Description of the vector G-bundles over G-spaces with quasi-free proper action of discrete group G

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January 21, 2009

Abstract

We give a description of the vector G-bundles over G-spaces with quasi-free proper action of discrete group G in terms of the classifying space.

1 The setting of the problem

This problem naturally arises from the Conner-Floyd's description ([2]) of the bordisms with the action of a group G using the so-called fix-point construction. This construction reduces the problem of describing the bordisms to two simpler problems: a) description of the fixed-point set (or, more generally, the stationary point set), which happens to be a submanifold attached with the structure of its normal bundle and the action of the same group G, however, this action could have stationary points of lower rank; b) description of the bordisms of lower rank with an action of the group G. We assume that the group G is discrete.

Lets ξ be an G-equivariant vector bundle with base M.

$$\begin{array}{c} \xi \\ \downarrow \\ M \end{array} \tag{1}$$

Lets $H < G$ be a normal finite subgroup. Assume that the action of the group G over the base M reduces to the factor group $G_0 = G/H$:

$$\begin{array}{ccc} G \times M & \longrightarrow & M \\ \downarrow & & \| \\ G_0 \times M & \longrightarrow & M \end{array} \tag{2}$$

suppose, additionally, that the action $G_0 \times M \longrightarrow M$ is free and there is no more fixed points of the action of the group H in the total space of the bundle ξ.

*Partly supported by the grant of RFFI No.08-01-00034-a, NSh-1562.2008.1, Program 2.1.1/5031

So, we have the following commutative diagram

$$\begin{array}{ccc} G \times \xi & \longrightarrow & \xi \\ \downarrow & & \downarrow \\ G_0 \times M & \longrightarrow & M \end{array} \tag{3}$$

Definition 1 *As in [6, p. 210], we shall say that the described action of the group G is quasi-free over the base with normal stationary subgroup H.*

Reducing the action to the subgroup H, we obtain the simpler diagram:

$$\begin{array}{ccc} H \times \xi & \longrightarrow & \xi \\ \downarrow & & \downarrow \\ M & = & M \end{array} \tag{4}$$

Following [4], let $\rho_k : H \longrightarrow \mathbf{U}(V_k)$ be the series of all the irreducible (unitary) representation of the finite group H. Then the H-bundle ξ can be presented as the finite direct sum:

$$\xi \approx \bigoplus_k \left(\xi_k \bigotimes V_k\right), \tag{5}$$

where the action of the group H over the bundles ξ_k is trivial, V_k denotes the trivial bundle with fiber V_k and with fiberwise action of the group H, defined using the linear representation ρ_k.

Lemma 1 *The group G acts on every term of the sum (5) separately.*

Proof. Consider now the action of the group G over the total space of the bundle ξ. Fix a point $x \in M$. The action of the element $g \in G$ is fiberwise, and maps the fiber ξ_x to the fiber ξ_{gx}:

$$\Phi(x, g) : \xi_x \longrightarrow \xi_{gx}.$$

Also, for a par of elements $g_1, g_2 \in G$ we have:

$$\Phi(x, g_1 g_2) = \Phi\left(g_2 x, g_1\right) \circ \Phi\left(x, g_2\right), \tag{6}$$

$$\Phi(x, g_1 g_2) : \xi_x \quad \overset{\Phi(x,g_2)}{\longrightarrow} \quad \xi_{g_2 x} \quad \overset{\Phi(g_2 x, g_1)}{\longrightarrow} \quad \xi_{g_1 g_2 x}$$

In particular, if $g_2 = h \in H < G$, then $g_2 x = hx = x$. So,

$$\Phi(x, gh) : \xi_x \quad \overset{\Phi(x,h)}{\longrightarrow} \quad \xi_x \quad \overset{\Phi(x,g)}{\longrightarrow} \quad \xi_{gx}$$

