\alpha \psi^{-1} \varphi_\alpha.$$

Then

$$h^{-1}_\beta \varphi'_{\beta\alpha} h_\alpha = (\varphi^{-1}_\beta \psi \varphi'_\beta) \varphi'_{\beta\alpha} (\varphi'^{-1}_\alpha \psi^{-1} \varphi_\alpha) =$$

$$= (\varphi^{-1}_\beta \psi \varphi'_\beta)(\varphi'^{-1}_\beta \varphi'_\alpha)(\varphi'^{-1}_\alpha \psi^{-1} \varphi_\alpha) =$$

$$= (\varphi^{-1}_\beta \psi)(\psi^{-1} \varphi_\alpha) = \varphi_{\beta\alpha}.$$

Conversely, if the relation

$$\varphi_{\beta\alpha} = h_\beta^{-1}\varphi'_{\beta\alpha}h_\alpha.$$

holds, put

$$\psi = \varphi_\alpha h_\alpha^{-1}\varphi_\alpha'^{-1}.$$

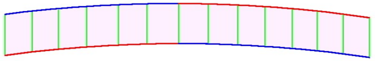
The definition ψ is valid for the subspaces $p'^{-1}(U_\alpha)$ covering E'. To prove that the right hand sides of the definition ψ coincide on the intersection $p'^{-1}(U_\alpha \cap U_\beta)$ the relations $\varphi_{\beta\alpha} = h_\beta^{-1}\varphi'_{\beta\alpha}h_\alpha$ are used:

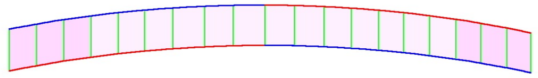
$$\begin{aligned}\varphi_\beta h_\beta^{-1}\varphi_\beta'^{-1} &= (\varphi_\alpha\varphi_\alpha^{-1}\varphi_\beta)h_\beta^{-1}(\varphi_\beta'^{-1}\varphi'_\alpha\varphi_\alpha'^{-1}) = \\ &= \varphi_\alpha\varphi_{\alpha\beta}h_\beta^{-1}\varphi_{\beta\alpha}'^{-1}\varphi_\alpha'^{-1} = \\ &= \varphi_\alpha h_\alpha^{-1}\varphi_\alpha'^{-1}.\end{aligned}$$

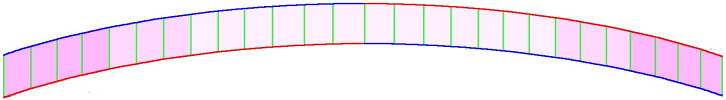
1. Let $E = B \times F$ and $p : E \longrightarrow B$ be projections onto the first factors. Then the atlas consists of one chart $U_\alpha = B$ and only one the transition function $\varphi_{\alpha\alpha} = \mathbf{Id}$ and the bundle is said to be *trivial*.

2. Let E be the Möbius band. One can think of this bundle as a square in the plane, $\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ with the points $(0, y)$ and $(1, 1 - y)$ identified for each $y \in [0, 1]$. The projection p maps the space E onto the segment $I_x = \{0 \leq x \leq 1\}$ with the endpoints $x = 0$ and $x = 1$ identified, that is, onto the circle S^1 . Let us show that the map p defines a locally trivial bundle. The atlas consists of two intervals (recall 0 and 1 are identified)

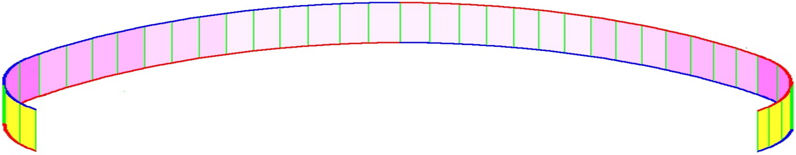
$$U_\alpha = \{0 < x < 1\},\ U_\beta = \{0 \leq x < \frac{1}{2}\} \cup \{\frac{1}{2} < x \leq 1\}.$$

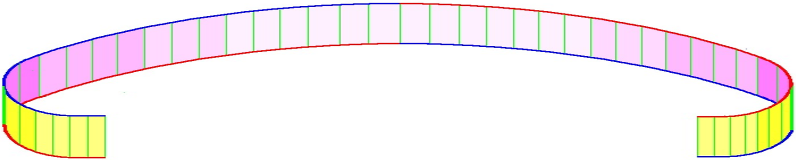


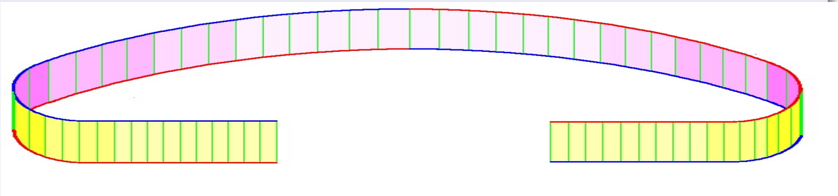




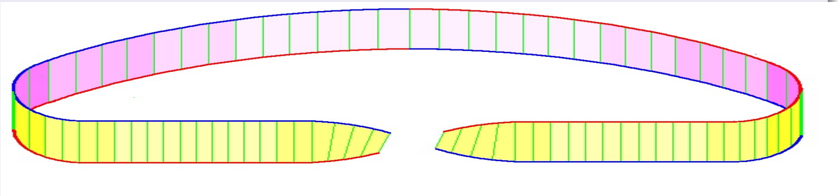
Locally trivial bundles
Example: Móbius band

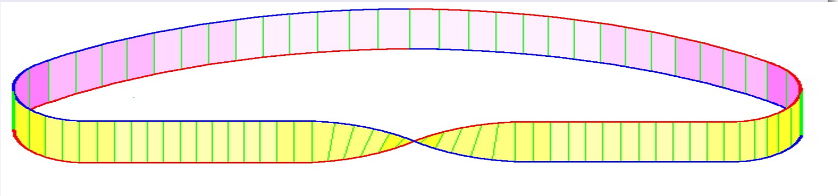






Locally trivial bundles
Example: Móbius band





The coordinate homeomorphisms may be defined as following:

$$\begin{cases} \varphi_\alpha : U_\alpha \times I_y \longrightarrow E, \\ \varphi_\alpha(x,y) = (x,y), \\ \varphi_\beta : U_\beta \times I_y \longrightarrow E, \\ \varphi_\beta(x,y) = (x,y) \quad \text{for} \quad 0 \leq x < \frac{1}{2}, \\ \varphi_\beta(x,y) = (x, 1-y) \quad \text{for} \quad \frac{1}{2} < x \leq 1. \end{cases}$$

The intersection of two charts $U_\alpha \cap U_\beta$ consists of union of two intervals $U_\alpha \cap U_\beta = (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. The transition function $\varphi_{\beta\alpha}$ have the following form

$$\begin{array}{l} \varphi_{\beta\alpha} = (x, y) \text{ for } \quad 0 < x < \frac{1}{2}, \\ \varphi_{\beta\alpha} = (x, 1 - y) \text{ for } \quad \frac{1}{2} < x < 1. \end{array}$$

The Möbius band is not isomorphic to a trivial bundle. Indeed, for a trivial bundle all transition functions can be chosen equal to the identity. Then there exist fiber preserving homeomorphisms

$$h_\alpha : U_\alpha \times I_y \longrightarrow U_\alpha \times I_y,$$
$$h_\beta : U_\beta \times I_y \longrightarrow U_\beta \times I_y,$$

such that

$$\varphi_{\beta\alpha} = h_\beta^{-1} h_\alpha$$

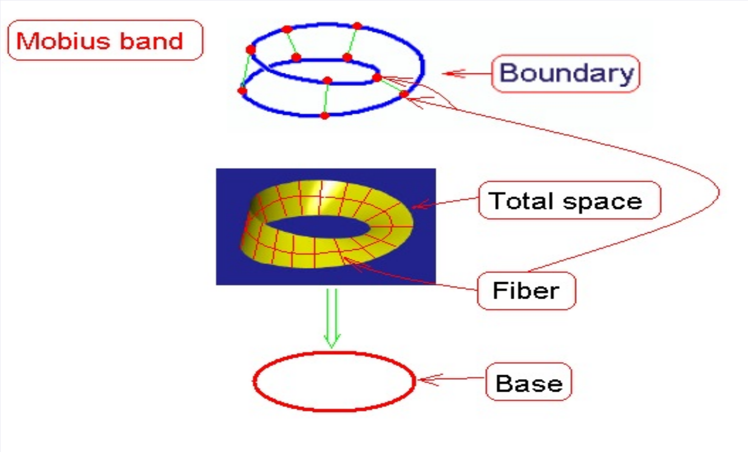
in its domain of definition $(U_\alpha \cap U_\beta) \times I_y$.

Then h_α, h_β are fiberwise homeomorphisms for fixed value of the first argument x giving homeomorphisms of interval I_y to itself. Each homeomorphism of the interval to itself maps end points to end points. So the functions

$$h_\alpha(x,0),\ h_\alpha(x,1),\ h_\beta(x,0),\ h_\beta(x,1)$$

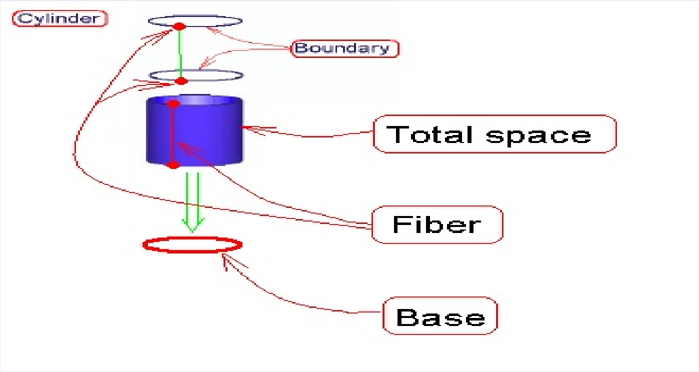
are constant functions, with values equal to zero or one (since the each chart is connected!).

The same is true for the functions $h_\beta^{-1}h_\alpha(x,0)$, that is $h_\beta^{-1}h_\alpha(x,0)$ also are constant functions. On the other hand the function $\varphi_{\beta\alpha}(x,0)$ is not constant because it equals zero for each $0 < x < \frac{1}{2}$ and equals one for each $\frac{1}{2} < x < 1$. Since $\varphi_{\beta\alpha}(x,0) = h_\beta^{-1}h_\alpha$ we have the contradiction. This contradiction shows that the Möbius band is not isomorphic to a trivial bundle.



Mobius band
Boundary
Total space
Fiber
Base

Locally trivial bundles
Example: Móbius band



3. Let E be the space of tangent vectors to two dimensional sphere $\mathbf{S}^2$ embedded in three dimensional Euclidean space $\mathbf{R}^3$. Let

$$p : E \longrightarrow \mathbf{S}^2$$

be the map associating each vector to its initial point. Let us show that p is a locally trivial bundle with fiber $\mathbf{R}^2$. Fix a point $s_0 \in \mathbf{S}^2$. Choose a Cartesian system of coordinates in $\mathbf{R}^3$ such that the point s_0 is the North Pole on the sphere (that is, the coordinates of s_0 equal $(0, 0, 1)$). Let U be the open subset of the sphere $\mathbf{S}^2$ defined by inequality $z > 0$. If $\vec{s} \in U,\ \vec{s} = (x, y, z)$, then

$$x^2 + y^2 + z^2 = 1,\ z > 0.$$

Let $\vec{e} = (\xi, \eta, \zeta)$ be a tangent vector to the sphere at the point $\vec{s}$. Then $(\vec{s}, \vec{e}) = 0$, or

$$x\xi + y\eta + z\zeta = 0,$$

that is,

$$\zeta = -(x\xi + y\eta)/z.$$

Define the map

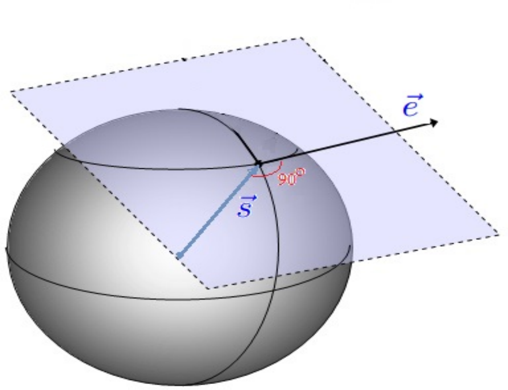
$$\varphi : U \times \mathbf{R}^2 \longrightarrow p^{-1}(U)$$

by the formula

$$\varphi(x, y, z, \xi, \eta) = (x, y, z, \xi, \eta, -(x\xi + y\eta)/z)$$

giving the coordinate homomorphism for the chart U containing the point $s_0 \in \mathbf{S}^2$. Thus the map p gives a locally trivial bundle. This bundle is called the *tangent bundle* of the sphere $\mathbf{S}^2$.

Locally trivial bundles
Example: Tangent bundle to sphere



Another way to prove that the map $p : E \longrightarrow \mathbf{S}^2$ is a locally trivial bundle consists in calculation of the differential of the map

$$Dp : TE \longrightarrow T\mathbf{S}^2.$$

The relations of cocyclicity

$$\varphi_{\alpha\alpha} = \mathbf{Id}, \quad \varphi_{\alpha\gamma}\varphi_{\gamma\beta}\varphi_{\beta\alpha} = \mathbf{Id}$$

and homology relations

$$\varphi_{\beta\alpha} = h_{\beta}^{-1}\varphi'_{\beta\alpha}h_{\alpha}$$

for the transition functions of a locally trivial bundle are similar to those involved in the calculation of one dimensional cohomology with coefficients in some algebraic sheaf. This analogy can be explain after a slight change of terminology and notation and the change will be useful for us for investigating the classification problem of locally trivial bundles.

Notice that a fiberwise homeomorphism of the Cartesian product of the base U and the fiber F onto itself

$$\varphi : U \times F \longrightarrow U \times F,$$

can be represented as a family of homeomorphisms of the fiber F onto itself, parametrized by points of the base B .

In other words, each fiberwise homeomorphism φ defines a map

$$\Phi : U \longrightarrow \mathbf{Homeo}\,(F),$$

where $\mathbf{Homeo}\,(F)$ is the group of all homeomorphisms of the fiber F . Furthermore, if we choose the right topology on the group $\mathbf{Homeo}\,(F)$ the map Φ becomes continuous.

Sometimes the opposite is true: the map

$$\Phi : U \longrightarrow \mathbf{Homeo}\ (F),$$

generates the fiberwise homeomorphism

$$\varphi : U \times F \longrightarrow U \times F,$$

with respect to the formula

$$\varphi(x, f) = (x, \Phi(x)(f)).$$

1-dimensional non commutative cohomology

So instead of $\varphi_{\alpha\beta}$ a family of functions

$$\Phi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathbf{Homeo}\ (F),$$

can be defined on the intersection $U_\alpha \cap U_\beta$ and having values in the group $\mathbf{Homeo}\ (F)$. In homological algebra the family of functions $\bar{\varphi}_{\alpha\beta}$ is called a *one dimensional cochain* with values in the sheaf of germs of functions with values in the group $\mathbf{Homeo}\ (F)$.

1-dimensional non commutative cohomology

The conditions

$$\varphi_{\alpha\alpha} = \mathbf{Id}, \quad \varphi_{\alpha\gamma}\varphi_{\gamma\beta}\varphi_{\beta\alpha} = \mathbf{Id},$$

means that

$$\begin{cases} \Phi_{\alpha\alpha}(x) = \mathbf{Id}, \quad x \in U_\alpha, \\ \Phi_{\alpha\gamma}(x)\Phi_{\gamma\beta}(x)\Phi_{\beta\alpha}(x) = \mathbf{Id}, \quad x \in U_\alpha \cap U_\beta \cap U_\gamma. \end{cases}$$

and we say that the cochain $\{\Phi_{\alpha\beta}\}$ is a *cocycle*.

