Transitive Lie algebroids - categorical point of view *†

A.S.Mishchenko
Moscow State University

Introduction

Transitive Lie algebroids have specific properties that allow to look at the transitive Lie algebroid as an element of the object of a homotopy functor. Roughly speaking each transitive Lie algebroids can be described as a vector bundle over the tangent bundle of the manifold which is endowed with additional structures. Therefore transitive Lie algebroids admits a construction of inverse image generated by a smooth mapping of smooth manifolds. The construction can be managed as a homotopy functor from the category of smooth manifolds to the transitive Lie algebroids. The intention of this article is to make a classification of transitive Lie algebroids and on this basis to construct a classifying space. The realization of the intention allows to describe characteristic classes of transitive Lie algebroids form the point of view a natural transformation of functors similar to the classical abstract characteristic classes for vector bundles.

1 Definitions and formulation of the problem

Given smooth manifold M let

$$E \xrightarrow{a} TM \xrightarrow{p_T} M$$

be a vector bundle over TM with fiber g, $p_E = p_T \cdot a$. So we have a commutative diagram of two vector bundles

$$\begin{array}{ccc} E & \xrightarrow{a} & TM \\ {\scriptstyle p_E}\downarrow & & \downarrow{\scriptstyle p_T} \\ M & \longrightarrow & M \end{array}$$

*2000 Mathematics Subject Classification: 57R20, 57R22
†Key words: Lie algebroid, flat bundle, transition function, characteristic classes, classifying space

The diagram is endowed with additional structure (commutator braces) and then is called ([1], definition 3.3.1, [2], definition 1.1.1) transitive Lie algebroid

$$\mathcal{A} = \left\{ \begin{array}{ccc} E & \xrightarrow{a} & TM \\ {\scriptstyle p_E}\downarrow & & \downarrow{\scriptstyle p_T} \\ M & \longrightarrow & M \end{array} ; \{\bullet,\bullet\} \right\}.$$

Let $f : M' \longrightarrow M$ be a smooth map. Then one can define an inverse image (pullback) of the Lie algebroid ([1], page 156, [2], definition 1.1.4), $f^{!!}(\mathcal{A})$. This means that given the finite dimensional Lie algebra g there is the functor $\mathcal{A}$ such that with any manifold M it assigns the family $\mathcal{A}(M)$ of all transitive Lie algebroids with fixed Lie algebra g.

In the dissertation [3] the following statement was proved: Each transitive Lie algebroid is trivial, that is there is a trivialization of vector bundles $E, TM, \ker a = \bar{g}$ such that

$$E \approx TM \oplus \bar{g},$$

and the Lie bracket is defined by the formula:

$$[(X,u),(Y,v)] = ([X,Y],[u,v] + X(v) - Y(u)).$$

Then using the construction of pullback and the idea by Allen Hatcher [4] one can prove that the functor $\mathcal{A}$ is homotopic functor. More exactly for two homotopic smooth maps $f_0, f_1 : M_1 \longrightarrow M_2$ and for the transitive Lie algebroid

$$(E \xrightarrow{a} TM_2 \longrightarrow M_2; \{\bullet,\bullet\})$$

two inverse images $f_0^{!!}(\mathcal{E})$ and $f_1^{!!}(\mathcal{A})$ are isomorphic.

Hence there is a final classifying space $\mathcal{B}_g$ such that the family of all transitive Lie algebroids with fixed Lie algebra g over the manifold M has one-to-one correspondence with the family of homotopy classes of continuous maps $[M, \mathcal{B}_g]$:

$$\mathcal{A}(M) \approx [M, \mathcal{B}_g].$$

Using this observation one can describe the family of all characteristic classes of a transitive Lie algebroids in terms of cohomologies of the classifying space $\mathcal{B}_g$. Really, from the point of view of category theory a characteristic class α is a natural transformation from the functor $\mathcal{A}$ to the cohomology functor H^*. This means that for the transitive Lie algebroid $\mathcal{E} = (E \xrightarrow{a} TM \longrightarrow M; \{\bullet,\bullet\})$ the value of the characteristic class $\alpha(\mathcal{E})$ is a cohomology class

$$\alpha(\mathcal{E}) \in H^*(M),$$

such that for smooth map $f : M_1 \longrightarrow M$ we have

$$\alpha(f_0^{!!}(\mathcal{E})) = f^*(\alpha(\mathcal{E})) \in H^*(M_1).$$

Hence the family of all characteristic classes $\{\alpha\}$ for transitive Lie algebroids with fixed Lie algebra g has a one-to-one correspondence with the cohomology group $H^*(\mathcal{B}_g)$.

On the base of these abstract considerations a natural problem can be formulated.

Problem. Given finite dimensional Lie algebra g describe the classifying space $\mathcal{B}_g$ for transitive Lie algebroids in more or less understandable terms.

Below we suggest a way of solution the problem and consider some trivial examples.