Analogously, if $g_1 = h \in H < G$, then $g_1 gx = hgx = gx$. So

$$\Phi(x, hg) : \xi_x \xrightarrow{\Phi(x,g)} \xi_{gx} \xrightarrow{\Phi(gx,h)} \xi_{gx}$$

According to [4] the operator $\Phi(x, h)$ does not depends on the point $x \in M$,

$$\Phi(x, h) = \Psi(h) : \bigoplus_k \left(\xi_{k,x} \bigotimes V_k\right) \longrightarrow \bigoplus_k \left(\xi_{k,x} \bigotimes V_k\right),$$

here, since the action of the group H is given over every space V_k using pairwise different irreducible representations ρ_k, we have

$$\Psi(h) = \bigoplus_k \left(\mathbf{Id} \bigotimes \rho_k(h)\right).$$

In this way, we obtain the following relation:

$$\Phi(x, gh) = \Phi(x, g) \circ \Psi(h) = \Phi(x, ghg^{-1}g) = \Psi(ghg^{-1}) \circ \Phi(x, g). \tag{7}$$

Lets write the operator $\Phi(x, g)$ using matrices to decompose the space ξ_x as the direct sum

$$\xi_x = \bigoplus_k \left(\xi_{k,x} \bigotimes V_k\right):$$

$$\Phi(x, g) = \begin{pmatrix} \Phi(x,g)_{1,1} & \cdots & \Phi(x,g)_{k,1} & \cdots \\ \vdots & \ddots & \vdots & \\ \Phi(x,g)_{1,k} & \cdots & \Phi(x,g)_{k,k} & \cdots \\ \vdots & & \vdots & \ddots \end{pmatrix} \tag{8}$$

If $k \neq l$ then $\Phi(x, g)_{k,l} = 0$, i.e. the matrix $\Phi(x, g)$ its diagonal,

$$\Phi(x, g) = \bigoplus_k \Phi(x, g)_{k,k} : \bigoplus_k \left(\xi_{k,x} \bigotimes V_k\right) \longrightarrow \bigoplus_k \left(\xi_{k,gx} \bigotimes V_k\right),$$

$$\Phi(x, g)_{k,k} : \left(\xi_{k,x} \bigotimes V_k\right) \longrightarrow \left(\xi_{k,gx} \bigotimes V_k\right),$$

as it was required to prove. ∎

2 Description of the particular case $\xi = \xi_0 \bigotimes V$

Here we will consider the particular case of a G-vector bundle $\xi = \xi_0 \otimes V$ with base M.

$$\begin{array}{c} \xi \\ \downarrow \\ M \end{array}$$

where the action of the group G is quasi-free over the base with finite normal stationary subgroup $H < G$.

We will assume that the group H acts trivially over the bundle ξ_0. By V we denote the trivial bundle with fiber V and with fiberwise action of the group H given by an irreducible linear representation ρ.

Definition 2 *A canonical model for the fiber in a G-bundle $\xi = \xi_0 \bigotimes V$ with fiber $F \otimes V$ is the product $G_0 \times (F \otimes V)$ with an action of the group G*

$$\begin{array}{ccc} G \times (G_0 \times (F \otimes V)) & \stackrel{\phi}{\longrightarrow} & G_0 \times (F \otimes V) \\ \downarrow & & \downarrow \\ G \times G_0 & \stackrel{\mu}{\longrightarrow} & G_0 \end{array}$$

where μ denotes the natural left action of G on its quotient G_0, and

$$\phi([g], g_1) : [g] \times (F \otimes V) \to [g_1 g] \times (F \otimes V)$$

is given by the formula

$$\phi([g], g_1) = \mathbf{Id} \otimes \rho(u(g_1 g) u^{-1}(g)). \tag{9}$$

where

$$u : G \longrightarrow H$$

is a homomorphism of right H-modules by multiplication, i.e.

$$u(gh) = u(g)h, \quad u(1) = 1, \quad g \in G, h \in H.$$

Lemma 2 *The definition (9) of the action of G is well-defined.*

Proof. It is enough to prove that that a) the formula (9) defines an action, i.e.

$$\phi([g], g_2 g_1) = \phi([g_1 g], g_2) \circ \phi([g], g_1),$$

and b) that the formula (9) does not depends on the chosen representative $gh \in [g]$:

$$\mathbf{Id} \otimes \rho(u(g_1 g) u^{-1}(g)) = \mathbf{Id} \otimes \rho(u(g_1 gh) u^{-1}(gh))$$

for every $g \in G$ and $h \in H$.