1-dimensional non commutative cohomology

The condition

$$\varphi_{\beta\alpha} = h_{\beta}^{-1}\varphi'_{\beta\alpha}h_{\alpha}.$$

means that there is a zero dimensional cochain

$$H_{\alpha} : U_{\alpha} \longrightarrow \textbf{Homeo}\,(F)$$

such that

$$\Phi_{\beta\alpha}(x) = H_{\beta}^{-1}(x)\Phi'_{\beta\alpha}(x)H_{\alpha}(x),\ x \in U_{\alpha} \cap U_{\beta}.$$

1-dimensional non commutative cohomology

Using the language of homological algebra the last condition means that cocycles $\{\Phi_{\beta\alpha}\}$ and $\{\Phi'_{\beta\alpha}\}$ are cohomologous. Thus the family of locally trivial bundles with fiber F and base B is in one to one correspondence with the one dimensional cohomology of the space B with coefficients in the sheaf of the germs of continuous $\mathbf{Homeo}\,(F)$ –valued functions for given open covering $\{U_\alpha\}$:

$$Bundles_F(B) \Leftrightarrow H^1(B; \underline{\mathbf{Homeo}}\,(F)).$$

1-dimensional non commutative cohomology

Despite obtaining a simple description of the family of locally trivial bundles in terms of homological algebra, it is ineffective since there is no simple method of calculating cohomologies of this kind. Nevertheless, this representation of the transition functions as a cocycle turns out very useful because of the situation described below.

First of all notice that using the new interpretation a locally trivial bundle is determined by the base B , the atlas $\{U_\alpha\}$ and the functions $\{\Phi_{\alpha\beta}\}$ taking the value in the group $G = \mathbf{Homeo}\,(F)$:

$$\{B, G, \{U_\alpha\}, \{\Phi_{\alpha\beta}\}\}.$$

The fiber F itself does not directly take part in the description of the bundle.

Action of structural group on the fiber

Hence, one can at first describe a locally trivial bundle as a family of functions $\{\Phi_{\alpha\beta}\}$ with values in some topological group G , and after that construct the total space of the bundle with fiber F by additionally defining an action of the group G on the space F ,

$$G \times F \longrightarrow F,$$

that is, defining a continuous homomorphism of the group G into the group $\mathbf{Homeo}\,(F)$:

$$G \longrightarrow \mathbf{Homeo}\,(F).$$

Structural subgroup

Secondly, the notion of locally trivial bundle can be generalized and structural of bundle made richer by requiring that both the transition functions $\Phi_{\alpha\beta}$ and the functions H_α are not arbitrary but take values in some subgroup of the homeomorphism group $\mathbf{Homeo}\,(F)$.

Changing of fiber

Thirdly, sometimes information about locally trivial bundle may be obtained by substituting some other fiber F' for the fiber F but using the 'same' transition functions. Thus we come to a new definition of a locally trivial bundle with additional structure — the group where the transition functions take their values.

Definition (Bundle with a structural group)

Let E , B , F be topological spaces and G be a topological group which acts continuously on the space F :

$$G \times F \longrightarrow F.$$

A continuous map

$$p : E \longrightarrow B$$

is said to be a *locally trivial bundle with fiber F and structural group G*

Definition (Bundle with a structural group)

if there is an atlas $\{U_\alpha\}$ and the coordinate homeomorphisms

$$\varphi_\alpha : p^{-1}(U_\alpha) \longrightarrow U_\alpha \times F$$

such that the transition functions

$$\varphi_{\beta\alpha} = \varphi_\beta \varphi_{\alpha^{-1}} : (U_\alpha \cap U_\beta) \times F \longrightarrow (U_\alpha \cap U_\beta) \times F$$

have the form:

Definition (Bundle with a structural group)

have the form:

$$\varphi_{\beta\alpha}(x, f) = (x, \Phi_{\beta\alpha}(x)f),$$

where $\Phi_{\beta\alpha} : (U_\alpha \cap U_\beta) \longrightarrow G$ are continuous functions satisfying the conditions

$$\begin{cases} \Phi_{\alpha\alpha}(x) \equiv 1, \quad x \in U_\alpha, \\ \Phi_{\alpha\beta}(x)\Phi_{\beta\gamma}(x)\Phi_{\gamma\alpha}(x) \equiv 1, \quad x \in U_\alpha \cap U_\beta \cap U_\gamma. \end{cases}$$

The functions $\Phi_{\alpha\beta}$ are also called the transition functions.

compatible with structural group isomorphisms

Let

$$\psi : E' \longrightarrow E$$

be an isomorphism of locally trivial bundles with structural group G . Let φ_α and $\varphi_\alpha^{'}$ be the coordinate homeomorphisms of the bundles $p : E \longrightarrow B$ and $p' : E' \longrightarrow B$, respectively.

compatible with structural group isomorphisms

One says that the isomorphism ψ is *compatible with structural group* G if the homomorphisms

$$\varphi_\alpha^{-1}\psi\varphi'_\alpha : U_\alpha \times F \longrightarrow U_\alpha \times F$$

are determined by continuous functions

$$H_\alpha : U_\alpha \longrightarrow G,$$

defined by relation

$$\varphi_\alpha^{-1}\psi\varphi'_\alpha(x, f) = (x, H_\alpha(x) f).$$

Isomorphic bundles

Thus two bundles with structural group G and transition functions $\Phi_{\beta\alpha}$ and $\Phi'_{\beta\alpha}$ are isomorphic, the isomorphism being compatible with structural group G , if

$$\Phi_{\beta\alpha}(x) = H_\beta(x)\Phi'_{\beta\alpha}(x)H_\alpha(x)$$

for some continuous functions $H_\alpha : U_\alpha \longrightarrow G$.

Equivalent bundles

So two bundles whose the transition functions satisfy the conditions

$$\Phi_{\beta\alpha}(x) = H_{\beta}(x)\Phi'_{\beta\alpha}(x)H_{\alpha}(x)$$

are called *equivalent bundles*.

Reducing of structural group

It is sometimes useful to increase or decrease structural group G. Two bundles which are not equivalent with respect of structural group G may become equivalent with respect to a larger structural group G', $G \subset G'$. When a bundle with structural group G admits transition functions with values in a subgroup H , it is said that structural group G is *reduced to the subgroup* H .

Trivial bundle

It is clear that if structural group of the bundle $p : E \longrightarrow B$ consists of only one element then the bundle is trivial. So to prove that the bundle is trivial, it is sufficient to show that its structural group G may be reduced to the trivial subgroup.

Change of structural group

More generally, if

$$\rho : G \longrightarrow G'$$

is a continuous homomorphism of topological groups and we are given a locally trivial bundle with structural group G and the transition functions

$$\Phi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G,$$

then

$\Downarrow$

Change of structural group

then a new locally trivial bundle may be constructed with structural group G' for which the transition functions are defined by

$$\Phi'_{\alpha\beta}(x) = \rho(\Phi_{\alpha\beta}(x)).$$

This operation is called *a change of structural group* (with respect to the homomorphism ρ).

Remark

Note that the fiberwise homeomorphism

$$\varphi : U \times F \longrightarrow U \times F$$

in general is not induced by continuous map

$$\Phi : U \longrightarrow \mathbf{Homeo}\,(F).$$

Remark

We will not analyze the problem and note only that later on in all our applications the fiberwise homeomorphisms will be induced by continuous maps

$$\Phi : U \longrightarrow \mathbf{Homeo}\,(F).$$

that is splitted into a composition of continuous maps into structural group G and a (continuous) homomorphism

$$G \longrightarrow \mathbf{Homeo}\,(F).$$

Special fiber

Now we can return to the third situation, that is, to the possibility to choosing a space as a special fiber of a locally trivial bundle with structural group G . Let us consider the fiber

$$F = G$$

with the action of G on F being that of left translation, that is, the element $g \in G$ acts on the F by the homeomorphism

$$g(f) = gf,\ f \in F = G.$$

Definition (Principal bundle)

A locally trivial bundle with structural group G is called principal G –bundle if $F = G$ and action of the group G on F ,

$$G \times F \longrightarrow F,$$

is defined by the left translations:

$$(g, f) \longrightarrow gf, \quad g \in G, \quad f \in F = G.$$

Important property

An important property of principal G –bundles is the consistency of the homeomorphisms with structural group G and it can be described not only in terms of the transition functions (the choice of which is not unique) but also in terms of equivariant properties of bundles.

Theorem (Right action)

Let

$$p : E \longrightarrow B$$

be a principal G –bundle,

$$\varphi_\alpha : U_\alpha \times G \longrightarrow p^{-1}(U_\alpha)$$

be the trivializations.

Right action

Then there is a right action of the group G on the total space E,

$$E \times G \longrightarrow E, \qquad E \circlearrowright \cdot g \quad : \qquad g \in G$$

such that:

1. the right action of the group G is fiberwise, that is,

$$p(x) = p(xg), \; x \in E, \; g \in G.$$

   or equivalently, the projection p is equivariant with respect to trivial action of the group G on the base B :

$$\overset{\cdot g}{\circlearrowright} E \xrightarrow{p} \overset{\cdot g}{\circlearrowright} B, \qquad b \cdot g = b, \qquad b \in B, \qquad g \in G.$$

Right action

2. the homeomorphism φ_α transforms the right action of the group G on the total space into right translations on the second factor, that is,

$$\varphi_\alpha(x, f)g = \varphi_\alpha(x, fg), \; x \in U_\alpha, \; f, g \in G.$$

Right action

$$
\begin{array}{ccccc}
\overset{\cdot g}{\circlearrowright} & & \overset{\cdot g}{\circlearrowright} & & \overset{\cdot g}{\circlearrowright} \\
E & \hookleftarrow & p^{-1}(U_\alpha) & \xleftarrow[\varphi_\alpha]{} & U_\alpha \times G \\
\downarrow & & \downarrow & & \downarrow \\
B & \hookleftarrow & U_\alpha & = & U_\alpha
\end{array}
$$

Proof.

According to the definitions, the transition functions $\varphi_{\beta\alpha} = \varphi_{\beta}\varphi_{\alpha}^{-1}$ have the following form

$$\varphi_{\beta\alpha}(x, f) = (x, \Phi_{\beta\alpha}(x)f),$$

where

$$\Phi_{\beta\alpha} : U_{\alpha} \cap U_{\beta} \longrightarrow G$$

are continuous functions satisfying the conditions

$$\begin{array}{ll} \Phi_{\alpha\alpha}(x) \equiv 1, & x \in U_{\alpha}, \\ \Phi_{\alpha\beta}(x)\Phi_{\beta\gamma}(x)\Phi_{\gamma\alpha}(x) \equiv 1, & x \in U_{\alpha} \cap U_{\beta} \cap U_{\gamma}. \end{array}$$

Proof.

Since an arbitrary point $z \in E$ can be represented in the form

$$z = \varphi_\alpha(x, f)$$

for some index α , the formula

$$\varphi_\alpha(x, f) \cdot g = \varphi_\alpha(x, f \cdot g),\ x \in U_\alpha,\ f, g \in G.$$

determines the continuous right action of the group G provided that this definition is independent of the choice of index α .

Proof.

So suppose that

$$z = \varphi_\alpha(x, f) = \varphi_\beta(x, f').$$

We need to show that the element $z \cdot g$ does not depend on the choice of index, that is,

$$\varphi_\alpha(x, fg) = \varphi_\beta(x, f'g).$$

or

$$(x, f' \cdot g) = \varphi_\beta^{-1}\varphi_\alpha(x, f \cdot g) = \varphi_{\beta\alpha}(x, f \cdot g)$$

or

$$f' \cdot g = \Phi_{\beta\alpha}(x) \cdot (f \cdot g) = (\Phi_{\beta\alpha}(x) \cdot f) \cdot g.$$

Proof.

However,

$$(x, f') = \varphi_\beta \varphi_\alpha^{-1}(x, f) = \varphi_{\beta\alpha}(x, f) = (x, \Phi_{\beta\alpha} f),$$

Hence

$$f' = \Phi_{\beta\alpha}(x)(f).$$

Proof.

Thus multiplying

$$f' = \Phi_{\beta\alpha}(x) \cdot f.$$

by g on the right gives

$$f' \cdot g = \Phi_{\beta\alpha}(x) \cdot (f \cdot g).$$

$\blacksquare$

The theorem allows us to consider principal G –bundles as having a right action on the total space.

Theorem (Equivariant map)

Let

$$\psi : E' \longrightarrow E$$

be a fiberwise map of principal G –bundles. This map is the isomorphism of locally trivial bundles with structural group G , that is, compatible with structural group G if and only if this map is equivariant (with respect to right actions of the group G on the total spaces).

Proof.

Let

$$p : E \longrightarrow B,$$
$$p' : E' \longrightarrow B$$

be locally trivial principal bundles both with structural group G and let φ_α , φ'_α be coordinate homeomorphisms.

Proof.

Then by the definition the map ψ is an isomorphism of locally trivial bundles with structural group G when

$$\varphi_\alpha^{-1}\psi\varphi'_\alpha(x,g) = (x, H_\alpha(x)g).$$

for some continuous functions

$$H_\alpha : U_\alpha \longrightarrow G.$$

Proof.

It is clear that the maps $\varphi_\alpha^{-1}\psi\varphi'_\alpha(x,g) = (x, H_\alpha(x)g)$ are equivariant since

$$(\varphi_\alpha^{-1}\psi\varphi'_\alpha(x,g))g_1 = (x, H_\alpha(x)g)g_1 =$$

$$= (x, H_\alpha(x)gg_1) = \varphi_\alpha^{-1}\psi\varphi'_\alpha(x, gg_1).$$

Hence the map ψ is equivariant with respect to the right actions of the group G on the total spaces E and E' .

Proof.

Conversely, let the map ψ be equivariant with respect to the right actions of the group G on the total spaces E and E' . Then the map $\varphi_\alpha^{-1}\psi\varphi'_\alpha$ is equivariant with respect to right translations of the second coordinate of the space $U_\alpha \times G$. Since the map $\varphi_\alpha\psi\varphi'^{-1}_\alpha$ is fiberwise, it has the following form

$$\varphi_\alpha\psi\varphi'^{-1}_\alpha(x, g) = (x, A_\alpha(x, g)).$$

Proof.

The equivariance of the map implies that

$$A_\alpha(x, gg_1) = A_\alpha(x, g)g_1$$

for any $x \in U_\alpha,\ g, g_1 \in G$. In particular, putting $g = e$ that

$$A_\alpha(x, g_1) = A_\alpha(x, e)g_1$$

Proof.

So putting

$$H_\alpha(x) = A_\alpha(x, e),$$

it follows that

$$A_\alpha(x, g) = H_\alpha(x)g$$

and

$$\varphi_\alpha^{-1}\psi\varphi'_\alpha(x, g) = (x, H_\alpha(x)g).$$

The last identity means that ψ is compatible with structural group G . ∎

Conclusion

Thus using the theorem, to show that two locally trivial bundles with structural group G (and the same base B) are isomorphic it necessary and sufficient to show that there exists an equivariant map of corresponding principal G –bundles (inducing the identity map on the base B).