2 Description of transitive Lie algebroids using transition functions

Consider the trivial transitive Lie algebroids

$$E \approx TM \oplus \bar{g}, \quad \bar{g} \approx M \times g,$$

and the Lie bracket is defined by the formula:

$$[(X,u),(Y,v)] = ([X,Y],[u,v] + X(v) - Y(u)),$$

where $X, Y \in \Gamma^\infty(TM)$ are smooth vector fields, $u, v \in \Gamma^\infty \bar{g}$ are smooth sections which are represented as smooth vector functions with values in the Lie algebra g. Consider a fiberwise isomorphism $\mathcal{A} : E \longrightarrow E$ that is identical on the second summands and generates the Lie algebra homomorphism $\mathcal{A} : \Gamma^\infty(E) \longrightarrow \Gamma^\infty(E)$. The isomorphism $\mathcal{A}$ can be written by formula:

$$\begin{array}{l} (v,Y) = \mathcal{A}(u,X); \\ (v,X) = (\varphi(x)(u(x)) + \omega(X), X), \end{array}$$

where $\varphi(x) : g \longrightarrow g$ is a fiberwise map of the bundle $\bar{g}$, and ω is a differential form with values in g. The isomorphism $\mathcal{A}$ can be expressed as a matrix

$$\left(\begin{array}{c} v(x) \\ Y \end{array} \right) = \left(\begin{array}{cc} \varphi(x) & \omega \\ 0 & 1 \end{array} \right) \cdot \left(\begin{array}{c} u(x) \\ X \end{array} \right)$$

From the property of that $\mathcal{A}$ is a Lie algebra homomorphism:

$$\mathcal{A}([(X,u),(Y,v)]) = [\mathcal{A}(X,u), \mathcal{A}(Y,v)]$$

one has that

$$\begin{array}{l} \varphi(x)([u_1(x), u_2(x)]) = [\varphi(x)(u_1(x)), \varphi(x)(u_2(x))]), \\ d\omega(X_1, X_2) + [\omega(X_1), \omega(X_2)] = 0, \\ d\varphi(X)(u) = [\varphi(u), \omega(X)]. \end{array} \tag{1}$$

Consider an atlas of charts on the manifold M, $\{\mathfrak{U}_\alpha\}$, $\bigcup\limits_\alpha U_\alpha = M$, and the trivializations $E_\alpha \overset{\Phi_\alpha}{\approx} TU_\alpha \otimes (U_\alpha \times g)$ of the Lie algebroid E over each chart U_α with the Lie brackets defined by the formula

$$[(X,u),(Y,v)] = ([X,Y],[u,v] + X(v) - Y(u)),$$

for $X, Y \in \Gamma^\infty(TU_\alpha)$, $u, v \in \Gamma^\infty(U_\alpha \times g)$.

On the intersection of two charts $U_{\alpha\beta} = U_\alpha \cap U_\beta$ we have the transition function

$$\Phi_{\beta\alpha} = \Phi_\beta \Phi_\alpha^{-1} : TU_{\alpha\beta} \otimes (U_{\alpha\beta} \times g) \longrightarrow TU_{\alpha\beta} \otimes (U_{\alpha\beta} \times g)$$

which have the matrix form

$$\begin{pmatrix} v(x) \\ Y \end{pmatrix} = \Phi_{\beta\alpha} \begin{pmatrix} u(x) \\ X \end{pmatrix} = \begin{pmatrix} \varphi_{\beta\alpha}(x) & \omega_{\beta\alpha} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} u(x) \\ X \end{pmatrix}.$$

For another choice of trivializations Φ'_α the correspondent transition functions $\Phi'_{\beta\alpha}$ satisfy the homology condition:

$$\Phi'_{\beta\alpha} = \mathrm{H}_\beta \cdot \Phi_{\beta\alpha} \cdot \mathrm{H}_\alpha^{-1}$$

$$\begin{pmatrix} \varphi'_{\beta\alpha}(x) & \omega'_{\beta\alpha} \\ 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \eta_\beta(x) & \mu_\beta \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \varphi_{\beta\alpha}(x) & \omega_{\beta\alpha} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \eta_\alpha^{-1}(x) & -\eta_\alpha^{-1}\mu_\alpha \\ 0 & 1 \end{pmatrix}$$

or

$$\begin{pmatrix} \varphi'_{\beta\alpha}(x) & \omega'_{\beta\alpha} \\ 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \eta_\beta(x)\varphi_{\beta\alpha}(x)\eta_\alpha^{-1}(x) & -\eta_\beta(x)\varphi_{\beta\alpha}(x)\eta_\alpha^{-1}(x)\mu_\alpha + \eta_\beta(x)\omega_{\beta\alpha} + \mu_\beta \\ 0 & 1 \end{pmatrix},$$

or

$$\varphi'_{\beta\alpha}(x) = \eta_\beta(x)\varphi_{\beta\alpha}(x)\eta_\alpha^{-1}(x),$$

$$\omega'_{\beta\alpha} = -\eta_\beta(x)\varphi_{\beta\alpha}(x)\eta_\alpha^{-1}(x)\mu_\alpha + \eta_\beta(x)\omega_{\beta\alpha} + \mu_\beta.$$