In fact,

$$\begin{array}{l} \phi([g], g_2 g_1) = \mathbf{Id} \otimes \rho(u(g_2 g_1 g) u^{-1}(g)) = \\ \\ \mathbf{Id} \otimes \rho(u(g_2 g_1 g) u(g_1 g) u^{-1}(g_1 g) u^{-1}(g)) = \\ = \mathbf{Id} \otimes \rho(u(g_2 g_1 g) u(g_1 g)) \circ \mathbf{Id} \otimes \rho(u^{-1}(g_1 g) u^{-1}(g)) = \\ \\ = \phi([g_1 g], g_2) \circ \phi([g], g_1), \end{array}$$

what proves a), and, recalling the equation $u(gh) = u(g)h$ for every $g \in G$ and $h \in H$, it is clear that

$$u(g_1 gh) u^{-1}(gh) = u(g_1 g) h h^{-1} u^{-1}(g) = u(g_1 g) u^{-1}(g),$$

which is a sufficient condition for b) to be true. ∎

As it is well known, for the actions we are studying, we can always consider over the base M an atlas of equivariant charts $\{O_\alpha\}$,

$$M = \bigcup_\alpha O_\alpha,$$

$$[g]O_\alpha = O_\alpha, \qquad \forall [g] \in G_0.$$

If the atlas is fine enough, then every chart can be presented as a disjoint union of its subcharts:

$$O_\alpha = \bigsqcup_{[g]\in G_0} [g]U_\alpha \approx U_\alpha \times G_0,$$

i.e. $[g]U_\alpha \cap [g']U_\alpha = \emptyset$ if $[g] \neq [g']$, and when $\alpha \neq \beta$, if $U_\alpha \cap [g_{\alpha\beta}]U_\beta \neq \emptyset$, then the element $g_{\alpha\beta}$ is the only one for which that intersection is non-empty, i.e. if $[g] \neq [g_{\alpha\beta}]$, then $U_\alpha \cap [g]U_\beta = \emptyset$, i.e.

$$O_\alpha \cap O_\beta \approx (U_\alpha \cap [g_{\alpha\beta}]U_\beta) \times G_0,$$

for every α, β. We use these facts and notations to formulate the next theorem.

Theorem 1 *The bundle $\xi = \xi_0 \bigotimes V$ is locally homeomorphic to the cartesian product of some chart U_α by the canonical model. More precisely, for a fine enough atlas, there exist G-equivariant trivializations*

$$\psi_\alpha : O_\alpha \times (F \otimes V) \to \xi|_{O_\alpha} \tag{10}$$

where

$$O_\alpha \times (F \otimes V) \approx U_\alpha \times (G_0 \times (F \otimes V))$$

and the diagram

$$\begin{array}{ccc} \xi|_{O_\alpha} & \xrightarrow{g} & \xi|_{O_\alpha} \\ \uparrow \psi_\alpha & & \uparrow \psi_\alpha \\ U_\alpha \times (G_0 \times (F \otimes V)) & \xrightarrow{\mathbf{Id} \times \phi(g)} & U_\alpha \times (G_0 \times (F \otimes V)) \end{array} \tag{11}$$

is commutative where $g \in G$, $\mathbf{Id} : U_\alpha \to U_\alpha$, and $\phi(g)$ denotes the canonical action.