Trivial bundle

In particular, if one of the bundles is trivial, for instance, $E' = B \times G$, then to construct an equivariant map $\psi : E' \longrightarrow E$ it is sufficient to define a continuous map ψ on the subspace $\{(x, e) : x \in B,\} \subset E' = B \times G$ into E . Then using equivariance, the map ψ is extended by formula

$$\psi(x, g) = \psi(x, e)g.$$

Trivial bundle

The map $\{(x,e) : x \in B\} \xrightarrow{\psi} E'$ can be considered as a map

$$s : B \longrightarrow E$$

satisfying the property

$$ps(x) = x, \; x \in B.$$

Cross–section

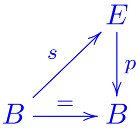
The map

$$s : B \longrightarrow E$$

with the property

$$ps(x) = x, \; x \in B.$$

is called a *cross–section* of the bundle. Each cross-section generates the commutative diagram



Cross–section

So each trivial principal bundle has cross–sections. For instance, the map $B \longrightarrow B \times G$ defined by $x \longrightarrow (x, e)$ is a cross–section. Conversely, if a principal bundle has a cross–section s then this bundle is isomorphic to the trivial principal bundle. The corresponding isomorphism $\psi : B \times G \longrightarrow E$ is defined by the formula

$$\psi(x, g) = s(x)g, \; x \in B, \; g \in G.$$

Equivariant map

Let us relax our restrictions on equivariant mappings of principal bundles with structural group G . Consider arbitrary equivariant mappings of total spaces of principal G –bundles with arbitrary bases.

Equivariant map

Each fiber of a principal G –bundle is an orbit of the right action of the group G on the total space and hence for each equivariant mapping

$$\psi : E' \longrightarrow E$$

of total spaces, each fiber of the bundle

$$p' : E' \longrightarrow B'$$

maps to a fiber of the bundle

$$p : E \longrightarrow B.$$

Equivariant map

In other words, the mapping ψ induces a mapping of bases

$$\chi : B' \longrightarrow B$$

and the following diagram is commutative

$$\begin{array}{ccc} E' & \xrightarrow{\psi} & E \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{\chi} & B \end{array}$$

Equivariant map

Let $U_\alpha \subset B$ be a chart in the base B and let U'_β be a chart such that

$$\chi(U'_\beta) \subset U_\alpha.$$

The mapping $\varphi_\alpha \psi \varphi'^{-1}_\beta$ makes the following diagram commutative

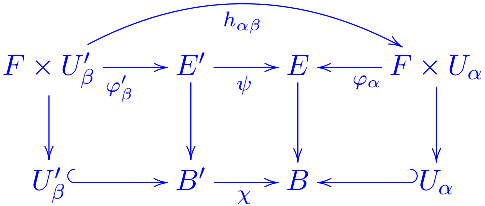
$$\begin{array}{ccc}
U'_\beta \times G & \xrightarrow{\varphi_\alpha \psi \varphi'^{-1}_\beta} & U_\alpha \times G \\
\Big\downarrow p' \varphi'^{-1}_\beta & & \Big\downarrow p \varphi^{-1}_\alpha \\
U'_\beta & \xrightarrow{\chi} & U_\alpha
\end{array}$$

Consider a commutative diagram for two locally trivial bundles which are equipped with the family of the coordinate homeomorphisms whose transition functions belong to the structural group G .

$$\begin{CD} E' @>\psi>> E \\ @V{p'}VV @VV{p}V \\ B' @>\chi>> B \end{CD}$$

We say that the diagram is compatible with structural group G if there is atlases of charts $\{U_\alpha, \varphi_\alpha : F \times U_\alpha \longrightarrow E\}$ and $\{U'_\beta, \varphi'_\beta : F \times U'_\beta \longrightarrow E'\}$ such that

- $$\chi(U'_\beta) \subset U_\alpha,$$
- The map $h_{\alpha\beta} = \varphi_\alpha \psi \varphi'^{-1}_\beta$ in the diagram



satisfies the condition

$$h_{\alpha\beta}(f, b') = (H_{\alpha\beta}(b')f, \chi(b')), \quad H_{\alpha\beta} : U'_\beta \longrightarrow G.$$

Definition (pullback bundle)

Consider a commutative diagram for two locally trivial bundles which are equipped with the family of the coordinate homeomorphisms whose transition functions belong to the structural group G .

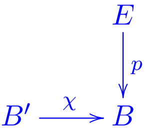
$$\begin{array}{ccc} E' & \xrightarrow{\psi} & E \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{\chi} & B \end{array}$$

Assume that the diagram is compatible with structural group G. Then the bundle $p' : E' \longrightarrow B'$ is called *pullback bundle* or *inverse image* of the bundle $p : E \longrightarrow B$:

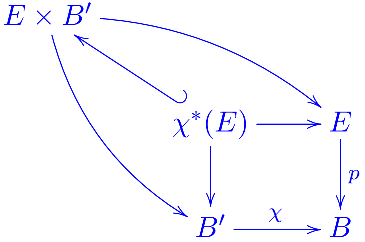
$$E' = \chi^*(E).$$

Construction of pullback bundle

Given a map



Put $\chi^*(E) = \{(e, b') \in E \times B' : \chi(b') = p(e)\} \subset E \times B'$:



Transition functions of pullback bundle

Given the pullback bundle

$$\begin{array}{ccc} \chi^*(E) & \xrightarrow{\chi^*} & E \\ \downarrow & & \downarrow p \\ B' & \xrightarrow{\chi} & B \end{array}$$

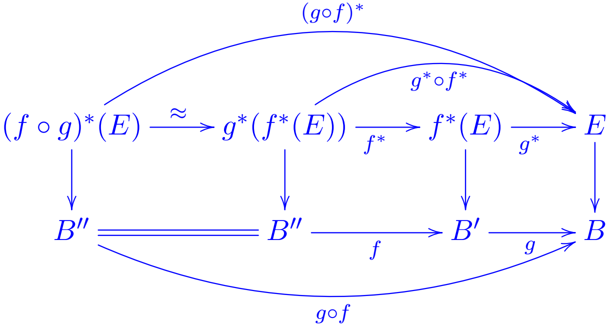
The transition functions are defined as followings

$$U'_\alpha = \chi^{-1}(U_\alpha),$$

$$\Phi'_{\alpha\beta}(b') = \Phi_{\alpha\beta}(\chi(b')) \in G, \quad b' \in U'_\alpha.$$

$$\mathbf{Id}^*(E) \approx E.$$

$$(f \circ g)^*(E) \approx g^*(f^*(E)),$$



Theorem (Homotopy invariance of pullback)

Let $f, g : B' \longrightarrow B$ be two continuous maps of compact spaces that are homotopic, $f \sim g$, $p : E \longrightarrow B$ be a locally trivial bundle with structural group G . Then bundles f^*E and $g^*(E)$ are isomorphic

$$f^*(E) \approx g^*(E).$$

We follow the book by A.Hatcher

A. Hatcher *Vector bundles and K-theory* http://www.math.cornell.edu/ hatcher/#VBKT,(2009)
(Theorem 1.6.)

Proof.

Without loss of generality it is sufficient to prove

Proposition (cartesian product with unit segment)

Let $p: E \longrightarrow B \times [0,1]$. Then restrictions $E|_{B\times\{0\}}$ and $E|_{B\times\{1\}}$ are isomorphic.

Proof.

Let $\{U_\alpha, \varphi_\alpha\}$ be an atlas of charts on the space $B \times [0,1]$. Passing to refinement we can assume that each chart has the form $U_{\alpha,j} = V_\alpha \times (\frac{j}{n}, \frac{j+2}{n})$. So we can assume that $U_\alpha = V_\alpha \times [0,1]$.

Proof.

Let f_α be a partition of unit which is subordinate to the atlas of chart V_α that is $\textbf{supp}\ f_\alpha \subset V_\alpha$. We should compare two bundles $E|_{B\times\{0\}}$ and $E|_{\textbf{Graph}\,(f_\alpha)}$, where $\textbf{Graph}\,(f_\alpha)$ is the graph of the function f_α ,

$$\textbf{Graph}\,(f_\alpha) = \{(b, f_\alpha(b)) : b \in B\} \subset B \times [0,1].$$

there is natural homeomorphism

$$q : B \times \{0\} \longrightarrow \textbf{Graph}\,(f_\alpha),$$

$$q(b,0) = (b, f_\alpha(b)) \in \textbf{Graph}\,(f_\alpha), \quad (b,0) \in B \times \{0\}.$$

Proof.

Consider a refined atlas of charts:

$$W_\alpha = V_\alpha, \quad W_\beta = V_\beta \setminus \textbf{supp } f_\alpha, \quad \beta \neq \alpha.$$

So $W_\beta \cap W_\gamma \cap \textbf{supp } f_\alpha = \emptyset$. Then on the intersections $W_\beta \cap W_\gamma$ the transition functions of the bundle $E|_{\textbf{Graph } f_\alpha}$ coincide with transition functions of the bundle $E|_{B\times\{0\}}$. Hence

$$E|_{\textbf{Graph } f_\alpha} \approx E|_{B\times\{0\}}.$$

The statement can be proved by the induction on the number of charts.

Theorem (The classification theorem)

Let us consider a principal G –bundle,

$$\begin{array}{c} E_G \\ \downarrow p_G \\ B_G \end{array}$$

such that all homotopy groups of the total space E_G are trivial:

$$\pi_i(E_G) = 0,\ 0 \leq i < \infty.$$

Let B be a CW complex. Then any principal G –bundle $p : E \longrightarrow B$ is isomorphic to the inverse image of the bundle $p_G : E_G \longrightarrow B_G$, with respect to a continuous mapping $f : B \longrightarrow B_G$.

$$\begin{array}{ccc} E & \xrightarrow{f^*} & E_G \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle f_G} \\ B & \xrightarrow[f]{} & B_G \end{array}$$

The classification theorem

Two inverse images of the bundle

$$p_G : E_G \longrightarrow B_G,$$

with respect to the mappings

$$f, g : B \longrightarrow B_G$$

are isomorphic if and only if the mappings f and g are homotopic.

Corollary (Description of all bundles)

The family of all isomorphism classes of principal G –bundles over the base B is in one to one correspondence with the family of homotopy classes of continuous mappings from B to B_G :

$$\mathbf{B}undle_G(B) \approx [B, B_G].$$

Corollary (homotopy invariance)

If two cellular spaces B and B' are homotopy equivalent then the families of all isomorphism classes of principal G –bundles over the bases B and B' are in one to one correspondence. This correspondence is defined by inverse image with respect to a homotopy equivalence

$$B \longrightarrow B'.$$

Definition (Vector bundle)

A locally trivial bundle

$$\xi : \begin{array}{c} E \\ \downarrow p \\ B \end{array}$$

is called *vector bundle* if the fiber F is homeomorphic to a vector space $F \approx \mathbf{R}^n$ and the structural group is the group of all linear automorphisms of the space $\mathbf{R}^n$, $G \approx \mathbf{GL}(n, \mathbf{R})$. By definition the dimension of the vector bundle is equal to n ,

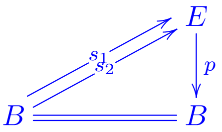
$$\dim \xi = n.$$

First of all notice that each fiber $p^{-1}(x),\ x\in B$ has the structure of vector space which does not depend on the choice of coordinate homeomorphism. In other words, the operations of addition and multiplication by scalars is independent of the choice of coordinate homeomorphism.
Indeed, since structural group G is $\mathbf{GL}(n,\mathbf{R})$ the transition functions

$$\varphi_{\alpha\beta}:(U_\alpha\cap U_\beta)\times\mathbf{R}^n\longrightarrow(U_\alpha\cap U_\beta)\times\mathbf{R}^n$$

are linear mappings with respect to the second factor. Hence a linear combination of vectors goes to the linear combination of images with the same coefficients.

Denote by $\Gamma(B, \xi)$ the set of all sections of the vector bundle ξ . Then the set $\Gamma(B, \xi)$ becomes an (infinite dimensional) vector space. To define a structure of vector space on the $\Gamma(B, \xi)$ consider two sections s_1 , s_2 :



Put

$$(s_1 + s_2)(x) = s_1(x) + s_2(x),\ x \in B,$$

$$(\lambda s_1)(x) = \lambda(s_1(x)),\ \lambda \in R,\ x \in B.$$

These formulas define on the set $\Gamma(B,\xi)$ the structure of vector space. Notice that an arbitrary section $s:B\longrightarrow E$ can be described in local terms. Let $\{U_\alpha\}$ be an atlas, $\varphi_\alpha:\mathbf{R}^n\times U_\alpha\longrightarrow p^{-1}(U_\alpha)$ be coordinate homeomorphisms, $\varphi_{\beta\alpha}=\varphi_\beta^{-1}\varphi_\alpha$. Then the compositions

$$\varphi_\alpha^{-1}\cdot s:U_\alpha\xrightarrow{s}E\xleftarrow{\varphi_\alpha}U_\alpha\times\mathbf{R}^n$$

are sections of trivial bundles over U_α and determine vector valued functions $s_\alpha:U_\alpha\longrightarrow\mathbf{R}^n$ by the formula

$$(x,s_\alpha(x))=(\varphi_\alpha^{-1}\cdot s)(x),\ x\in U_\alpha.$$

On the intersection of two charts $U_\alpha \cap U_\beta$ the functions $s_\alpha(x)$ satisfy the following compatibility condition

$$s_\beta(x) = \Phi_{\beta\alpha}(x)(s_\alpha(x)).$$

Conversely, if one has a family of vector valued functions $s_\alpha : U_\alpha \longrightarrow \mathbf{R}^n$ which satisfy the compatibility condition $s_\beta(x) = \Phi_{\beta\alpha}(x)(s_\alpha(x))$, then the formula

$$s(x) = \varphi_\alpha(x, s_\alpha(x))$$

determines the mapping $s : B \longrightarrow E$ uniquely (that is, independent of the choice of chart U_α).
The map s is a section of the bundle ξ .

Direct sum

There are natural operations induced by the direct sum and tensor product of vector spaces on the family of vector bundles over a common base B . Firstly, consider the operation of direct sum of vector bundles. Let ξ_1 and ξ_2 be two vector bundles with fibers V_1 and V_2 , respectively.
Denote the transition functions of these bundles in a common atlas of charts by $\Phi^1_{\alpha\beta}(x)$ and $\Phi^2_{\alpha\beta}(x)$.

Direct sum

Notice that values of the transition function $\Phi^1_{\alpha\beta}(x)$ lie in the group $\mathbf{GL}(V_1)$ whereas the values of the transition function $\Phi^2_{\alpha\beta}(x)$ lie in the group $\mathbf{GL}(V_2)$. Hence the transition functions $\Phi^1_{\alpha\beta}(x)$ and $\Phi^2_{\alpha\beta}(x)$ can be considered as matrix–values functions of orders $n_1 = \dim V_1$ and $n_2 = \dim V_2$, respectively. Both of them should satisfy the conditions

$$\Phi_{\alpha\alpha}(x) \equiv 1, \; x \in U_\alpha,$$
$$\Phi_{\alpha\beta}(x)\Phi_{\beta\gamma}(x)\Phi_{\gamma\alpha}(x) \equiv 1, \; x \in U_\alpha \cap U_\beta \cap U_\gamma.$$

Direct sum

We form a new space $V = V_1 \oplus V_2$. The linear transformation group $\mathbf{GL}(V)$ is the group of matrices of order $n = n_1 + n_2$ which can be decomposed into blocks with respect to decomposition of the space V into the direct sum $V_1 \oplus V_2$. Then the group $\mathbf{GL}(V)$ has the subgroup $\mathbf{GL}(V_1) \oplus \mathbf{GL}(V_2)$ of matrices which have the following form:

$$A = \left\| \begin{array}{cc} A_1 & 0 \\ 0 & A_2 \end{array} \right\| = A_1 \oplus A_2,\ A_1 = \mathbf{GL}(V_1),\ A_2 = \mathbf{GL}(V_2).$$

Direct sum

Then we can construct new the transition functions

$$\Phi_{\alpha\beta}(x) = \Phi^1_{\alpha\beta}(x) \oplus \Phi^2_{\alpha\beta}(x) = \left\| \begin{matrix} \Phi^1_{\alpha\beta}(x) & 0 \\ 0 & \Phi^2_{\alpha\beta}(x) \end{matrix} \right\|.$$

These transition functions satisfy the same conditions

$$\Phi_{\alpha\alpha}(x) \equiv 1,\ x \in U_\alpha,$$
$$\Phi_{\alpha\beta}(x)\Phi_{\beta\gamma}(x)\Phi_{\gamma\alpha}(x) \equiv 1,\ x \in U_\alpha \cap U_\beta \cap U_\gamma,$$

that is, they define a vector bundle with fiber $V = V_1 \oplus V_2$.

