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Construction of Fredholm Representations and a Modification of the Higson–Roe Corona

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Abstract. The Fredholm representation theory is well adapted to the construction of homotopy invariants of non-simply-connected manifolds by means of the generalized Hirzebruch formula $[\sigma(M)] = \langle L(M)\,\mathrm{ch}_A\, f^*\xi, [M]\rangle \in \mathbf{K}^0_A(\mathrm{pt}) \otimes \mathbf{Q}$, where $A = C^*[\pi]$ is the C^*-algebra of the group π, $\pi = \pi_1(M)$. The bundle $\xi \in \mathbf{K}^0_A(B\pi)$ is the canonical A-bundle generated by the natural representation $\pi \longrightarrow A$.

Recently, the first author constructed a natural family of Fredholm representations that lead to a symmetric vector bundle on the completion of the fundamental group with a modification of the Higson–Roe corona, provided that the completion is a closed manifold.

In the present paper, a homology version of symmetry is discussed for the case in which the completion, with a modification of the Higson–Roe corona, is a manifold with boundary. The results were developed during the visit of the first author to Ancona on March, 2007. The last version is supplemented by details considering the case of manifolds with boundary.

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The Fredholm representation theory is well adapted to the construction of homotopy invariants of non-simply-connected manifolds by means of the generalized Hirzebruch formula

$$[\sigma(M)] = \langle L(M)\,\mathrm{ch}_A\, f^*\xi, [M]\rangle \in \mathbf{K}^0_A(\mathrm{pt}) \otimes \mathbf{Q}, \tag{1}$$

where $A = C^*[\pi]$ is the group C^*-algebra of the group π, $\pi = \pi_1(M)$. The bundle $\xi \in \mathbf{K}^0_A(B\pi)$ is the canonical A-bundle generated by the natural representation $\pi \longrightarrow A$. The mapping $f\colon M \longrightarrow B\pi$ induces an isomorphism of fundamental groups. The element $[\sigma(M)] \in \mathbf{K}^0_A(\mathrm{pt})$ is the noncommutative signature of the manifold M; here we assume that $\mathbf{Z}[\frac{1}{2}][\pi] \subset A$.

Let $\rho = (T_1, F, T_2)$ be a Fredholm representation of the group π, i.e., a pair of unitary representations $T_1, T_2\colon \pi \longrightarrow B(H)$ and a Fredholm operator $F\colon H \longrightarrow H$ such that

$$FT_1(g) - T_2(g)F \in \mathrm{Comp}(H), \quad g \in \pi. \tag{2}$$

Replacing the algebra $B(H)$ by the Calkin algebra $\mathcal{K} = B(H)/\,\mathrm{Comp}(H)$, one obtains a representation $\widehat{\rho}$ of $\pi \times \mathbf{Z}$ in the Calkin algebra,

$$\widehat{\rho}\colon \pi \times \mathbf{Z} \longrightarrow \mathcal{K}, \qquad \widehat{\rho}(g,n) = T_2(g)F^n = F^n T_1(g), \quad g \in \pi, \quad n \in \mathbf{Z},$$
$$\rho_*\colon K_A(X) \xrightarrow{\mathrm{Id}\otimes\beta} K_{A\widehat{\otimes}C(S^1)}(X \times S^1) \xrightarrow{\widehat{\rho}} K_{\mathcal{K}}(X \times S_1). \tag{3}$$

Here $\beta \in K_{C(S^1)}(S^1)$ stands for the canonical element related to the regular representation of $\mathbf{Z}$.

Combining (3) with the Hirzebruch formula (1), one proves the homotopy invariance of the corresponding higher signature.

1. CONSTRUCTION OF FREDHOLM REPRESENTATIONS

Let T be a sum of finitely many copies of the regular representation of π and let Φ be the block-diagonal operator defined as a matrix-valued function $F(g)$, $g \in \pi$,

$$F(g)\colon V \longrightarrow V. \tag{4}$$

Let $H = \bigoplus_{g\in\pi} V_g$, $\quad V_g \equiv V$ and $T_h\colon H \longrightarrow H$, $V_g \longrightarrow V_{hg}$. The condition '$\Phi$ is a Fredholm operator' means that

$$\|F(g)\| \leqslant C, \quad \|F^{-1}(g)\| \leqslant C \tag{5}$$

for any $g \in \pi$, possibly except for finitely many elements. Condition (2) means that

$$\lim_{|g|\longrightarrow\infty} \|F(g) - F(hg)\| = 0. \tag{6}$$

If the pair
$$\rho = (T, \Phi) \tag{7}$$
satisfies conditions (5) and (6), then ρ is said to be a *Fredholm representation* of π.