Proof. Using an atlas as in the remarks at the beginning of the theorem, we shall construct the trivialization (10) starting from an arbitrary trivialization

$$\psi_\alpha : U_\alpha \times (F \otimes V) \to \xi|_{U_\alpha}$$

in such a way, that the diagram

$$\begin{array}{ccc} \xi|_{U_\alpha} & \xrightarrow{g} & \xi|_{[g]U_\alpha} \\ \uparrow \psi_\alpha & & \uparrow \psi_\alpha \\ U_\alpha \times (F \otimes V) & \longrightarrow & [g]U_\alpha \times (F \otimes V) \end{array}$$

commutes for every $g \in [g]$, where the left and upper arrows are given and we have to construct the down and right arrows.

From such a construction, the equivariance will follow automatically and the proof of the theorem reduces to show that the constructed down arrow coincides with that on (11).

Evidently, for a given trivialization $\psi_\alpha : U_\alpha \times (F \otimes V) \to \xi|_{U_\alpha}$, there are several ways to define a trivialization $\psi_\alpha : [g]U_\alpha \times (F \otimes V) \to \xi|_{[g]U_\alpha}$, since there are several elements $g \in G$ sending $\xi|_{U_\alpha}$ to $\xi|_{[g]U_\alpha}$.

Thus, consider a set-theoretic cross-section

$$p' : G_0 \longrightarrow G,$$

to the projection p in the exact sequence of groups

$$\mathbf{1} \longrightarrow H \longrightarrow G \xrightarrow{p} G_0,$$

$$p \circ p' = \mathbf{Id} : G_0 \xrightarrow{p'} G \xrightarrow{p} G_0.$$

Put

$$g' = p' \circ p : G \longrightarrow G.$$

Without loss of generality, we can take $g'(1) = 1$.

In this case

$$g'(g) = gu^{-1}(g),$$

where

$$u : G \longrightarrow H$$

is a homomorphism of right H-modules by multiplication, i.e.

$$u(gh) = u(g)h, \quad g \in G, h \in H.$$

In particular, this means that

$$g'(gh) = g'(g), \quad h \in H.$$

Lets

$$\tilde{\psi}_\alpha : U_\alpha \times F \longrightarrow \xi_0|_{U_\alpha}$$

be some trivialization. We define the trivialization ψ_α in (10) by the rule: if $[g]x_\alpha \in [g]U_\alpha$, i.e. $x_\alpha \in U_\alpha$, then, the map

$$\psi_\alpha([g]x_\alpha) : [g]x_\alpha \times (F \otimes V) \longrightarrow \xi_{[g]x_\alpha} \otimes V$$

is given by the formula

$$\begin{aligned} \psi_\alpha([g]x_\alpha) &= \Phi(x_\alpha, g'(g)) \circ \left(\tilde{\psi}_\alpha(x_\alpha) \otimes \mathbf{Id}\right) = \\ &= \Phi(x_\alpha, gu^{-1}(g)) \circ \left(\tilde{\psi}_\alpha(x_\alpha) \otimes \mathbf{Id}\right). \end{aligned} \tag{12}$$

where, from the first equality, it is clear that the definition does not depend on the representative $g \in [g]$.

In particular, for $[g] = 1$, we recover the initial trivialization

$$\psi_\alpha(x_\alpha) = \tilde{\psi}_\alpha(x_\alpha) \otimes \mathbf{Id}$$

since $\Phi(x, g'(1)) = \Phi(x, 1) = 1$.