Definition of direct sum

The bundle constructed above is called the *direct sum of vector bundles* ξ_1 and ξ_2 and is denoted by $\xi = \xi_1 \oplus \xi_2$.

Geometric construction of direct sum

The direct sum operation can be constructed in a geometric way. Namely, let $p_1 : E_1 \longrightarrow B$ be a vector bundle ξ_1 and let $p_2 : E_2 \longrightarrow B$ be a vector bundle ξ_2 . Consider the Cartesian product of total spaces $E_1 \times E_2$ and the projection

$$p = p_1 \times p_2 : E_1 \times E_2 \longrightarrow B \times B.$$

It is clear that p is vector bundle with the fiber $V = V_1 \oplus V_2$.

Geometric construction of direct sum

Consider the diagonal $\Delta \subset B \times B$, that is, the subset $\Delta = \{(x, x) : x \in B\}$. The diagonal Δ is canonically homeomorphic to the space B . The restriction of the bundle p to $\Delta \approx B$ is a vector bundle over B . The total space E of this bundle is the subspace $E \subset E_1 \times E_2$ that consists of the vectors (y_1, y_2) such that

$$p_1(y_1) = p_2(y_2).$$

It is easy to check that $\{U_{\alpha_1} \times U_{\alpha_2}\}$ gives an atlas of charts for the bundle p .

Geometric construction of direct sum

The transition functions $\varphi_{(\beta_1\beta_2)(\alpha_1\alpha_2)}(x,y)$ on the intersection of two charts $(U_{\alpha_1}\times U_{\alpha_2})\cap(U_{\beta_1}\times U_{\beta_2})$ have the following form:

$$\varphi_{(\beta_1\beta_2)(\alpha_1\alpha_2)}(x,y)=\left\|\begin{array}{cc}\varphi^1_{\beta_1\alpha_1}(x) & 0\\ 0 & \varphi^2_{\beta_2\alpha_2}(y)\end{array}\right\|.$$

Hence on the diagonal $\Delta\approx B$ the atlas consists of sets $U_\alpha\approx\Delta\cap(U_\alpha\times U_\alpha)$.

Geometric construction of direct sum

Then the transition functions for the restriction of the bundle p on the diagonal have the following form:

$$\varphi_{(\beta\beta)(\alpha\alpha)}(x,x) = \left\| \begin{array}{cc} \varphi^1_{\beta\alpha}(x) & 0 \\ 0 & \varphi^2_{\beta\alpha}(x) \end{array} \right\|.$$

So these transition functions coincide with the transition functions defined for the direct sum of the bundles ξ_1 and ξ_2 .

Tensor product

Now let us proceed to the definition of tensor product of vector bundles. As before, let ξ_1 and ξ_2 be two vector bundles with fibers V_1 and V_2 and let $\Phi^1_{\alpha\beta}(x)$ and $\Phi^2_{\alpha\beta}(x)$ be the transition functions of the vector bundles ξ_1 and ξ_2 ,
$\Phi^1_{\alpha\beta}(x) \in \mathbf{GL}(V_1),\ \Phi^2_{\alpha\beta}(x) \in \mathbf{GL}(V_2),\ x \in V_\alpha \cap V_\beta.$
Let $V = V_1 \otimes V_2$ be the tensor product of the vector spaces V_1 and V_2 .

Tensor product

Then form the tensor product (Kronecker product) $A_1 \otimes A_2 \in \mathbf{GL}(V_1 \otimes V_2)$ of the two matrices $A_1 \in \mathbf{GL}(V_1)$, $A_2 \in \mathbf{GL}(V_2)$. Put

$$\Phi_{\alpha\beta}(x) = \Phi^1_{\alpha\beta}(x) \otimes \Phi^2_{\alpha\beta}(x).$$

Definition of tensor product

Now we have obtained a family of the matrix value functions $\Phi_{\alpha\beta}(x)$ which satisfy the conditions of cocylicity. The corresponding vector bundle ξ with fiber $V = V_1 \otimes V_2$ and transition functions $\Phi_{\alpha\beta}(x)$ will be called *the tensor product* of bundles ξ_1 and ξ_2 and denoted by

$$\xi = \xi_1 \otimes \xi_2.$$

Remark

What is common in the construction of the operations of direct sum and operation of tensor product? Both operations can be described as the result of applying the following sequence of operations to the pair of vector bundles ξ_1 and ξ_2 :

- Pass to the principal $\mathbf{GL}(V_1)$ – and $\mathbf{GL}(V_2)$ – bundles;
- Construct the principal $(\mathbf{GL}(V_1) \times \mathbf{GL}(V_2))$ – bundle over the Cartesian square $B \times B$;
- Restrict to the diagonal Δ , homeomorphic to the space B .

Remark

- Finally, form a new principal bundle by means of the relevant representations of structural group $\mathbf{GL}(V_1) \times \mathbf{GL}(V_2)$ in the groups $\mathbf{GL}(V_1 \oplus V_2)$ and $\mathbf{GL}(V_1 \otimes V_2)$, respectively.

The difference between the operations of direct sum and tensor product lies in choice of the representation of the group $\mathbf{GL}(V_1) \times \mathbf{GL}(V_2)$. By using different representations of structural groups, further operations of vector bundles can be constructed, and algebraic relations holding for representations induce corresponding algebraic relations vector bundles.

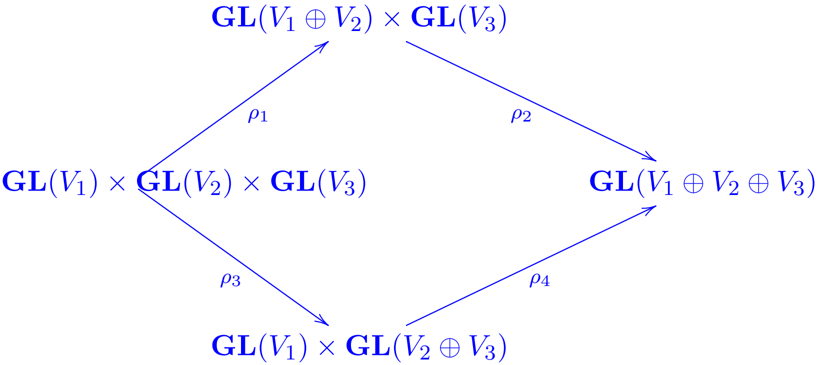
Associativity of the direct sum

In particular, for the operations of direct sum and tensor product the following well known relations hold:

$$(\xi_1 \oplus \xi_2) \oplus \xi_3 = \xi_1 \oplus (\xi_2 \oplus \xi_3).$$

Associativity of the direct sum

This relation is a consequence of the commutative diagram



Associativity of the direct sum

where

$$\begin{aligned} \rho_1(A_1, A_2, A_3) &= (A_1 \oplus A_2, A_3), \\ \rho_2(B, A_3) &= B \oplus A_3, \\ \rho_3(A_1, A_2, A_3) &= (A_1, A_2 \oplus A_3), \\ \rho_4(A_1, C) &= A_1 \oplus C. \end{aligned}$$

Associativity of the direct sum

Then

$$\begin{aligned}\rho_2\rho_1(A_1, A_2, A_3) &= (A_1 \oplus A_2) \oplus A_3,\\ \rho_4\rho_3(A_1, A_2, A_3) &= A_1 \oplus (A_2 \oplus A_3).\end{aligned}$$

It is clear that

$$\rho_2\rho_1 = \rho_4\rho_3$$

since the relation

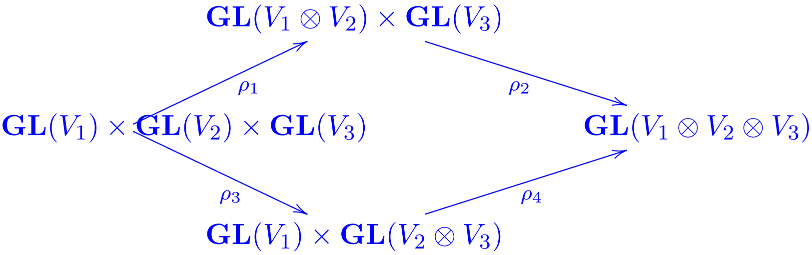
$$(A_1 \oplus A_2) \oplus A_3 = A_1 \oplus (A_2 \oplus A_3)$$

is true for matrices.

Associativity for tensor products

$$(\xi_1 \otimes \xi_2) \otimes \xi_3 = \xi_1 \otimes (\xi_2 \otimes \xi_3).$$

This relation is a consequence of the following commutative diagram



Associativity for tensor products

The commutativity of the diagram is implied from the relation

$$(A_1 \otimes A_2) \otimes A_3 = A_1 \otimes (A_2 \otimes A_3)$$

for matrices.

Distributivity

$$(\xi_1 \oplus \xi_2) \otimes \xi_3 = (\xi_1 \otimes \xi_3) \oplus (\xi_2 \otimes \xi_3).$$

This property is implied by the corresponding relation

$$(A_1 \oplus A_2) \otimes A_3 = (A_1 \otimes A_3) \oplus (A_2 \otimes A_3).$$

for matrices.

Trivial vector bundle

Denote the trivial vector bundle with the fiber $\mathbf{R}^n$ by $\bar{n}$. The total space of trivial bundle is homeomorphic to the Cartesian product $B \times R_n$ and it follows that

$$\bar{n} = \bar{1} \oplus \bar{1} \oplus \cdots \oplus \bar{1} (n \text{ times}).$$

and

$$\xi \otimes \bar{1} = \xi,$$

$$\xi \otimes \bar{n} = \xi \oplus \xi \oplus \cdots \oplus \xi (n \text{ times}).$$

Hom

Let $V = \mathbf{Hom}\ (V_1, V_2)$ be the vector space of all linear mappings from the space V_1 to the space V_2 . For infinite dimensional Banach spaces we will assume that all linear mappings considered are bounded. Then there is a natural representation of the group $\mathbf{GL}(V_1) \times \mathbf{GL}(V_2)$ into the group $\mathbf{GL}(V)$ which to any pair $A_1 \in \mathbf{GL}(V_1),\ A_2 \in \mathbf{GL}(V_2)$ associates the mapping

$$\rho(A_1, A_2) : \mathbf{Hom}\ (V_1, V_2) \longrightarrow \mathbf{Hom}\ (V_1, V_2)$$

by the formula

$$\rho(A_1, A_2)(f) = A_2 \circ f \circ A_1^{-1}.$$

Hom

Then following the general method of constructing operations for vector bundles one obtains for each pair of vector bundles ξ_1 and ξ_2 with fibers V_1 and V_2 and transition functions $\varphi^1_{\alpha\beta}(x)$ and $\varphi^2_{\alpha\beta}(x)$ a new vector bundle with fiber V and transition functions

$$\varphi_{\alpha\beta}(x) = \rho(\varphi^1_{\alpha\beta}(x), \varphi^2_{\alpha\beta}(x))$$

This bundle is denoted by **HOM** (ξ_1, ξ_2) .

Dual vector bundles

When $V_2 = \mathbf{R}^1$, the space $\mathbf{Hom}\,(V_1, R^1)$ is denoted by V_1^* . Correspondingly, when $\xi_2 = \bar{1}$ the bundle $\mathbf{HOM}\,(\xi, \bar{1})$ will be denoted by ξ^* and called the *dual bundle*. It is easy to check that the bundle ξ^* can be constructed from ξ by means of the representation of the group $\mathbf{GL}(V)$ to itself by the formula

$$A \longrightarrow (A^t)^{-1},\ A \in \mathbf{GL}(V).$$

Bilinear mapping

There is a bilinear mapping

$$V \times V^* \xrightarrow{\beta} \mathbf{R}^1,$$

which to each pair (x, h) associates the value $h(x)$.
Consider the representation of the group $\mathbf{GL}(V)$ on the space $V \times V^*$ defined by matrix

$$A \longrightarrow \begin{Vmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{Vmatrix}.$$

Bilinear mapping

Then structural group $\mathbf{GL}(V \times V^*)$ of the bundle $\xi \oplus \xi^*$ is reduced to the subgroup $\mathbf{GL}(V)$. The action of the group $\mathbf{GL}(V)$ on the $V \times V^*$ has the property that the mapping

$$V \times V^* \xrightarrow{\beta} \mathbf{R}^1,$$

is equivariant with respect to trivial action of the group $\mathbf{GL}(V)$ on $\mathbf{R}^1$.

Bilinear mapping

This fact means that the value of the form h on the vector x does not depend on the choice of the coordinate system in the space V . Hence there exists a continuous mapping

$$\bar{\beta} : \xi \oplus \xi^{*} \longrightarrow \bar{1},$$

which coincides with β on each fiber.

Exterior powers

Let $\Lambda_k(V)$ be the k-th exterior power of the vector space V . Then to each transformation $A : V \longrightarrow V$ is associated the corresponding exterior power of the transformation

$$\Lambda_k(A) : \Lambda_k(V) \longrightarrow \Lambda_k(V),$$

that is, there is the natural representation

$$\Lambda_k : \mathbf{GL}(V) \longrightarrow \mathbf{GL}(\Lambda_k(V)).$$

Exterior powers

The corresponding operation for vector bundles is called *the operation of the k-th exterior power* and the result denoted by $\Lambda_k(\xi)$. Similar to vector spaces, for vector bundles one has

- $\Lambda_1(\xi) = \xi$,
- $\Lambda_k(\xi) = 0$ for $k > \dim \xi$,
- $\Lambda_k(\xi_1 \oplus \xi_2) = \oplus_{\alpha=0}^{k} \Lambda_\alpha(\xi_1) \otimes \Lambda_{k-\alpha}(\xi_2)$,

where by definition $\Lambda_0(\xi) = \bar{1}$.

Generating function

It is convenient to write these relations using the generating function. Let us introduce the formal polynomial

$$\Lambda_t(\xi) = \Lambda_0(\xi) + \Lambda_1(\xi)t + \Lambda_2(\xi)t^2 + \cdots + \Lambda_n(\xi)t^n.$$

Then

$$\Lambda_t(\xi_1 \oplus \xi_2) = \Lambda_t(\xi_1) \otimes \Lambda_t(\xi_2).$$

and this formula should be interpreted as follows: the degrees of the formal variable are added and the coefficients are vector bundles formed using the operations of tensor product and direct sum.

Linear maps of vector bundles

Consider two vector bundles ξ_1 and ξ_2 where

$$\xi_i = \{p_i : E_i \longrightarrow B,\ V_i \text{ is fiber}\}.$$

Consider a fiberwise continuous mapping

$$f : E_1 \longrightarrow E_2.$$

The map f will be called a *linear map of vector bundles* or *homomorphism of bundles* if f is linear on each fiber.
The family of all such linear mappings will be denoted by

$$\mathbf{Hom}\,(\xi_1, \xi_2).$$

Linear maps of vector bundles

Then the following relation holds:

$$\mathbf{Hom}\,(\xi_1, \xi_2) = \Gamma(B, \mathbf{HOM}\,(\xi_1, \xi_2)).$$

By intuition, this relation is evident since elements from both the left-hand and right-hand sides are families of linear transformations from the fiber V_1 to the fiber V_2 , parametrized by points of the base B .

Linear maps of vector bundles

To prove the relation

$$\mathbf{Hom}\,(\xi_1, \xi_2) = \Gamma(B, \mathbf{HOM}\,(\xi_1, \xi_2)),$$

let us express elements from both the left-hand and right-hand sides of the relation in terms of local coordinates. Consider an atlas $\{U_\alpha\}$ and coordinate homeomorphisms $\varphi^1_\alpha,\ \varphi^2_\alpha$ for bundles $\xi_1,\ \xi_2$.