Consider the universal covering $\widetilde{B\pi}$ of the classifying space $B\pi$ endowed with the left action of π. In accordance with the construction in [2], the vector bundle generated by the representation ρ on the space $B\pi$ can be seen as an equivariant continuous family of Fredholm operators on the space $E\pi = \widetilde{B\pi}$. The equivariance property refers to the diagonal action on the Cartesian product
$$T_h\colon E\pi \times H, \qquad (x, \xi) \longrightarrow (hx, T_h(\xi)). \tag{8}$$
Namely, let the space $B\pi$ be endowed with a simplicial structure and let $E\pi = \widetilde{B\pi}$ be endowed with the induced simplicial structure induced by the covering $E\pi = \widetilde{B\pi} \xrightarrow{p} B\pi$. Let $\{x_i\}$ be a family of vertices of $E\pi = \widetilde{B\pi}$, one for each orbit of the action of π. Then each simplex σ of $E\pi = \widetilde{B\pi}$ is completely defined by its vertices $\sigma = (h_0 x_{i_0}, \dots, h_n x_{i_n}), \quad h_0, \dots, h_n \in \pi$. Any point $x \in \sigma$ is uniquely defined as a convex linear combination of vertices $x = \sum_{k=0}^{n} \lambda_k h_k x_{i_k}$. Then the equivariant family of Fredholm operators corresponding to the Fredholm representation ρ (7) is defined by the formula
$$\Phi_x = \Phi_x(\rho) = \sum_{k=0}^{n} \lambda_k \Phi_{h_k x_{i_k}} = \sum_{k=0}^{n} \lambda_k T_{h_k} \Phi_{x_{i_k}} T_{h_k}^{-1} = \sum_{k=0}^{n} \lambda_k T_{h_k} \Phi T_{h_k}^{-1}. \tag{9}$$
Hence
$$(\Phi_x)_g = \sum_{k=0}^{n} \lambda_k F_{h_k^{-1} g}. \tag{10}$$
It is clear that the family (9) is equivariant. Indeed, $hx = \sum_{k=0}^{n} \lambda_k h h_k x_{i_k}$. Hence
$$\Phi_{hx} = \sum_{k=0}^{n} \lambda_k T_{hh_k} \Phi T_{hh_k}^{-1} = T_h \Big(\sum_{k=0}^{n} \lambda_k T_{h_k} \Phi T_{h_k}^{-1} \Big) T_h^{-1} = T_h \Phi_x T_h^{-1}.$$
Relations (10), (6), and (5) imply that the operators (9) are Fredholm ones.

On the other hand, the operators (4) generate the continuous family $F_x\colon V \longrightarrow V$, $x \in E\pi$, where $F_x = \sum_{k=0}^{n} \lambda_k F(h_k^{-1})$. This family can be regarded as a linear mapping of the trivial bundle,
$$F_x : E\pi \times V \longrightarrow E\pi \times V. \tag{11}$$
Consider the universal covering $p\colon E\pi \longrightarrow B\pi$. Write $\mathcal{K}^i(E\pi) = \lim_{\leftarrow} K_c^i(p^{-1}(X))$, where the inverse limit is taken with respect to the family of all compact subsets $X \subset B\pi$.

Theorem 1. *The mapping* (11) *defines the element* $F(\rho) \in \mathcal{K}^0(E\pi)$.