Using this trivialization the action of the group G can be carried to the cartesian product $O_\alpha \times (F \otimes V)$:

$$\Phi_\alpha(g) : O_\alpha \times (F \otimes V) \longrightarrow O_\alpha \times (F \otimes V)\,.$$

Lets $x_\alpha \in U_\alpha$, $g \in G$, then

$$\Phi_\alpha([g]x_\alpha, g_1) : [g]x_\alpha \times (F \otimes V) \longrightarrow [g_1 g]x_\alpha \times (F \otimes V)$$

is given by the formula

$$\Phi_\alpha([g]x_\alpha, g_1) = (\psi_\alpha([g_1 g]x_\alpha))^{-1}\, \Phi([g]x_\alpha, g_1)\psi_\alpha([g]x_\alpha).$$

Applying (12), we obtain

$$\begin{aligned}
\Phi_\alpha([g]x_\alpha, g_1) = \quad & \left(\Phi(x_\alpha, g_1 g u^{-1}(g_1 g)) \circ \left(\tilde{\psi}_\alpha(x_\alpha) \otimes \mathbf{Id}\right)\right)^{-1} \circ \\
& \circ \Phi([g]x_\alpha, g_1) \circ \Phi(x_\alpha, g u^{-1}(g)) \circ \left(\tilde{\psi}_\alpha(x_\alpha) \otimes \mathbf{Id}\right) = \\
& = \left(\tilde{\psi}_\alpha(x_\alpha) \otimes \mathbf{Id}\right)^{-1} \circ \\
& \circ \Phi(x_\alpha, g_1 g u^{-1}(g_1 g))^{-1} \circ \Phi([g]x_\alpha, g_1) \circ \Phi(x_\alpha, g u^{-1}(g)) \circ \\
& \circ \left(\tilde{\psi}_\alpha(x_\alpha) \otimes \mathbf{Id}\right) = \\
& = \left(\tilde{\psi}_\alpha(x_\alpha) \otimes \mathbf{Id}\right)^{-1} \circ \\
& \circ \Phi(x_\alpha, u^{-1}(g_1 g))^{-1} \circ \Phi(x_\alpha, g_1 g)^{-1} \circ \Phi([g]x_\alpha, g_1) \circ \\
& \circ \Phi(x_\alpha, g) \circ \Phi(x_\alpha, u^{-1}(g)) \circ \\
& \circ \left(\tilde{\psi}_\alpha(x_\alpha) \otimes \mathbf{Id}\right) = \\
& = \left(\tilde{\psi}_\alpha(x_\alpha) \otimes \mathbf{Id}\right)^{-1} \circ \\
& \circ \Phi(x_\alpha, u^{-1}(g_1 g))^{-1} \circ \Phi(x_\alpha, u^{-1}(g)) \circ \\
& \circ \left(\tilde{\psi}_\alpha(x_\alpha) \otimes \mathbf{Id}\right);
\end{aligned}$$

$$\begin{aligned}\Phi_\alpha([g]x_\alpha, g_1) = \; & \left(\tilde{\psi}_\alpha(x_\alpha)\otimes \mathbf{Id}\right)^{-1}\circ \\ & \circ (\mathbf{Id}\otimes \rho(u(g_1 g)))\circ \left(\mathbf{Id}\otimes \rho(u^{-1}(g))\right)\circ \\ & \circ \left(\tilde{\psi}_\alpha(x_\alpha)\otimes \mathbf{Id}\right) = \\ & = \left(\tilde{\psi}_\alpha(x_\alpha)\otimes \mathbf{Id}\right)^{-1}\circ \\ & \circ \left(\mathbf{Id}\otimes \left(\rho(u(g_1 g)u^{-1}(g))\right)\right)\circ \\ & \circ \left(\tilde{\psi}_\alpha(x_\alpha)\otimes \mathbf{Id}\right) = \\ & = \mathbf{Id}\otimes \rho(u(g_1 g)u^{-1}(g)).\end{aligned}$$

The operator

$$\Phi_\alpha([g]x_\alpha, g_1) = \mathbf{Id}\otimes \rho(u(g_1 g)u^{-1}(g)) = \phi(g_1, [g]).$$

does not depend on the point $x_\alpha \in U_\alpha$. So, the theorem is proved. ∎

By $\mathrm{Aut}_G\,(G_0\times(F\otimes V))$ we denote the group of equivariant automorphisms of the space $G_0\times(F\otimes V)$ as a vector G-bundle with base G_0, fiber $F\otimes V$ and canonical action of the group G.