Linear maps of vector bundles

By means of the mapping $f : E_1 \longrightarrow E_2$ we construct a family of mappings:

$$(\varphi_\alpha^2)^{-1} \cdot f \cdot \varphi_\alpha^1 : U_\alpha \times V_1 \longrightarrow U_\alpha \times V_2,$$

defined by the formula:

$$\left((\varphi_\alpha^2)^{-1} \cdot f \cdot \varphi_\alpha^1\right)(x, h) = (x, f_\alpha(x)(h)),$$

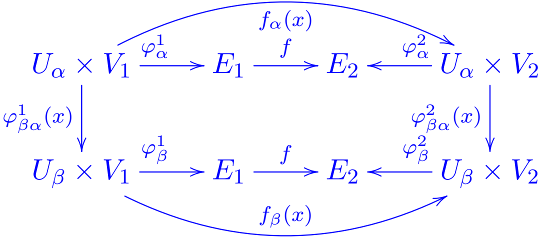
for the continuous family of linear mappings

$$f_\alpha(x) : V_1 \longrightarrow V_2, \quad x \in U_\alpha.$$

Linear maps of vector bundles

On the intersection of two charts $U_\alpha \cap U_\beta$ two functions $f_\alpha(x)$ and $f_\beta(x)$ satisfy the following condition

$$\varphi^2_{\beta\alpha}(x) f_\alpha(x) = f_\beta(x) \varphi^1_{\beta\alpha}(x),$$



Linear maps of vector bundles

This means that

$$f_\beta(x) = \varphi^2_{\beta\alpha}(x) f_\alpha(x) \varphi^1_{\alpha\beta}(x),$$

or

$$f_\beta(x) = \varphi_{\beta\alpha}(x)(f_\alpha(x)),$$

where $\varphi_{\beta\alpha}(x)$ is the transition function of the bundle **HOM** (ξ_1, ξ_2) which is defined by the formula

$$\varphi_{\beta\alpha}(x)(f) = \varphi^2_{\beta\alpha}(x) f \varphi^1_{\alpha\beta}(x).$$

Linear maps of vector bundles

In other words, the family of functions

$$f_\alpha(x) \in V = \mathbf{Hom}\,(V_1, V_2),\ x \in U_\alpha$$

satisfies the condition

$$f_\beta(x) = \varphi_{\beta\alpha}(x)(f_\alpha(x)),$$

that is, determines a section of the bundle $\mathbf{HOM}\,(\xi_1, \xi_2)$.

$$f \in \Gamma(B, \mathbf{HOM}\,(\xi_1, \xi_2)).$$

Linear maps of vector bundles

Conversely, given a section of the bundle $\mathbf{HOM}\ (\xi_1, \xi_2)$,

$$f \in \Gamma(B, \mathbf{HOM}\ (\xi_1, \xi_2)),$$

that is, a family of functions $f_\alpha(x)$ satisfying condition

$$f_\beta(x) = \varphi_{\beta\alpha}(x)(f_\alpha(x)),$$

defines a linear mapping from the bundle ξ_1 to the bundle ξ_2 ,

$$f : \xi_1 \longrightarrow \xi_2.$$

Linear maps of vector bundles

In particular, if

$$\xi_1 = \bar{1},\ V_1 = \mathbf{R}^1$$

then

$$\mathbf{Hom}\ (V_1, V_2) = V_2.$$

Hence

$$\mathbf{HOM}\ (\bar{1}, \xi_2) = \xi_2.$$

Hence

$$\Gamma(\xi_2) = \mathbf{Hom}\ (\bar{1}, \xi_2),$$

that is, the space of all sections of vector bundle ξ_2 is identified with the space of all linear mappings from the one dimensional trivial bundle $\bar{1}$ to the bundle ξ_2 .

Bilinear form

The second example of mappings of vector bundles gives an analogue of bilinear form for vector bundles. Bilinear form an a linear space is a mapping

$$V \times V \longrightarrow \mathbf{R}^1,$$

which is linear with respect to each argument. Consider a continuous family of bilinear forms parametrized by points of base.

Bilinear form

This gives us a definition of bilinear form on vector bundle, namely, a fiberwise continuous mapping

$$f : \xi \oplus \xi \longrightarrow \bar{1}$$

which is bilinear in each fiber and is called a *bilinear form on the bundle* ξ .

Bilinear form

Just as on a linear space, a bilinear form on the vector bundle

$$f : \xi \oplus \xi \longrightarrow \bar{1}$$

induces a linear mapping from the vector bundle ξ to its dual bundle ξ^*

$$\bar{f} : \xi \longrightarrow \xi^*,$$

such that f decomposes into the composition

$$\xi \oplus \xi \xrightarrow{\bar{f} \oplus \mathbf{Id}} \xi^* \oplus \xi \xrightarrow{\beta} \bar{1},$$

Bilinear form

where

$$\mathbf{Id} : \xi \longrightarrow \xi$$

is the identity mapping and

$$\xi \oplus \xi \xrightarrow{\bar{f} \oplus \mathbf{Id}} \xi^* \oplus \xi$$

is the direct sum of mappings $\bar{f}$ and $\mathbf{Id}$ on each fiber.

Definition

Definition of scalar product

When the bilinear form f is symmetric, positive and nondegenerate we say that f is a *scalar product on the bundle* ξ .

Theorem (Existence of scalar product)

Let ξ be a finite dimensional vector bundle over a compact base space B . Then there exists a scalar product on the bundle ξ , that is, a nondegenerate, positive, symmetric bilinear form on the ξ .

Proof.

We must construct a fiberwise mapping

$$f : \xi \oplus \xi \longrightarrow \bar{1}$$

which is bilinear, symmetric, positive, nondegenerate form in each fiber. This means that if $x \in B$, $v_1, \ v_2 \in p^{-1}(x)$ then the value $f(v_1, v_2)$ can be identified with a real number such that

$$f(v_1, v_2) = f(v_2, v_1)$$

and $f(v, v) > 0$ for any $v \in p^{-1}(x), \ v \neq 0$.

Scalar product

Consider the weaker condition

$$f(v, v) \geq 0.$$

Then we obtain a nonnegative bilinear form on the bundle ξ . If $f_1,\ f_2$ are two nonnegative bilinear forms on the bundle ξ then the sum $f_1 + f_2$ and a linear combination $\varphi_1 f_1 + \varphi_2 f_2$ for any two nonnegative continuous functions φ_1 and φ_2 on the base B gives a nonnegative bilinear form as well.

Scalar product

Let $\{U_\alpha\}$ be an atlas for the bundle ξ . The restriction $\xi|_{U_\alpha}$ is a trivial bundle and is therefore isomorphic to a Cartesian product $U_\alpha \times V$ where V is fiber of ξ . Therefore the bundle $\xi|_{U_\alpha}$ has a nondegenerate positive definite bilinear form

$$f_\alpha : \xi|_{U_\alpha} \oplus \xi|_{U_\alpha} \longrightarrow \bar{1}.$$

In particular, if $v \in p^{-1}(x),\ x \in U_\alpha$ and $v \neq 0$ then

$$f_\alpha(v, v) > 0.$$

Scalar product

Consider a partition of unity $\{g_\alpha\}$ subordinate to the atlas $\{U_\alpha\}$. Then

$$0 \le g_\alpha(x) \le 1,$$

$$\sum_\alpha g_\alpha(x) \equiv 1,$$

$$\textbf{supp } g_\alpha \subset U_\alpha.$$

Scalar product

We extend the form f_α by formula

$$\bar{f}_\alpha(v_1, v_2) = \begin{cases} g_\alpha(x) f_\alpha(v_1, v_2) & v_1, v_2 \in p^{-1}(x) \quad x \in U_\alpha, \\ 0 & v_1, v_2 \in p^{-1}(x) \quad x \notin U_\alpha. \end{cases}$$

Scalar product

It is clear that the form defines a continuous nonnegative form on the bundle ξ . Put

$$f(v_1, v_2) = \sum_{\alpha} f_{\alpha}(v_1, v_2).$$

The form $f(v_1, v_2)$ is then positive definite. Actually, let $0 \neq v \in p^{-1}(x)$. Then there is an index α such that

$$g_{\alpha}(x) > 0.$$

Scalar product

This means that

$$x \in U_\alpha \text{ and } f_\alpha(v, v) > 0.$$

Hence

$$\bar{f}_\alpha(v, v) > 0$$

and

$$f(v, v) > 0.$$

Reduction

to $\mathbf{O}(n)$

Theorem (Reduction to $\mathbf{O}(n)$)

For any vector bundle ξ over a compact base space B with $\dim \xi = n$, structural group $\mathbf{GL}(n, \mathbf{R})$ reduces to subgroup $\mathbf{O}(n)$. PROOF. Let us give another geometric interpretation of the property that the bundle ξ is locally trivial. Let U_α be a chart and let

$$\varphi_\alpha : U_\alpha \times V \longrightarrow p^{-1}(U_\alpha)$$

be a trivializing coordinate homeomorphism.

Proof.

Then any vector $v \in V$ defines a section of the bundle ξ over the chart U_α

$$\sigma : U_\alpha \longrightarrow p^{-1}(U_\alpha),$$
$$\sigma(x) = \varphi_\alpha(x, v) \in p^{-1}(U_\alpha).$$

If $v_1, \dots, v_n$ is a basis for the space V then corresponding sections

$$\sigma_k^\alpha(x) = \varphi_\alpha(x, v_k)$$

form a system of sections such that for each point $x \in U_\alpha$ the family of vectors $\sigma_1^\alpha(x), \dots, \sigma_n^\alpha(x) \in p^{-1}(x)$ is a basis in the fiber $p^{-1}(x)$.

Proof.

Conversely, if the system of sections

$$\sigma_1^\alpha, \dots, \sigma_n^\alpha : U_\alpha \longrightarrow p^{-1}(U_\alpha)$$

forms basis in each fiber then we can recover a trivializing coordinate homeomorphism

$$\varphi_\alpha(x, \sum_i \lambda_i v_i) = \sum_i \lambda_i \sigma_i^\alpha(x) \in p^{-1}(U_\alpha).$$

Proof.

From this point of view, the transition function $\varphi^{\beta\alpha} = \varphi_\beta^{-1}\varphi_\alpha$ has an interpretation as a change of basis matrix from the basis $\{\sigma_1^\alpha(x), \ldots, \sigma_n^\alpha(x)\}$ to $\{\sigma_1^\beta(x), \ldots, \sigma_n^\beta(x)\}$ in the fiber $p^{-1}(x),\ x \in U_\alpha \cap U_\beta$. Thus the theorem will be proved if we construct in each chart U_α a system of sections $\{\sigma_1^\alpha, \ldots, \sigma_n^\alpha\}$ which form an orthonormal basis in each fiber with respect to a inner product in the bundle ξ .

Proof.

Then the transition matrices from one basis $\{\sigma_1^\alpha(x), \ldots, \sigma_n^\alpha(x)\}$ to another basis $\{\sigma_1^\beta(x), \ldots, \sigma_n^\beta(x)\}$ will be orthonormal, that is, $\varphi_{\beta\alpha}(x) \in \mathbf{O}(n)$. The proof of the theorem will be completed by the following lemma.

Lemma (Orthonormal basis)

Let ξ be a vector bundle, f a scalar product in the bundle ξ and $\{U_\alpha\}$ an atlas for the bundle ξ . Then for any chart U_α there is a system of sections $\{\sigma_1^\alpha, \ldots, \sigma_n^\alpha\}$ orthonormal in each fiber $p^{-1}(x),\ x \in U_\alpha$.

Proof.

The proof of the lemma simply repeats the Gramm-Schmidt method of construction of orthonormal basis. Let

$$\tau_1, \ldots, \tau_n : U_\alpha \longrightarrow p^{-1}(U_\alpha)$$

be an arbitrary system of sections forming a basis in each fiber $p^{-1}(x), \ x \in U_\alpha$.

Proof.

Since for any $x \in U_\alpha$,

$$\tau_1(x) \neq 0$$

one has

$$f(\tau_1(x), \tau_1(x)) > 0.$$

Put

$$\tau_1'(x) = \frac{\tau_1(x)}{\sqrt{f(\tau_1(x), \tau_1(x))}}.$$

Proof.

The new system of sections $\tau_1', \tau_2, \ldots, \tau_n$ forms a basis in each fiber. Put

$$\tau_2''(x) = \tau_2(x) - f(\tau_2(x), \tau_1'(x))\tau_1'(x).$$

The new system of sections $\tau_1', \tau_2'', \tau_3(x), \ldots, \tau_n$ forms a basis in each fiber.

Proof.

The vectors $\tau_1'(x)$ have unit length and are orthogonal to the vectors $\tau_2''(x)$ at each point $x \in U_\alpha$. Put

$$\tau_2'(x) = \frac{\tau_2''(x)}{\sqrt{f(\tau_2''(x), \tau_2''(x))}}.$$

Proof.

Again, the system of sections $\tau_1', \tau_2', \tau_3(x), \ldots, \tau_n$ forms a basis in each fiber and, moreover, the vectors τ_1', τ_2' are orthonormal. Then we rebuild the system of sections by induction. Let the sections

$$\tau_1', \ldots, \tau_k', \tau_{k+1}(x), \ldots, \tau_n$$

form a basis in each fiber and suppose that the sections $\tau_1', \ldots, \tau_k'$ be are orthonormal in each fiber.

Proof.

Put

$$\tau''_{k+1}(x) = \tau_{k+1}(x) - \sum_{i=1}^{k} f(\tau_{k+1}(x), \tau'_i(x))\tau'_i(x),$$

$$\tau'_{k+1}(x) = \frac{\tau''_{k+1}(x)}{\sqrt{f(\tau''_{k+1}(x), \tau''_{k+1}(x))}}.$$

Proof.

It is easy to check that the system $\tau'_1, \dots, \tau'_{k+1}, \tau_{k+2}(x), \dots, \tau_n$ forms a basis in each fiber and the sections $\tau'_1, \dots, \tau'_{k+1}$ are orthonormal. The lemma is proved by induction. Thus the proof of the theorem is finished. ∎

Remark

In the lemma we proved a stronger statement: if $\{\tau_1, \ldots, \tau_n\}$ is a system of sections of the bundle ξ in the chart U_α which is a basis in each fiber $p^{-1}(x)$ and if in addition vectors $\{\tau_1, \ldots, \tau_k\}$ are orthonormal then there are sections $\{\tau'_{k+1}, \ldots, \tau'_n\}$ such that the system

$$\{\tau_1, \ldots, \tau_k, \tau'_{k+1}, \ldots, \tau'_n\}$$

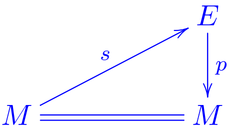
is orthonormal in each fiber. In other words, if a system of orthonormal sections can be extended to basis then it can be extended to orthonormal basis.

Remark

In theorems the condition of compactness of the base B can be replaced by the condition of paracompactness. In the latter case we should first choose a locally finite atlas of charts.