Consider the direct image of the bundle (11) over $B\pi$,
$$A \longrightarrow B\pi, \tag{12}$$
where the fiber is the direct sum of the fibers of the bundle (11) over each orbit of the action of the group π on the space $E\pi$. The total space A is defined as $A = \big\{(u, \xi) : u \in B\pi, \xi \in \oplus_{x \in u}(x \times V)\big\}$. Let
$$\widetilde{A} \longrightarrow E\pi \tag{13}$$
be the inverse image of the bundle (12). The total space $\widetilde{A}$ is defined as $\widetilde{A} = \big\{(x, \xi) : x \in E\pi, \xi \in \oplus_{y \in [x]}(y \times V)\big\} = \big\{(x, \xi), x \in E\pi, \xi \in \oplus_{g \in \pi}(gx \times V)\big\}$. Define the action of the group π on the total space $\widetilde{A}$ by the formula $f_h(x, \xi) = (hx, \eta)$, where $\xi = \oplus \xi_g \in \oplus_{g \in \pi}(gx \times V)$, $\eta = \oplus \eta_g \in \oplus_{g \in \pi}(ghx \times V)$, and $\eta_g = \xi_{gh}$. It is clear that $A = \widetilde{A}/\pi$. On the other hand, there is an isomorphism φ between the bundle (13) and the bundle (8),
$$\varphi\colon E\pi \times \bigoplus_{g \in \pi} V_g \longrightarrow \widetilde{A}, \qquad \varphi(x, \oplus \xi_g) = (x, \oplus \xi_{g^{-1}}). \tag{14}$$
This isomorphism is equivariant. By means of this isomorphism, the mapping (11) goes to the mapping
$$\widetilde{F} : \widetilde{A} \longrightarrow \widetilde{A}, \qquad \widetilde{F}(x, \oplus \xi_g) = (x, \oplus F_{gx}(\xi_g)) = \Big(x, \oplus \sum_{k=0}^{n} \lambda_k F_{h_k^{-1} g^{-1}}(\xi_g) \Big). \tag{15}$$

It is clear that the mapping (15) goes to (9) under the isomorphism (14).

Thus, the following theorem holds.

Theorem 2. *Consider the Fredholm representation of* π *of the form* (7). *Let* $\xi_\rho \in \mathbf{K}(B\pi)$ *be the element defined by the mapping* (9). *Then* $p_!(F(\rho)) = \xi_\rho \in \mathbf{K}^0(B\pi)$, *where* $p_! : \mathcal{K}^0(E\pi) \longrightarrow \mathbf{K}^0(B\pi)$ *is the direct image in* K*-theory.*

Consider the action of π on the Cartesian product $E\pi \times V$ given by the left action on the first factor and the identity action on the other one.

Consider a metric on the space $E\pi$ such that $r(xg, yg) \longrightarrow 0$ and $|g| \longrightarrow \infty$. Let $\overline{E\pi}$ be the completion of the space $E\pi$ (with respect to the metric r). Then any continuous mapping $f\colon (\overline{E\pi}, \overline{E\pi}\backslash E\pi) \longrightarrow (B(V), U(V))$ defines a continuous family of Fredholm representations $\rho(x)$, $x \in E\pi$.

By Theorem 1, the family $\rho(x)$ generates an equivariant family $F_{x,y}\colon E\pi \times E\pi \times V \longrightarrow E\pi \times E\pi \times V$, and therefore an element $F(\rho(x)) \in \mathcal{K}^0\left((E\pi \times E\pi)/\pi\right)$. Let $p_!\colon \mathcal{K}^0\left((E\pi \times E\pi)/\pi\right) \longrightarrow \mathbf{K}^0(B\pi \times B\pi)$ be the direct image in K-theory. Then $p_!(F(\rho(x))) = \xi_{\rho(x)} \in \mathbf{K}^0(B\pi \times B\pi)$.

A symmetric property holds for the element $\xi_{\rho(x)}$, $(1 \otimes u)\xi_{\rho(x)} = (u \otimes 1)\xi_{\rho(x)} \in \mathbf{K}^0(B\pi \times B\pi)$, $u \in \mathbf{K}^0(B\pi)$.

2. SYMMETRIC COHOMOLOGY CLASSES IN $H^*(M \times M)$

If the space $B\pi$ is a compact manifold and the space $E\pi$ is compactified to a disk with an extension of the action of π, we obtain a new proof of the Novikov conjecture proven in [3].

For this purpose, consider a closed orientable compact manifold M and a cohomology class $w \in H^*(M \times M; \mathbf{Q})$. Assume that w has the symmetric property,

$$w \cdot (1 \otimes x) = (x \otimes 1) \cdot w, \quad \forall x \in H^*(M; \mathbf{Q}). \tag{16}$$

Our aim is to describe symmetric elements w of this kind. Let x_i, $0 \leqslant i \leqslant N$, be a homogeneous basis in $H^*(M; \mathbf{Q})$, $x_0 = 1 \in H^0(M; \mathbf{Q})$, $x_N \in H^n(M; \mathbf{Q})$, $\dim M = n$, and $\langle x_N, [M]\rangle = 1$.

