

A Short Course in Differential Geometry and Topology

A.T. Fomenko and A.S. Mishchenko



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Preface

A Short Course on Differential Geometry and Topology by Professor A.T. Fomenko and Professor A.S. Mishchenko is based on the course taught at the Faculty of Mechanics and Mathematics of Moscow State University. It is intended for students of mathematics, mechanics and physics and also provides a useful reference text for postgraduates and researchers specialising in modern geometry and its applications.

The course is structured in seven chapters and covers the basic material on general topology, nonlinear coordinate systems, theory of smooth manifolds, theory of curves and surfaces, transformation groups, tensor analysis and Riemannian geometry, theory of integration and homologies, fundamental groups and variational problems of Riemannian geometry. All of the chapters are highly illustrated and the text is supplemented by a large number of definitions, examples, problems, exercises to test the concepts introduced. The material has been carefully selected and is presented in a concise manner which is easily accessible to students.

Chapter 1: Introduction to Differential Geometry

This chapter consists of four sections and includes definitions, examples, problems and illustrations to aid the reader.

- 1.1 Curvilinear Coordinate Systems. Simplest Examples
- 1.2 The Length of a Curve in Curvilinear Coordinates
- 1.3 Geometry on the Sphere and Plane
- 1.4 Pseudo-Sphere and Lobachevskii Geometry

Chapter 2: General Topology

This chapter is an introduction to topology and the development of topology. Topology is a field of mathematics which studies the properties of geometric that are not changed under a "deformation" or other transformations similar to deformations. General topology arose as a result of studying the most general properties of geometric spaces and their transformations related to the convergence and continuity properties. The chapter consists of four sections and includes examples, definitions, problems and illustrations.

- 2.1 Definitions and Simplest Properties of Metric and Topological Spaces
- 2.2 Connectedness. Separation Axioms
- 2.3 Compact Spaces
- 2.4 Functional Separability. Partition of Unity

Chapter 3: Smooth Manifolds: General Theory

This chapter covers the general theory of smooth manifolds, introduces the manifold as a special concept in geometry and includes definitions, examples and problems to test understanding.

- 3.1 Concept of a Manifold
- 3.2 Assignment of Manifolds in Equations
- 3.3 Tangent Vectors. Tangent Space
- 3.4 Submanifolds

Chapter 4: Smooth Manifolds: Examples

This chapter continues the study of smooth manifolds and focuses on examples.

- 4.1 Theory of Curves on the Plane and in the Three-Dimensional Space
- 4.2 Surfaces
- 4.3 Transformation Groups
- 4.4 Dynamical Systems
- 4.5 Classification of Two-Dimensional Surfaces
- 4.6 Two-Dimensional Manifolds on Riemann Surfaces

Chapter 5: Tensor Analysis and Riemannian Geometry

This chapter studies local properties of smooth manifolds and includes definitions, illustrations, exercises and examples throughout.

- 5.1 General Concept of Tensor Field on a Manifold
- 5.2 Simplest Examples of Tensor Fields
- 5.3 Connection and Covariant Differentiation
- 5.4 Parallel Translation. Geodesies
- 5.5 Curvature Tensor

Chapter 6: Homology Theory

This chapter focuses on the properties of manifolds on which other functions and mapping depend. Examples and illustrations are included throughout the chapter.

- 6.1 Calculus of Differential Forms
- 6.2 Integration of Exterior Forms
- 6.3 Degree of a Mapping and its Applications

PREFACE

Chapter 7: Simplest Variational Problems of Riemannian Geometry

This chapter begins with the general concept of functional and its variation and subsequent sections cover extremality of geodesies, minimal surfaces, calculus of variations and symplectic geometry.

- 7.1 Concept of Functional. Extremal Functions. Euler Equation
- 7.2 Extremality of Geodesies
- 7.3 Minimal Surfaces
- 7.4 Calculus of Variations and Symplectic Geometry

A Short Course in Differential Geometry and Topology is intended for students of mathematics, mechanics and physics and also provides a useful reference text for postgraduates and researchers specialising in modern geometry and its applications.

Selected Problems in Differential Geometry and Topology

A.T. Fomenko, A.S. Mischenko and Y.P. Solovyev

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A Short Course in Differential Geometry and Topology

A.T. Fomenko and A.S. Mishchenko Moscow State University

This volume is intended for graduates and research students in mathematics and physics. It covers general topology, nonlinear co-ordinate systems, theory of smooth manifolds, theory of curves and surfaces, transformation groups, tensor analysis and Riemannian geometry, theory of integration and homologies, fundamental groups and variational principles in Riemannian geometry. The text is presented in a form that is easily accessible to students and is supplemented by a large number of examples, problems, drawings and appendices.

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