Definition (Space of sections of vector bundle)

Let $\xi: E \xrightarrow{p} M$ be a vector bundle with fixed smooth structure. Let $\Gamma^\infty(M, \xi)$ be the space of all smooth sections of the vector bundle ξ . The space $\Gamma^\infty(M, \xi)$ has a natural structure of a module over the algebra $\mathbf{C}^\infty(M)$.
Consider a section s :



and a function $f \in \mathcal{C}^\infty(M)$. Put

$$(f \cdot s)(x) = f(x) \cdot s(x), \; x \in M.$$

Algebraic properties

- 1-dimensional trivial vector bundle, $\xi \approx \bar{1}$, gives the module of section

$$\Gamma^{\infty}(M, \bar{1}) \approx \mathbf{C}^{\infty}(M).$$

- If the vector bundle splits into a direct sum, $\xi \approx \eta \oplus \zeta$, then

$$\Gamma^{\infty}(M, \xi) \approx \Gamma^{\infty}(M, \eta) \oplus \Gamma^{\infty}(M, \zeta).$$

Theorem (Free module of sections of trivial bundle)

If the vector bundle $\xi: \ E \xrightarrow{p} M$ is trivial,

$$\begin{array}{ccc} E & \longrightarrow & M \times \mathbf{R}^n \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

$\xi = \bar{1} \oplus \bar{1} \oplus \cdots \oplus \bar{1}$, then the module of sections $\Gamma^\infty(M, \xi)$ is isomorphic to free $\mathbf{C}^\infty(M)$ –module,

$$\Gamma^\infty(M, \xi) \approx (\mathbf{C}^\infty(M))^n = \mathbf{C}^\infty(M) \oplus \mathbf{C}^\infty(M) \oplus \cdots \oplus \mathbf{C}^\infty(M).$$

Theorem (Whitney theorem)

Assume that M is compact manifold, and ξ is a vector bundle. Then there is a vector bundle η such that

$$\xi \oplus \eta \approx \bar{N} \approx M \times \mathbf{R}^N.$$

Consequence (Projective module)

The module $\Gamma^\infty(M,\xi)$ is a projective finitely generated module.

Definition (Differential Form)

Definition

Definition

Definition

Definition

Characteristic classes

Motivation

We showed that, generally speaking, any bundle can be obtained as an inverse image or pull back of a universal bundle by a continuous mapping of the base spaces. In particular, isomorphisms of vector bundles over X are characterized by homotopy classes of continuous mappings of the space X to the classifying space $\mathbf{BO}(n)$ (or $\mathbf{BU}(n)$ for complex bundles).

Motivation

But it is usually difficult to describe homotopy classes of maps from X into $\mathbf{BO}(n)$.
Instead, it is usual to study certain invariants of vector bundles defined in terms of the homology or cohomology groups of the space X .

Definition

Following this idea, we use the term *characteristic class* for a correspondence α which associates to each n -dimensional vector bundle ξ over X a cohomology class $\alpha(\xi) \in H^*(X)$ with some fixed coefficient group for the cohomology groups.

Definition

In addition, we require functoriality : if

$$f : X \longrightarrow Y$$

is a continuous mapping, η an n –dimensional vector bundle over Y , and $\xi = f^*(\eta)$ the pull-back vector bundle over X , then

$$\alpha(\xi) = f^*(\alpha(\eta)),$$

where f^* denotes the induced natural homomorphism of cohomology groups

$$f^* : H^*(Y) \longrightarrow H^*(X).$$

Comments

If we know the cohomology groups of the space X and the values of all characteristic classes for given vector bundle ξ , then might hope to identify the bundle ξ , that is, to distinguish it from other vector bundles over X . In general, this hope is not justified. Nevertheless, the use of characteristic classes is a standard technique in topology and in many cases gives definitive results.

Let us pass on to study properties of characteristic classes.

Theorem (Description of characteristic classes)

The family of all characteristic classes of n –dimensional real (complex) vector bundles is in one–to–one correspondence with the cohomology ring $H^*(\mathbf{BO}(n))$ (respectively, with $H^*(\mathbf{BU}(n))$).

Proof

Let ξ_n be the universal bundle over the classifying space $\mathbf{BO}(n)$ and α a characteristic class. Then $\alpha(\xi_n) \in H^*(\mathbf{BO}(n))$ is the associated cohomology class. Conversely, if $x \in H^*(\mathbf{BO}(n))$ is arbitrary cohomology class then a characteristic class α is defined by the following rule: if $f : X \longrightarrow \mathbf{BO}(n)$ is continuous map and $\xi = f^*(\xi_n)$ put

$$\alpha(\xi) = f^*(x) \in H^*(X).$$

Let us check that this correspondence gives a characteristic class.

Proof.

If

$$g : X \longrightarrow Y$$

is continuous map and

$$h : Y \longrightarrow \mathbf{BO}(n)$$

is a map such that

$$\eta = h^*(\xi_n),\ \xi = g^*(\eta),$$

Proof.

then

$$\alpha(\xi) = \alpha((hg)^*(\xi_n)) = (hg)^*(x) = g^*(h^*(x)) = \\ = g^*(\alpha(h^*(\xi_n))) = g^*(\alpha(\eta)).$$

If

$$f : \mathbf{BO}(n) \longrightarrow \mathbf{BO}(n)$$

is the identity mapping then

$$\alpha(\xi_n) = f^*(x) = x.$$

Hence the class α corresponds to the cohomology class x . ∎

Sequence of characteristic classes

We now understand how characteristic classes are defined on the family of vector bundles of a fixed dimension. The characteristic classes on the family of all vector bundles of any dimension should be as follows: a characteristic class is a sequence

$$\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots\}$$

where each term α_n is a characteristic class defined on vector bundles of dimension n .

Definition (Stable characteristic classes)

A class α of the form

$$\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_n, \dots\}$$

is said to be *stable* if the following condition holds:

$$\alpha_{n+1}(\xi \oplus \mathbf{1}) = \alpha_n(\xi),$$

for any n –dimensional vector bundle ξ .

Stable characteristic classes

In accordance with the theorem one can think of α_n as a cohomology class

$$\alpha_n \in H^*(\mathbf{BO}(n)).$$

Let

$$\varphi : \mathbf{BO}(n) \longrightarrow \mathbf{BO}(n+1)$$

be the natural mapping for which

$$\varphi^*(\xi_{n+1}) = \xi_n \oplus \mathbf{1}.$$

Stable characteristic classes

This mapping φ is induced by the natural inclusion of groups

$$\mathbf{O}(n) \subset \mathbf{O}(n+1).$$

Then the condition

$$\alpha_{n+1}(\xi \oplus \mathbf{1}) = \alpha_n(\xi)$$

is equivalent to:

$$\varphi^*(\alpha_{n+1}) = \alpha_n.$$

Stable characteristic classes

Consider the sequence

$$\mathbf{BO}(1)\longrightarrow\mathbf{BO}(2)\longrightarrow\ldots\longrightarrow\mathbf{BO}(n)\longrightarrow\mathbf{BO}(n+1)\longrightarrow\ldots$$

and the direct limit

$$\mathbf{BO}=\varinjlim\mathbf{BO}(n).$$

Let

$$H^*(\mathbf{BO})=\varprojlim H^*\left(\mathbf{BO}(n)\right).$$

Stable characteristic classes

Condition

$$\varphi^*(\alpha_{n+1}) = \alpha_n$$

means that the family of stable characteristic classes is in one-to-one correspondence with the cohomology ring $H^*(\mathbf{BO})$.

Now we consider the case of cohomology with integer coefficients.

Theorem (Integer-valued characteristic classes)

The ring $H^*(\mathbf{BU}(n);\ \mathbf{Z})$ of integer cohomology classes is isomorphic to the polynomial ring $\mathbf{Z}[c_1, c_2, \ldots, c_n]$, where

$$c_k \in H^{2k}(\mathbf{BU}(n);\ \mathbf{Z}).$$

Theorem sequential

The generators $\{c_1, c_2, \ldots, c_n\}$ can be chosen such that

- the natural mapping

$$\varphi : \mathbf{BU}(n) \longrightarrow \mathbf{BU}(n+1)$$

satisfies the conditions

$$\begin{array}{rcl} \varphi^*(c_k) & = & c_k, \ k = 1, 2, \ldots, n, \\ \varphi^*(c_{n+1}) & = & 0; \end{array}$$

Theorem sequential

- for a direct sum of vector bundles we have the relations

$$\begin{aligned} c_k(\xi \oplus \eta) &= c_k(\xi) + c_{k-1}(\xi)c_1(\eta) + \\ &+ c_{k-2}(\xi)c_2(\eta) + \cdots + c_1(\xi)c_{k-1}(\eta) + c_k(\eta) = \\ &= \sum_{\alpha+\beta=k} c_\alpha(\xi)c_\beta(\eta), \end{aligned}$$

where $c_0(\xi) = 1$.

Stable characteristic class

The condition $\varphi^*(c_k) = c_k$ means that the sequence

$$\{ \underbrace{0, \ldots, 0}_{(k-1)\ times}, c_k, c_k, \ldots, c_k, \ldots \}$$

is a stable characteristic class which will also be denoted by c_k . This notation was used in next relations. If $\dim \xi < k$ then $c_k(\xi) = 0$.

Generating function

Formula

$$c_k(\xi \oplus \eta) = \sum_{\alpha+\beta=k} c_\alpha(\xi) c_\beta(\eta)$$

can be written in a simpler way. Define the generating function as a formal series

$$c = 1 + c_1 + c_2 + \dots + c_k + \dots$$

The formal series has a well defined value on any vector bundle ξ since in the infinite sum in the formal series only a finite number of the summands will be nonzero:

$$c(\xi) = 1 + c_1(\xi) + c_2(\xi) + \dots + c_k(\xi), \text{ if } \dim \xi = k.$$

Generating function

Hence from

$$c_k(\xi \oplus \eta) = \sum_{\alpha+\beta=k} c_\alpha(\xi)c_\beta(\eta)$$

we see that

$$c(\xi \oplus \eta) = c(\xi)c(\eta).$$

Conversely, the relations $c_k(\xi \oplus \eta) = \sum_{\alpha+\beta=k} c_\alpha(\xi)c_\beta(\eta)$ may be obtained from $c(\xi \oplus \eta) = c(\xi)c(\eta)$ by considering the homogeneous components.

Filtration

The spectral sequence for locally trivial bundles is constructed using a filtration of a space X . The construction of the spectral sequence described below can be applied not only to cohomology theory but to any generalized cohomology theory. Thus let

$$X_0 \subset X_1 \subset \cdots \subset X_N = X$$

be a increasing filtration of the space X .

Homology exact sequence

Consider the cohomology exact sequence for a pair (X_p, X_{p-1}) induced by the sequence

$$X_{p-1} \xrightarrow{i} X_p \xrightarrow{j} (X_p, X_{p-1}) :$$

$$\ldots \longrightarrow H^{p+q-1}(X_p) \xrightarrow{i^*} H^{p+q-1}(X_{p-1}) \xrightarrow{\partial}$$

$$\xrightarrow{\partial} H^{p+q}(X_p, X_{p-1}) \xrightarrow{j^*} H^{p+q}(X_p) \xrightarrow{i^*} H^{p+q}(X_{p-1}) \xrightarrow{\partial}$$

$$\xrightarrow{\partial} H^{p+q+1}(X_p, X_{p-1}) \xrightarrow{j^*} \ldots$$

Put

$$\begin{aligned} D &= \bigoplus_{p,q} D^{p,q} = \bigoplus_{p,q} H^{p+q}(X_p) \\ E &= \bigoplus_{p,q} E^{p,q} = \bigoplus_{p,q} H^{p+q}(X_p, X_{p-1}). \end{aligned}$$

Homology exact sequence

After summation by all p and q , the sequences

$$\ldots \longrightarrow \bigoplus D^{p,q-1} \xrightarrow{i^*} \bigoplus D^{p-1,q} \xrightarrow{\partial}$$

$$\xrightarrow{\partial} \bigoplus E^{p,q} \xrightarrow{j^*} \bigoplus D^{p,q} \xrightarrow{i^*} \bigoplus D^{p-1,q+1} \xrightarrow{\partial}$$

$$\xrightarrow{\partial} \bigoplus E^{p,q+1} \xrightarrow{j^*} \ldots,$$

can be written briefly as

$$\ldots \longrightarrow D \xrightarrow{i^*} D \xrightarrow{\partial} E \xrightarrow{j^*} D \xrightarrow{i^*} D \xrightarrow{\partial} E \longrightarrow \ldots,$$

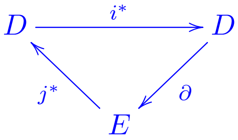
Homology exact sequence

where the bigradings of the homomorphisms ∂ , j^* , i^* are as follows:

$$\begin{aligned} \deg i^* &= (-1,1) \\ \deg j^* &= (0,0) \\ \deg \partial &= (1,0). \end{aligned}$$

Exact triangle

The sequence can be written as an exact triangle



Exact triangle

Put

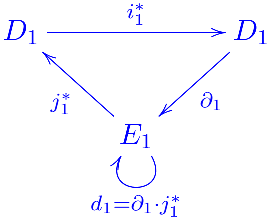
$$d = \partial j^* : E \longrightarrow E.$$

Then the bigrading of d is $\deg d = (1, 0)$ and it is clear that $d^2 = 0$. Rename all objects as

$$\begin{aligned} D_1 &= D, \\ E_1 &= E, \\ i_1 &= i^*, \\ j_1 &= j^*, \\ \partial_1 &= \partial, \\ d_1 &= d. \end{aligned}$$

The first exact triangle

The sequence can be rewritten as the first exact triangle



The second exact triangle

Now put

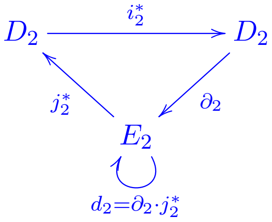
$$\begin{aligned} D_2 &= i_1(D_1) \subset D_1, \\ E_2 &= H(E_1, d_1). \end{aligned}$$

The grading of D_2 is inherited from the grading as an image, that is, $D_2 = \bigoplus D_2^{p,q}, \quad D_2^{p,q} = i_1(D_1^{p+1,q-1})$. The grading of E_2 is inherited from E_1 . Then we put

$$\begin{aligned} i_2 &= i_1|_{D_2} : D_2 \longrightarrow D_2, \\ \partial_2 &= \partial_1 i_1^{-1}, \\ j_2 &= j_1. \end{aligned}$$

The second exact triangle

All maps i_2 , j_2 and ∂_2 are well defined and form a new second exact triangle. This triangle is said to be derived from the first triangle.



Well defined maps

We can check that i_2 , j_2 and ∂_2 are well defined. In fact, if $x \in D_2$ then $i_1(x) \in D_2$. If $x \in D_1$ then $x = i_1(y)$ and

$$\partial_2(x) = [\partial_1(y)] \in H(E_1, d_1).$$

The latter inclusion follows from the identity

$$d_1\partial_1(y) = \partial_1 j_1 \partial_1(y) = 0.$$

If $i_1(y) = 0$, the exactness of the first triangle gives $y = j_1(x)$ and then

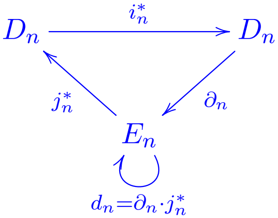
$$\partial_2(x) = [\partial_1 j_1(z)] = [d_1(z)] = 0.$$

Well defined maps

Hence ∂_2 is well defined. Finally, if $x \in E_1,\ d_1(x) = 0$ then $\partial_1 j_1(x) = 0$. Hence $j_1(x) = i_1(z) \in D_2$. If $x = d_1(y),\ x = \partial_1 j_1(z)$ then $j_1(x) = j_1 \partial_1 j_1(z) = 0$. Hence j_2 is well defined.