Corollary 1 *The transition functions on the intersection*

$$O_\alpha\cap O_\beta \approx (U_\alpha\cap[g_{\alpha\beta}]U_\beta)\times G_0,$$

i.e. the homomorphisms $\Psi_{\alpha\beta}$ on the diagram

$$\begin{array}{ccc}(U_\alpha\cap[g_{\alpha\beta}]U_\beta)\times(G_0\times(F\otimes V)) & \stackrel{\Psi_{\alpha\beta}}{\longrightarrow} & (U_\alpha\cap[g_{\alpha\beta}]U_\beta)\times(G_0\times(F\otimes V)) \\ \big\downarrow & & \big\downarrow \\ (U_\alpha\cap[g_{\alpha\beta}]U_\beta)\times G_0 & \stackrel{\mathbf{Id}}{\longrightarrow} & (U_\alpha\cap[g_{\alpha\beta}]U_\beta)\times G_0\end{array} \tag{13}$$

are equivariant with respect to the canonical action of the group G over the product of the base by the canonical model, i.e.

$$\Psi_{\alpha\beta}(x)\circ\phi(g_1,[g]) = \phi(g_1,[g])\circ\Psi_{\alpha\beta}(x)$$

for every $x\in U_\alpha\cap[g_{\alpha\beta}]U_\beta,\ g_1\in G,\ [g]\in G_0$, In other words,

$$\Psi_{\alpha\beta}(x)\in \mathrm{Aut}_G\,(G_0\times(F\otimes V)).$$

Now we give a more accurate description of the group $\mathrm{Aut}_G\,(G_0\times(F\otimes V))$. By definition, an element of the group $\mathrm{Aut}_G\,(G_0\times(F\otimes V))$ is an equivariant mapping $\mathbf{A}^a$, such that the pair $(\mathbf{A}^a, a)$ defines a commutative diagram

$$\begin{array}{ccc}(G_0\times(F\otimes V)) & \stackrel{\mathbf{A}^a}{\longrightarrow} & G_0\times(F\otimes V) \\ \big\downarrow & & \big\downarrow \\ G_0 & \stackrel{a}{\longrightarrow} & G_0,\end{array}$$

which commutes with the canonical action, i.e. the map $a \in \mathrm{Aut}_G(G_0)$ satisfies the condition

$$a \in \mathrm{Aut}_G(G_0) \approx G_0, \quad a[g] = [ga],\ [g] \in G_0,$$

and the mapping $\mathbf{A}^a = (A^a[g])_{[g]\in G_0}$,

$$A^a[g] : [g] \times (F \otimes V) \to [ga] \times (F \otimes V)$$

satisfies a commutation condition with respect to the action of the group G:

$$\begin{array}{ccc} [g] \times (F \otimes V) & \stackrel{A^a[g]}{\longrightarrow} & [ga] \times (F \otimes V) \\ \Big\downarrow \phi(g_1, [g]) & & \Big\downarrow \phi(g_1, [ga]) \\ [g_1 g] \times (F \otimes V) & \stackrel{A^a[g_1 g]}{\longrightarrow} & [g_1 ga] \times (F \otimes V) \end{array},$$

$$\phi(g_1, [ga]) \circ A^a[g] = A^a[g_1 g] \circ \phi(g_1, [g]) \tag{14}$$

i.e.

$$(\mathbf{Id} \otimes \rho(u(g_1 ga)u^{-1}(ga)))A^a[g] = A^a[g_1 g](\mathbf{Id} \otimes \rho(u(g_1 g)u^{-1}(g))) \tag{15}$$

where $[g] \in G_0, \quad g_1 \in G$.