The multiplication tensor λ^k_{ij} is defined by the formula $x_i \cdot x_j = \lambda^k_{ij} x_k$, where $\lambda^k_{i0} = \lambda^k_{0i} = \delta^k_i$ and $\lambda^N_{ij} = \langle x_i \cdot x_j, [M]\rangle$.

The associativity of the multiplication, $(x_i \cdot x_j) \cdot x_k = x_i \cdot (x_j \cdot x_k)$, means that $\lambda^l_{ij}\lambda^s_{lk}x_s = (\lambda^l_{ij}x_l) \cdot x_k = (x_i \cdot x_j) \cdot x_k = x_i \cdot (x_j \cdot x_k) = x_i \cdot (\lambda^l_{jk}x_l) = \lambda^s_{il}\lambda^l_{jk}x_s$, i.e.,

$$\lambda^l_{ij}\lambda^s_{lk} = \lambda^s_{il}\lambda^l_{jk}. \tag{17}$$

The element w can be represented as $w = \mu^{ij} x_i \otimes x_j$. Then condition (16) implies

$$\mu^{il} x_i \otimes x_l \cdot x_k = \mu^{lj} x_k \cdot x_l \otimes x_j, \qquad \mu^{il} x_i \otimes (\lambda^j_{lk} x_j) = \mu^{lj}(\lambda^i_{kl} x_i) \otimes x_j, \quad \text{or} \quad \mu^{il}\lambda^j_{lk} = \mu^{lj}\lambda^i_{kl}. \tag{18}$$

Assume that

$$\mu^{Nj} = \mu^{jN} = \delta^j_0. \tag{19}$$

Then it follows from (18) that $\mu^{il}\lambda^N_{lk} = \mu^{lN}\lambda^i_{kl}$ or $\mu^{il}\lambda^N_{lk} = \delta^l_0\lambda^i_{kl} = \lambda^i_{k0} = \delta^i_k$. This means that the matrix $\|\mu^{ij}\|$ is inverse to the matrix $\|\lambda^N_{ij}\|$,

$$\|\mu^{ij}\| = \|\lambda^N_{ij}\|^{-1}. \tag{20}$$

Relations (18) imply a part of the associativity relations (17),

$$\lambda^N_{i'i}\mu^{il}\lambda^j_{lk} = \lambda^N_{i'i}\mu^{lj}\lambda^i_{kl}, \qquad \delta^{i'}_l\lambda^j_{lk} = \lambda^N_{i'i}\mu^{lj}\lambda^i_{kl}, \qquad \lambda^j_{i'k} = \mu^{lj}\lambda^i_{kl}\lambda^N_{i'i},$$

$$\lambda^N_{jj'}\lambda^j_{i'k} = \lambda^N_{jj'}\mu^{lj}\lambda^i_{kl}\lambda^N_{i'i}, \quad \lambda^N_{jj'}\lambda^j_{i'k} = \delta^l_{j'}\lambda^i_{kl}\lambda^N_{i'i}, \quad \lambda^N_{jj'}\lambda^j_{i'k} = \lambda^i_{kj'}\lambda^N_{i'i}, \quad \lambda^j_{i'k}\lambda^N_{jj'} = \lambda^N_{i'i}\lambda^i_{kj'}.$$

Compare with (17); this gives $\lambda^l_{ij}\lambda^s_{lk} = \lambda^s_{il}\lambda^l_{jk}$.

As a consequence of (20), one can obtain relations for the symmetric elements of the form $w = (x \otimes 1)(\mu^{ij} x_i \otimes x_j)(1 \otimes y) = (\mu^{ij} x_i \otimes x_j)(1 \otimes xy)$.