The elements η_β and μ_β satisfy similar (1) conditions:

$$\eta_\beta(x)([u_1(x), u_2(x)]) = [\eta_\beta(x)(u_1(x)), \eta_\beta(x)(u_2(x))]),$$

$$d\mu_\beta(X_1, X_2) + [\mu_\beta(X_1), \mu_\beta(X_2)] = 0,$$

$$d\eta_\beta(X)(u) = [\eta_\beta(u), \mu_\beta(X)].$$

3 Case of commutative Lie algebra g

In commutative case the conditions (1) have for simple form:

$$\varphi_{\beta\alpha}(x)([u_1(x), u_2(x)]) = [\varphi_{\beta\alpha}(x)(u_1(x)), \varphi_{\beta\alpha}(x)(u_2(x))]),$$

$$d\omega_{\beta\alpha}(X_1, X_2) = 0, \tag{2}$$

$$d\varphi_{\beta\alpha}(X)(u) = 0.$$

Hence

$$\varphi_{\beta\alpha}(x) = \mathbf{const}\ .$$

This means that the vector bundle $\bar{g}$ is flat and the family $\omega = \{\omega_{\beta\alpha}\}$ defines a Cech cochain

$$\omega \in C^1(\mathfrak{U}, \Omega^1(\bar{g}))$$

in the bigraded Cech complex

$$C^{*,*} = \left\{\bigoplus C^i(\mathfrak{U}, \Omega^j(\bar{g}); d = d' + d''\right\}$$

where $\mathfrak{U} = \{U_\alpha\}$ is the atlas of charts.

One has

$$d'(\omega) = 0; \quad d''(\omega) = 0.$$

Hence ω defines cohomology class

$$[\omega] \in H^2(M; \bar{g}).$$

Therefore we have the following

Theorem 1 *The classification of all transitive Lie algebroids with fixed commutative Lie algebra g over the manifold M is determined by a flat Lie algebra bundle $\bar{g}$ over M and a 2-dimensional cohomology class $[\omega] \in H^2(M; \bar{g})$.*

4 Some general properties

In common case we can say that a little bit about the transition functions on the level of homology groups $H_*(g)$ of the Lie algebra g. Since each transition function $\varphi_{\beta\alpha}(x)$ is the homomorphism of the Lie algebra g, that is $\varphi_{\beta\alpha}(x) \in \mathbf{Aut}(g)$, the cocycle $\{\varphi_{\beta\alpha}(x)\}$ generate associated bundles with fibers $H_*(g)$, say, $\overline{H_*(g)}$, and bundles with fibers $H^*(g)$,$\overline{H^*(g)}$. The properties (1) imply that all bundles $\overline{H_*(g)}$ and $\overline{H^*(g)}$ are flat. In particular the differential forms $\omega_{\beta\alpha} \in \Omega^1(U_{\alpha\beta}; \bar{g})$ generates the cocycle

$$\overline{\omega} = \{\overline{\omega}_{\beta\alpha}\} \in C^1(\mathfrak{U}, \overline{H_1(g)}) = \bigoplus_{\alpha\beta} \Omega^1(U_{\alpha\beta}; \overline{H_1(g)}),$$

that is

$$d'(\overline{\omega}) = 0,$$

$$d''(\overline{\omega}) = 0.$$

This means that the cocycle $\overline{\omega}$ induces a cohomology class

$$[\overline{\omega}] \in H^2\left(M; \overline{H_1(g)}\right).$$

The foregoing consideration creates a conjecture that classification of the transitive Lie algebroid E induces by two things: the Lie algebra bundle with structural group $\widetilde{\mathbf{Aut}(g)}$ with special topology and the cohomology class $[\overline{\omega}] \in H^2\left(M; \overline{H_1(g)}\right)$. The special topology in the group $\mathbf{Aut}(g)$ is defined as a minimal topology, which is more fine topology than the classical topology in $\mathbf{Aut}(g)$ and such that all homomorphisms

$$\mathbf{Aut}(g) \longrightarrow \mathbf{Aut}(H_k(g))_{discrete}$$

are continuous.

This paper is partly supported by grants RBRF No 05-01-00923-a,07-01-91555-NNIO-a, 10-01-92601-KO-a and project No. RNP.2.1.1.5055

Список литературы

[1] Mackenzie, K.C.H., *General Theory of Lie Groupoids and Lie Algebroids,* Cambridge University Press,(2005)

[2] Kubarski, J., *The Chern-Weil homomorphism of regular Lie algebroids*, Publications du Department de Mathematiques, Universite Claude Bernard - Lyon-1, (1991) pp.4–63

[3] W.Walas Algebry Liego-Rineharta i pierwsze klasy charakterystyczne. *PhD manuscript*, Lodz, Poland, 2007.

[4] Allen Hatcher, Vector bundles and K-theory, *Available at http://www.math.cornell.edu/~hatcher/VBKT/VBpage.html*, page 7, 2003.