Series of exact triangles

Repeating the process one can construct a series of exact triangles



Series of exact triangles

The bigradings of homomorphisms i_n , j_n , ∂_n and $d_n = \partial_n j_n$ are as follows:

$$\begin{array}{rcl} \deg i_n & = & (-1, 1),) \\ \deg j_n & = & (0, 0),) \\ \deg \partial_n & = & (n, -n+1), \\ \deg d_n & = & (n, -n+1). \end{array}$$

Definition of spectral sequence

The sequence

$$(E_n, d_n)$$

is called the *spectral sequence in the cohomology theory* associated with a filtration

$$X_0 \subset X_1 \subset \cdots \subset X_N = X.$$

Theorem (Convergence of spectral sequence)

The spectral sequence (E_n, d_n) converges to the graded groups associated to the group $H^*(X)$ by the filtration $X_0 \subset X_1 \subset \cdots \subset X_N = X$:

Conventional sign

$E_n^{p,q} \Rightarrow E_\infty^{p,q}$

$$E_\infty^{p,q} \approx \frac{\mathbf{Ker}\,(H^{p+q}(X) \longrightarrow H^{p+q}(X_{p-1}))}{\mathbf{Ker}\,(H^{p+q}(X) \longrightarrow H^{p+q}(X_p))}.$$

Proof

According to the definition

$$D = \bigoplus_{p,q} D^{p,q} = \bigoplus_{p,q} H^{p+q}(X_p)$$
$$E = \bigoplus_{p,q} E^{p,q} = \bigoplus_{p,q} H^{p+q}(X_p, X_{p-1}).$$

Hence $E_1^{p,q} = 0$ for $p > N$. Hence for $n > N$, $d_n = 0$, that is,

$$E_n^{p,q} = E_{n+1}^{p,q} = \cdots = E_\infty^{p,q}.$$

Proof

By definition, we have $D_n^{p,q} = \mathbf{Im}\ \left(H^{p+q}(X_{p+n}) \longrightarrow H^{p+q}(X_p)\right)$. Hence for $n > N$, $D_n^{p,q} = \mathbf{Im}\ \left(H^{p+q}(X) \longrightarrow H^{p+q}(X_p)\right)$. Hence the homomorphism i_n^* is an epimorphism:

$$\begin{array}{ccccc} H^{p+q}(X) & \xrightarrow{epi} & \mathbf{Im}\ \left(H^{p+q}(X) \longrightarrow H^{p+q}(X_p)\right) & \hookrightarrow & H^{p+q}(X_p) \\ \| & & \downarrow{\scriptstyle i_n^*} & & \downarrow{\scriptstyle i^*} \\ H^{p+q}(X) & \xrightarrow{epi} & \mathbf{Im}\ \left(H^{p+q}(X) \longrightarrow H^{p+q}(X_{p-1})\right) & \hookrightarrow & H^{p+q}(X_{p-1}) \end{array}$$

Proof

Hence the exact triangle turns into the exact sequence

$$\cdots \xrightarrow{\partial_n} E_n^{p,q} \xrightarrow{j_n} D_n^{p,q} \xrightarrow{i_n} D_n^{p-1,q+1} \xrightarrow{\partial_n} \cdots$$

$$\cdots \to 0 \to E_n^{p,q}, \qquad D_n^{p-1,q+1} \to 0 \to \cdots$$

Hence

$$E_n^{p,q} = \mathbf{Ker}\ i_n = \frac{\mathbf{Ker}\ \left(H^{p+q}(X) \longrightarrow H^{p+q}(X_{p-1})\right)}{\mathbf{Ker}\ \left(H^{p+q}(X) \longrightarrow H^{p+q}(X_p)\right)}$$

∎

Locally trivial bundle

Let $p : Y \longrightarrow X$ be a locally trivial bundle with fibre F . Then the fundamental group $\pi_1(X, x_0)$ acts on the fiber F in the sense that there is natural homomorphism

$$\rho : \pi_1(X, x_0) \longrightarrow [F, F],$$

where $[F, F]$ is the family of homotopy equivalences of F . Then the homomorphism ρ induces an action of the group $\pi_1(X, x_0)$ on the groups $H^*(F)$.

Theorem (The second term of the spectral sequence)

Let $p:Y\longrightarrow X$ be a locally trivial bundle with fiber F and the trivial action of $\pi_1(X,x_0)$ on the cohomology groups of the fiber. Then the spectral sequence generated by filtration $Y_k=p^{-1}([X]^k)$, where $[X]^k$ is k –dimensional skeleton of X , converges to the groups associated to $H^*(Y)$ and the second term has the following form:

$$E_2^{p,q}=H^p\left(X,H^q(F)\right).$$

Locally trivial bundle. Proof.

The first term of the spectral sequence E_1 is defined to be $E_1^{p,q} = H^{p+q}(Y_p, Y_{p-1})$. Since the locally trivial bundle is trivial over each cell, the pair (Y_p, Y_{p-1}) has the same cohomology as the union $\bigcup_j(\sigma_j^p \times F, \partial\sigma_j^p \times F)$, that is,

$$E_1^{p,q} \approx \bigoplus_j H^{p,q}(\sigma_j^p \times F, \partial\sigma_j^p \times F) = \bigoplus_j H^q(F).$$

Hence, we can identify the term E_1 with the cochain group

$$E_1^{p,q} = C^p\left(X; H^q(F)\right)$$

with coefficients in the group $H^q(F)$.

Locally trivial bundle. Proof.

What we need to establish is that the differential d_1 coincides with the coboundary homomorphism in the chain groups of the space X. This coincidence follows from the exact sequence

$$\ldots \longrightarrow H^{p-1+q}(X_{p-1}) \xrightarrow{\partial} H^{p+q}(X_p, X_{p-1}) \xrightarrow{j^*}$$

$$\longrightarrow H^{p+q}(X_p) \xrightarrow{i^*} H^{p+q}(X_{p-1}) \longrightarrow \ldots$$

Locally trivial bundle. Proof.

Notice that the coincidence

$$E_1^{p,q} = C^p\left(X; H^q(F)\right)$$

only holds if the fundamental group of the base X acts trivially in the K –groups of the fiber F . In general, the term E_1 is isomorphic to the chain group with a local system of coefficients defined by the action of the fundamental group $\pi_1(X, x_0)$ in the group $H^q(F)$.

Multiplicative spectral sequence

We say that the spectral sequence is *multiplicative* if all groups $E_s = \bigoplus_{p,q} E_s^{p,q}$ are bigraded rings, the differentials d_s are derivations, that is,

$$d_s(xy) = (d_s x)y + (-1)^{p+q} x(d_s y), x \in E_s^{p+q},$$

and the homology of d_s is isomorphic to E_{s+1} as a ring.

Theorem (Multiplicative spectral sequence)

The spectral sequence locally trivial bundles is multiplicative. The ring structure of E_∞ is isomorphic to the ring structure of groups associated with the filtration generated by the skeletons of the base X .

Proof of the theorem

Let us pass now to the proof of the theorem. The method we use for the calculation of cohomology groups of the space $\mathbf{BU}(n)$ involves spectral sequences for bundles. Firstly, using spectral sequences, we calculate the cohomology groups of unitary group $\mathbf{U}(n)$.

Since

$$\begin{aligned} H^0(\mathbf{S}^n) &= \mathbf{Z}, \\ H^n(\mathbf{S}^n) &= \mathbf{Z}, \\ H^k(\mathbf{S}^n) &= \mathbf{0}, \text{ when } k \neq 0 \text{ and } k \neq n, \end{aligned}$$

the cohomology ring

$$H^*(\mathbf{S}^n) = \oplus_k H^k(\mathbf{S}^n) = H^0(\mathbf{S}^n) \oplus H^n(\mathbf{S}^n)$$

is a free exterior algebra over the ring of integers $\mathbf{Z}$ with a generator $a_n \in H^n(\mathbf{S}^n)$. The choice of the generator a_n is not unique: one can change a_n for $(-a_n)$. We write

$$H^*(\mathbf{S}^n) = \Lambda(a_n).$$

Now consider the bundle $\mathbf{U}(2)\longrightarrow\mathbf{S}^3$ with fibre $\mathbf{S}^1$.
The second term of spectral sequence for this bundle is

$$E_2^{*,*} = \sum_{p,q} E_2^{p,q} = H^*(\mathbf{S}^3, H^*(\mathbf{S}^1)) =$$

$$= H^*\left(\mathbf{S}^3\right) \otimes H^*(\mathbf{S}^1\right) = \Lambda(a_3) \otimes \Lambda(a_1) = \Lambda(a_1, a_3).$$

The differential d_2 vanishes except possibly on the generator

$$1 \otimes a_1 \in E_2^{0,1} = H^0\left(\mathbf{S}^3, H^1(\mathbf{S}^1)\right).$$

But then

$$d_2(1 \otimes a_1) \in E_2^{2,0} = H^2\left(\mathbf{S}^3, H^0(\mathbf{S}^1)\right) = \mathbf{0}.$$

Hence

$$\begin{array}{rcl} d_2(1 \otimes a_1) & = & 0, \\ d_2(a_3 \otimes 1) & = & 0, \\ d_2(a_3 \otimes a_1) & = & d_2(a_3 \otimes 1)a_1 - a_3 d_2(1 \otimes a_1) = 0. \end{array}$$

Hence d_2 is trivial and therefore

$$E_3^{p,q} = E_2^{p,q}.$$

Similarly $d_3 = 0$ and

$$E_4^{p,q} = E_3^{p,q} = E_2^{p,q}.$$

Continuing, $d_n = 0$ and

$$\begin{array}{rcl} E_{n+1}^{*,*} & = & E_n^{*,*} = \cdots = E_2^{*,*} = \Lambda(a_1, a_3), \\ E_\infty^{*,*} & = & \Lambda(a_1, a_3). \end{array}$$

The cohomology ring $H^*(\mathbf{U}(2))$ is associated to the ring $\Lambda(a_1, a_3)$, that is, the ring $H^*(\mathbf{U}(2))$ has a filtration for which the resulting factors are isomorphic to the homogeneous summands of the ring $\Lambda(a_1, a_3)$. In each dimension, $n = p + q$, the groups $E_\infty^{p,q}$ vanish except for a single value of p, q .

Hence

$$\begin{aligned} H^0(\mathbf{U}(2)) &= E_\infty^{0,0} = \mathbf{Z}, \\ H^1(\mathbf{U}(2)) &= E_\infty^{1,0} = \mathbf{Z}, \\ H^3(\mathbf{U}(2)) &= E_\infty^{0,3} = \mathbf{Z}, \\ H^4(\mathbf{U}(2)) &= E_\infty^{1,3} = \mathbf{Z}. \end{aligned}$$

Let

$$u_1 \in H^1(\mathbf{U}(2)), \ u_3 \in H^3(\mathbf{U}(2))$$

be generators which correspond to a_1 and a_3 , respectively. As $a_1 a_3$ is a generator of the group $E_\infty^{1,3}$, the element $u_1 u_3$ is a generator of the group $H^4(\mathbf{U}(2))$.

It is useful to illustrate our calculation as in the figure, where the nonempty cells show the positions of the generators the groups $E_s^{p,q}$ for each fixed s –level of the spectral sequence.

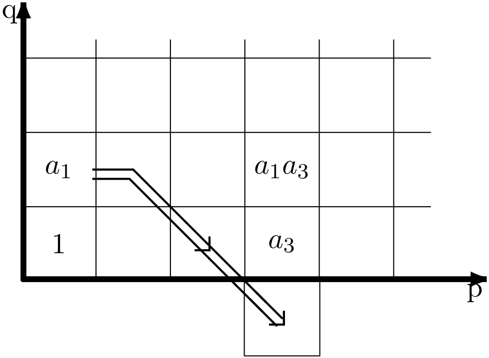


Figure: Spectral sequence for $\mathbf{U}(2)$

For brevity the tensor product sign $\otimes$ is omitted. The arrow denotes the action of the differential d_s for $s = 2, 3$. Empty cells denote trivial groups.
Thus we have shown

$$H^*(\mathbf{U}(2)) = \Lambda(u_1, u_3).$$

Proceeding inductively, assume that

$$H^*(\mathbf{U}(n-1)) = \Lambda(u_1, u_3, \dots, u_{2n-3}),\ u_{2k-1} \in H^{2k-1}(\mathbf{U}(n-1)), \quad 1 \le k \le n-1,$$

and consider the bundle

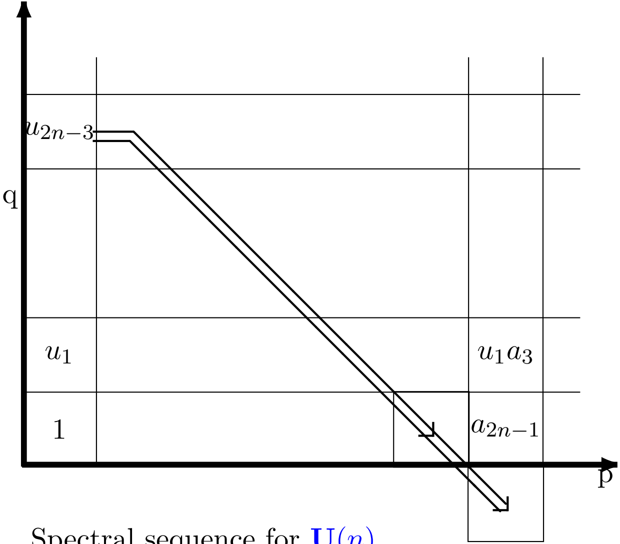
$$\mathbf{U}(n) \longrightarrow \mathbf{S}^{2n-1}$$

with fiber $\mathbf{U}(n-1)$.

The E_2 member of the spectral sequence has the following form:

$$\begin{aligned} E_2^{*,*} &= H^*\left(\mathbf{S}^{2n-1}, H^*\left(\mathbf{U}(n-1)\right)\right) = \\ &= \Lambda(a_{2n-1}) \otimes \Lambda(u_1, \ldots, u_{2n-3}) = \Lambda(u_1, u_3, \ldots, u_{2n-3}, a_{2n-1}). \end{aligned}$$

(see the figure).



Spectral sequence for $\mathbf{U}(n)$

The first possible nontrivial differential is d_{2n-1} . But

$$d_{2n-1}(u_k) = 0,\ k = 1, 3, \ldots, 2n-3,$$

and each element $x \in E^{0,q}$ decomposes into a product of the elements u_k . Thus $d_{2n-1}(x) = 0$. Similarly, all subsequent differentials d_s are trivial.

Thus

$$\begin{aligned} E_{\infty}^{p,q} = \cdots = E_{s}^{p,q} = \cdots = E_{2}^{p,q} = & \\ = & \; H^{p}\left(\mathbf{S}^{2n-1}, H^{q}\left(\mathbf{U}(n-1)\right)\right), \\ E_{\infty}^{*,*} = & \; \Lambda(u_1, u_3, \ldots, u_{2n-3}, u_{2n-1}). \end{aligned}$$

Let now show that the ring $H^*(\mathbf{U}(n))$ is isomorphic to exterior algebra $\Lambda(u_1, u_3, \ldots, u_{2n-3}, u_{2n-1})$. Since the group $E_\infty^{*,*}$ has no torsion, there are elements $v_1, v_3, \ldots, v_{2n-3} \in H^*(\mathbf{U}(n))$ which go to $u_1, u_3, \ldots, u_{2n-3}$ under the inclusion $\mathbf{U}(n-1) \subset \mathbf{U}(n)$.

All the v_k are odd dimensional. Hence elements of the form $v_1^{\varepsilon_1} v_3^{\varepsilon_3} \cdots v_{2n-3}^{\varepsilon_{2n-3}}$ where $\varepsilon_k = 0, 1$ generate a subgroup in the group $H^*(\mathbf{U}(n))$ mapping isomorphically onto the group $H^*(\mathbf{U}(n-1))$. The element $a_{2n-1} \in E_\infty^{2n-1,0}$ has filtration $2n-1$.