3. MANIFOLDS WITH BOUNDARY

Assume now that a closed orientable compact manifold M has a nonempty boundary ∂M. Then the Poincaré duality leads to the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_{k+1}(M) \arrow[r, "j"] & H_{k+1}(M,\partial M) \arrow[r, "\delta"] & H_k(\partial M) \arrow[r, "i"] & {} \\
\cdots \arrow[r] & H^{n-k}(M,\partial M) \arrow[r, "j^*"] \arrow[u, "D"] & H^{n-k}(M) \arrow[r, "i^*"] \arrow[u, "D"] & H^{n-k}(\partial M) \arrow[r, "\delta^*"] \arrow[u, "D"] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, "i"] & H_k(M) \arrow[r, "j"] & H_k(M,\partial M) \arrow[r] & \cdots \\
{} \arrow[r, "\delta^*"] & H^{n+1-k}(M,\partial M) \arrow[r, "j^*"] \arrow[u, "D"] & H^{n+1-k}(M) \arrow[r] \arrow[u, "D"] & \cdots
\end{tikzcd}$$

The Poincaré duality relates the multiplication in cohomology with the evaluation on the fundamental class by the formula $\langle x \wedge y, [M]\rangle = (x, Dy)$ for any $x \in H^*(M)$ and $y \in H^*(M,\partial M)$ (or for any $y \in H^*(M)$ and $x \in H^*(M,\partial M)$), and the operation $\wedge$ defines a pairing

$$\wedge\colon H^i(M) \times H^j(M,\partial M) \longrightarrow H^{i+j}(M,\partial M). \tag{21}$$

Hence, the pairing (21) defines a module structure over the ring $H^*(M)$ on $H^*(M,\partial M)$,

$$y \wedge (x_1 \cdot x_2) = (y \wedge x_1) \cdot x_2 = \pm x_1 \cdot (y \wedge x_2) \quad \text{for} \quad y \in H^*(M) \quad \text{and} \quad x_1, x_2 \in H^*(M,\partial M).$$

Consider the Cartesian product $M \times M$. The boundary $\partial(M \times M)$ is given by the union

$$\partial(M \times M) = (M \times \partial M) \cup (\partial M \times M), \quad (M \times \partial M) \cap (\partial M \times M) = \partial M \times \partial M.$$

Consider a cohomology class $w \in H^*(M \times M, \partial M \times M; \mathbf{Q}) \approx H^*(M,\partial M;\mathbf{Q}) \otimes H^*(M;\mathbf{Q})$. Assume that w has the symmetric property

$$w \cdot (1 \otimes y) = (y \otimes 1) \cdot w \in H^*(M \times M, \partial M \times M), \quad \forall y \in H^*(M;\mathbf{Q}). \tag{22}$$

The result is similar to that in the case of manifolds without boundary:

Theorem 3. *Let $w \in H^*(M \times M, \partial M \times M)$ have the symmetric property* (22). *Let $x_i \in H^*(M,\partial M)$ and $y_j \in H^*(M)$ be bases and let $w = \mu^{ij} x_i \otimes y_j$. Then $\|\mu^{ij}\| = \|\lambda_{ij}^N\|^{-1}$, where $\lambda_{ij}^N = \langle y_i \wedge x_j, [M,\partial M]\rangle$.*

To prove this fact, consider homogenous bases in the cohomology groups $H^*(M;\mathbf{Q})$ and $H^*(M,\partial M;\mathbf{Q})$, say, $x_i \in H^*(M,\partial M)$ and $y_j \in H^*(M)$.

Let $y_0 = 1 \in H^0(M;\mathbf{Q}) \approx \mathbf{Q}$, $x_N \in H^n(M,\partial M;\mathbf{Q}) \approx \mathbf{Q}$, $\dim M = n$, and $\langle x_N, [M,\partial M]\rangle = 1$. The pairing (21) is defined by the formula $y_i \wedge x_j = \lambda_{ij}^k x_k$. If $y_i, y_k \in H^*(M)$, then $y_i \wedge y_k = \nu_{ik}^s y_s$ is such that $\nu_{0k}^s = \nu_{k0}^s = \delta_k^s$. Property (22) can be represented as $\mu^{ij} x_i \otimes y_j y_k = \mu^{ij} y_k \wedge x_i \otimes y_j$ or $\mu^{ij} \nu_{jk}^s x_i \otimes y_s = \mu^{ij} \lambda_{ki}^s x_s \otimes y_j$. Hence, $\mu^{ij} \nu_{jk}^s = \mu^{ls} \lambda_{kl}^i$.

In particular, if $i = N$, then $\mu^{Nj} \nu_{jk}^s = \mu^{ls} \lambda_{kl}^N$.

As in the case of manifolds without boundary, assume that the element w satisfies condition (19), $\mu^{Nj} = \delta_0^j$. Then $\delta_k^s = \mu^{ls} \lambda_{kl}^N$.

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