Hence the elements of the form $v_1^{\varepsilon_1} v_3^{\varepsilon_3} \cdots v_{2n-3}^{\varepsilon_{2n-3}} a_{2n-1}$ form a basis of the group $E_\infty^{2n-1,0}$. Thus the group $H^*(\mathbf{U}(n))$ has a basis consisting of the elements $v_1^{\varepsilon_1} v_3^{\varepsilon_3} \cdots v_{2n-3}^{\varepsilon_{2n-3}} a_{2n-1}^{\varepsilon_{2n-1}}$, $\varepsilon_k = 0, 1.$ Thus

$$H^*(\mathbf{U}(n)) = \Lambda = (v_1, v_3, \ldots, v_{2n-3}, v_{2n-1}).$$

$\blacksquare$

Cohomology of $\mathbf{BU}(1)$

Now consider the bundle

$$\mathbf{EU}(1) \longrightarrow \mathbf{BU}(1)$$

with the fibre $\mathbf{U}(1) = \mathbf{S}^1$. From the exact homotopy sequence

$$\pi_1(\mathbf{EU}(1)) \longrightarrow \pi_1(\mathbf{BU}(1)) \longrightarrow \pi_0(\mathbf{S}^1)$$

it follows that

$$\pi_1(\mathbf{BU}(1)) = \mathbf{0}.$$

At this stage we do not know the cohomology of the base, but we know the cohomology of the fibre

$$H^*\left(\mathbf{S}^1\right) = \Lambda(u_1),$$

and cohomology of the total space

$$H^*\left(\mathbf{EU}(1)\right) = 0.$$

This means that

$$E_\infty^{p,q} = \cap_s E_s^{p,q} = 0.$$

We know that

$$E_2^{p,q} = H^p\left(\mathbf{BU}(1)\right), H^q(\mathbf{S}^1),$$

and hence

$$E_2^{p,q} = 0 \text{ when } q \geq 2.$$

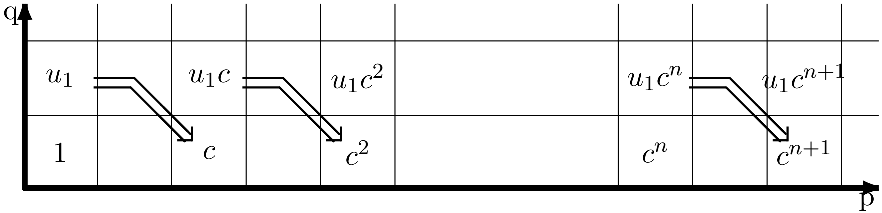


Figure: Spectral sequence for $\mathbf{BU}(1)$

In the figure nontrivial groups can only occur in the two rows with $q = 0$ and $q = 1$.

Moreover,

$$E_2^{p,1} \sim E_2^{p,0} \otimes u_1 \sim E_2^{p,0}.$$

But

$$E_2^{p,q} = 0 \text{ for } q \geq 2,$$

and it follows that

$$E_s^{p,q} = 0 \text{ for } q \geq 2.$$

Hence all differentials from d_3 on are trivial and so

$$E_3^{p,q} = \cdots = E_\infty^{p,q} = 0.$$

Also

$$E_3^{p,q} = H(E_2^{p,q}, d_2)$$

and thus the differential

$$d_2 : E_2^{p,1} \longrightarrow E^{p+2,0}$$

is an isomorphism. Putting

$$c = d_2(u_1),$$

we have

$$d_2(u_1 c^k) = d_2(u_1)c^k = c^{k+1}.$$

Hence the cohomology ring of the space $\mathbf{BU}(1)$ is isomorphic to the polynomial ring with a generator c of the dimension 2:

Now assume that

$$H^*\left(\mathbf{BU}(n-1)\right) = \mathbf{Z}[c_1, \ldots, c_{n-1}]$$

and consider the bundle

$$\mathbf{BU}(n-1) \longrightarrow \mathbf{BU}(n)$$

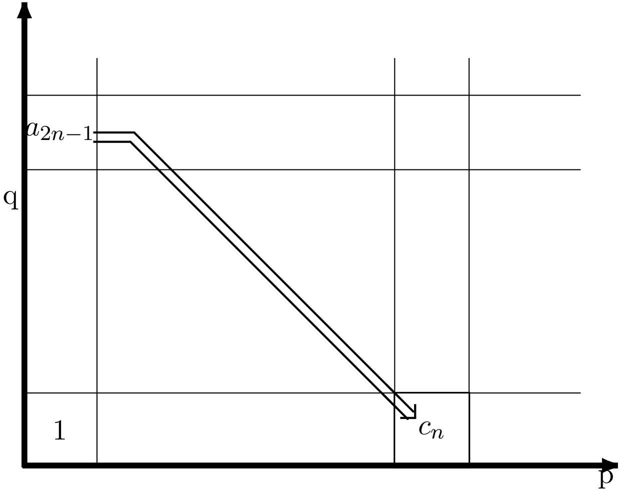
with the fiber $\mathbf{U}(n)/\mathbf{U}(n-1) = \mathbf{S}^{2n-1}$. The exact homotopy sequence gives us that

$$\pi_1\left(\mathbf{BU}(n)\right) = 0.$$

We know the cohomology of the fiber $\mathbf{S}^{2n-1}$ and the cohomology of the total space $\mathbf{BU}(n-1)$. The cohomology of the latter is not trivial but equals the ring $\mathbf{Z}[c_1, \ldots, c_{n-1}]$. Therefore, in the spectral sequence only the terms $E_s^{p,0}$ and $E_s^{p,2n-1}$ may be nontrivial and

$$E_2^{p,2n-1} = E_2^{p,0} \otimes a_{2n-1} = H^p(\mathbf{BU}(n)) \otimes = a_{2n-1}.$$

(see the next figure).


q
a_{2n-1}
1
c_n
p

Hence the only possible nontrivial differential is d_{2n} and therefore

$$E_2^{p,q} = \cdots = E_{2n}^{p,q},$$

$$H(E_{2n}^{p,q}, d_{2n}) = E_{2n+1}^{p,q} = \cdots = E_{\infty}^{p,q}$$

It is clear that if $n = p + q$ is odd then $E_{\infty}^{p,q} = 0$. Hence the differential

$$d_{2n} : E_{2n}^{0,2n-1} \longrightarrow E_{2n}^{2n,0}$$

is a monomorphism. If $p + q = k < 2n - 1$ then $E_2^{p,q} = E_{\infty}^{p,q}$. Hence for odd $k \leq 2n - 1$, the groups $E_{2n}^{k,0}$ are trivial.

Hence the differential

$$d_{2n} : E_{2n}^{k,2n-1} \longrightarrow E_{2n}^{k+2n,0}$$

is a monomorphism for $k \leq 2n$. This differential makes some changes in the term $E_{2n+1}^{k+2n,0}$ only for even $k \leq 2n$. Hence, for odd $k \leq 2n$, we have

$$E_{2n+1}^{k+2n,0} = 0.$$

Thus the differential

$$d_{2n} : E_{2n}^{k,2n-1} \longrightarrow E_{2n}^{k+2n,0}$$

is a monomorphism for $k \leq 4n$. By induction one can show that

$$E_{2n}^{k,0} = 0$$

for arbitrary odd k , and the differential

$$d_{2n} : E_{2n}^{k,2n-1} \longrightarrow E_{2n}^{k+2n,0}$$

is a monomorphism.

Hence

$$\begin{aligned} E_{2n+1}^{k,2n-1} &= 0 \\ E_{2n+1}^{k+2n,0} &= E_{2n}^{k2n,0}/E_{2n}^{k,2n-1} = E_{\infty}^{k+2n,0}. \end{aligned}$$

The ring $H^*(\mathbf{BU}(n-1))$ has no torsion and in the term $E_{\infty}^{p,q}$ only one row is nontrivial $(q=0)$. Hence the groups $E_{\infty}^{k+2n,0}$ have no torsion. This means that image of the differential d_{2n} is a direct summand.

Let

$$c_n = d_{2n}(a_{2n-1}).$$

and then

$$d_{2n}(xa_{2n-1}) = xc_{2n}.$$

It follows that the mapping

$$H^*(\mathbf{BU}(n)) \longrightarrow H^*(\mathbf{BU}(n))$$

defined by the formula

$$x \longrightarrow c_n x$$

is a monomorphism onto a direct summand and the quotient ring is isomorphic to the ring $H^*(\mathbf{BU}(n-1)) = \mathbf{Z}[c_1, \ldots, c_{n-1}]$
.

Thus

$$H^*(\mathbf{BU}(n)) = \mathbf{Z}[c_1, \dots, c_{n-1}, c_n].$$

Torus

Consider now the subgroup

$$\mathbf{T}^n = \mathbf{U}(1) \times \cdots \times \mathbf{U}(1) \ (n \text{ times }) \subset \mathbf{U}(n)$$

of diagonal matrices. The natural inclusion $\mathbf{T}^n \subset \mathbf{U}(n)$ induces a mapping

$$j_n : \mathbf{BT}^n \longrightarrow \mathbf{BU}(n).$$

But

$$\mathbf{BT}^n = \mathbf{BU}(1) \times \cdots \times \mathbf{BU}(1),$$

and hence

$$H^*(\mathbf{BT}^n) = \mathbf{Z}[t_1, \ldots, t_n].$$

Torus

Lemma

The homomorphism

$$j_n^* : \mathbf{Z}[c_1, \ldots, c_n] \longrightarrow \mathbf{Z}[t_1, \ldots, t_n]$$

induced by the mapping (406) is a monomorphism onto the direct summand of all symmetric polynomials in the variables $(t_1, \ldots, t_n)$.

Torus

PROOF. Let

$$\alpha : \mathbf{U}(n) \longrightarrow \mathbf{U}(n)$$

be the inner automorphism of the group induced by permutation of the basis of the vector space on which the group $\mathbf{U}(n)$ acts. The automorphism α acts on diagonal matrices by permutation of the diagonal elements. In other words, α permutes the factors in the group $\mathbf{T}^n$.

Torus

The same is true for the classifying spaces and the following diagram

$$\begin{array}{ccc} \mathbf{BT}^n & \xrightarrow{j_n} & \mathbf{BU}(n) \\ \downarrow \alpha & & \downarrow \alpha \\ \mathbf{BT}^n & \xrightarrow{j_n} & \mathbf{BU}(n) \end{array}$$

is commutative.

Torus

The inner automorphism α is homotopic to the identity since the group $\mathbf{U}(n)$ is connected. Hence, on the level of cohomology, the following diagram

$$\begin{array}{ccc} H^*(\mathbf{BT}^n) & \xleftarrow{j_n^*} & H^*(\mathbf{BU}(n)) \\ \uparrow \alpha^* & & \uparrow \alpha^* \\ H^*(\mathbf{BT}^n) & \xleftarrow{j_n^*} & H^*(\mathbf{BU}(n)) \end{array}$$

or

$$\begin{array}{ccc} \mathbf{Z}[c_1, \dots, c_n] & \xrightarrow{j_n^*} & \mathbf{Z}[t_1, \dots, t_n] \\ \downarrow \alpha^* & & \downarrow \alpha^* \\ \mathbf{Z}[c_1, \dots, c_n] & \xrightarrow{j_n^*} & \mathbf{Z}[t_1, \dots, t_n] \end{array}.$$

is commutative.

Torus

The left homomorphism α^* is the identity, whereas the right permutes the variables $(t_1, \ldots, t_n)$. Hence, the image of j_n^* consists of symmetric polynomials.

Torus

Now let us prove that the image of j_n^* is a direct summand. For this it is sufficient to show that the inclusion

$$\mathbf{U}(k) \times \mathbf{U}(1) \subset \mathbf{U}(k+1)$$

induces a monomorphism in cohomology onto a direct summand.

Torus

Consider the corresponding bundle

$$\mathbf{B}\left(\mathbf{U}(k)\times\mathbf{U}(1)\right)\longrightarrow\mathbf{BU}(k+1)$$

with fiber

$$\mathbf{U}(k+1)/\left(\mathbf{U}(k)\times\mathbf{U}(1)\right)=\mathbf{CP}^k.$$

The E_2 term of the spectral sequence is

$$E_2^{*,*}=H^*\left(\mathbf{BU}(k+1);H^*(\mathbf{CP}^k)\right).$$

Torus

For us it is important here that the only terms of (413) which are nontrivial occur when p and q are even. Hence all differentials d_s are trivial and

$$E_2^{p,q} = E_\infty^{p,q}.$$

Torus

None of these groups have any torsion. Hence the group

$$H^p\left(\mathbf{BU}(k+1);\mathbf{Z}\right) = E_\infty^{p,0} \subset H^*\left(\mathbf{BU}(k)\times\mathbf{BU}(1)\right)$$

is a direct summand.

Torus

It is very easy to check that the rank of the group $H^k(\mathbf{BU}(n))$ and the subgroup of symmetric polynomials of the degree k of variables $(t_1, \ldots, t_n)$ are the same. $\blacksquare$

Using Lemma, choose generators

$$c_1, \ldots, c_n \in H^*(\mathbf{BU}(n))$$

as inverse images of the elementary symmetric polynomials in the variables

$$t_1, \ldots, t_n \in H^*(\mathbf{BU}(1) \times \cdots \times \mathbf{BU}(1)).$$

Then the condition (418) follows from the fact that the element c_{n+1} is mapped by j_{n+1}^* to the product $t_1 \cdot \dots \cdot t_{n+1}$, which in turn is mapped by the inclusion (418) to zero.
the natural mapping

$$\varphi : \mathbf{BU}(n) \longrightarrow \mathbf{BU}(n+1)$$

satisfies the conditions

$$\begin{array}{lcl} \varphi^*(c_k) & = & c_k, \ k = 1, 2, \dots, n, \\ \varphi^*(c_{n+1}) & = & 0; \end{array}$$

Condition 345 follows from the properties of the elementary symmetric polynomials:

$$\sigma_k(t_1,\ldots,t_n,t_{n+1},\ldots,t_{n+m}) = \\ = \sum_{\alpha+\beta=k}\sigma_\alpha(t_1,\ldots,t_n)\sigma_\beta(t_{n+1},\ldots,t_{n+m}).$$

The proof of the theorem is finished. ∎

The generators $c_1, \ldots, c_n$ are not unique but only defined up to a choice of signs for the generators $t_1, \ldots, t_n$. Usually the sign of the t_k is chosen in such way that for the Hopf bundle over $\mathbf{CP}^1$ the value of c_1 on the fundamental circle is equal to 1 .

The characteristic classes c_k are called *Chern classes*. If X is complex analytic manifold then characteristic classes of the tangent bundle TX are simply called *characteristic classes of manifold* and one writes

$$c_k(X) = c_k(TX).$$

Definition

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Literature

- John Milnor *Differential Topology*, Princeton University, (1958)
- Andrew Wallace *Differential Topology, First Steps*, W.A.Benjamin, Inc., New York, Amsterdam, (1968)
- James R. Munkres *Elementary Differential Topology*, Ann. of Math. Stud., 54, Massachusets Institute of Technology, (1966)
- John Milnor *Topology from the Differentiable Viepoint* , The University Press of Vitginia, (1965)

Literature

- James R. Munkres *Analysis on Manifold,* Addison Wesley, Redwood City, CA , (1990)
- Bjorn Ian Dundas *Differential Topology,* Johns Hopkins University, (2013)
- A.S.Mishchenko *Topology of manifolds,* Harbin Institute of Technology, (2004)
- J.M.Lee *Introduction to Smooth manifolds,* University of Washington, Seattle, WA (2000)

Literature

- M.A. Kervaire *A Manifold which does not not admit any Differentiable Structure.* Commentarii Mathematici Helvetici vol. 34, (1960)
- E.E. Moise *Affine Structures in 3-Manifolds, V. The Triangulation Theorem and Hauptvermutung* Annals of Mathematics, Vol. 56, No. 1 (1952)
- R.C. Kirby, and L.C. Siebenmann *Foundational Essays on Topological Manifolds. Smoothings, and Triangulations. Princeton,* New Jersey: Princeton University Press (1977)

Literature

- T. Asselmeyuer-Maluga and C.H.Brans *Exotic Smoothness and Physics. Differential Topology and Spacetime Models*, World Scientific Publishing Co. Re. Ltd. (2007)
- Von Herm Heinrich Tietze *Uber Funktionen, die auf einer abgesehlossenen Menge stetig sind*, Journal fur die reine und angewandte Mathematik, vol.145, p.9-14 (1915)
- Paul Urysohn *Über die MÄchtigkeit der zusammenhaengenden Mengen*, Mathematische Annalen, vol.94, Issue 1, p.261-295 (1925)
- A. Hatcher *Vector bundles and K-theory* http://www.math.cornell.edu/ hatcher/#VBKT,(2009)