Lemma 3 *One has an exact sequence of groups*

$$\mathbf{1} \to GL(F) \longrightarrow \mathrm{Aut}_G\left(G_0 \times (F \otimes V)\right) \longrightarrow G_0 \to \mathbf{1}. \tag{16}$$

Proof. To define a projection

$$pr : \mathrm{Aut}_G\left(G_0 \times (F \otimes V)\right) \longrightarrow G_0$$

we send the fiberwise map

$$\mathbf{A}^a : G_0 \times (F \otimes V) \longrightarrow G_0 \times (F \otimes V)$$

to its restriction over the base $a : G_0 \to G_0$, i.e. $a \in \mathrm{Aut}_G(G_0) \approx G_0$. So, this is a well-defined homomorphism.

We need to show that pr is an epimorphism and that its kernel is isomorphic to $GL(F)$. Lets calculate the kernel.

For $[a] = [1]$ we have

$$(\mathbf{Id} \otimes \rho(u(g_1 g)u^{-1}(g)))A^1[g] = A^1[g_1 g](\mathbf{Id} \otimes \rho(u(g_1 g)u^{-1}(g))) \tag{17}$$

In the case $g_1 = h \in H$, we obtain

$$(\mathbf{Id} \otimes \rho(u(hg)u^{-1}(g)))A^1[g] = A^1[g](\mathbf{Id} \otimes \rho(u(hg)u^{-1}(g)))$$

Since the representation ρ is irreducible, by Schur's lemma, we have

$$A^1[g] = B^1[g] \otimes \mathbf{Id}.$$

On the other side, assuming in (17) that $g = 1$, we have

$$(\mathbf{Id} \otimes \rho(u(g)))A^1[1] = A^1[g](\mathbf{Id} \otimes \rho(u(g))),$$

i.e.

$$(\mathbf{Id} \otimes \rho(u(g)))(B^1[1] \otimes \mathbf{Id}) = (B^1[g] \otimes \mathbf{Id})(\mathbf{Id} \otimes \rho(u(g))),$$

or

$$(B^1[g] \otimes \mathbf{Id}) = (B^1[1] \otimes \mathbf{Id}).$$

So, the kernel $\ker pr$ is isomorphic to the group $GL(F)$.

In the generic case, i.e. $[a] \neq 1$, we can compute the operator $A^a[g]$ in terms of its value at the identity $A^a[1]$ from the formula (15): assuming $g = 1$, we obtain (changing g_1 by g):

$$(\mathbf{Id} \otimes \rho(u(ga)u^{-1}(a)))A^a[1] = A^a[g](\mathbf{Id} \otimes \rho(u(g))), \tag{18}$$

i.e.

$$A^a[g] = (\mathbf{Id} \otimes \rho(u(ga)u^{-1}(a)))A^a[1](\mathbf{Id} \otimes \rho(u^{-1}(g))), \tag{19}$$

Therefore, the operator is completely defined by its value

$$A^a[1] : [1] \times (F \otimes V) \to [a] \times (F \otimes V)$$

at the identity $g = 1$.

Now we describe the operator $A^a[1]$ in terms of the representation ρ and its properties.

We have a commutation rule with respect to the action of the subgroup H:

$$\begin{array}{ccc} [1] \times (F \otimes V) & \xrightarrow{A^a[1]} & [a] \times (F \otimes V) \\ \downarrow \phi(h,[1]) & & \downarrow \phi(h,[a]) \\ [1] \times (F \otimes V) & \xrightarrow{A^a[1]} & [a] \times (F \otimes V) \end{array},$$

Equivalently

$$A^a[1] \circ \phi(h, [1]) = \phi(h, [a]) \circ A^a[1],$$

i.e.

$$A^a[1] \circ (\mathbf{Id} \otimes \rho(h)) = (\mathbf{Id} \otimes \rho(g'^{-1}(a)hg'(a))) \circ A^a[1],$$

i.e.

$$A^a[1] \circ (\mathbf{Id} \otimes \rho(h)) = (\mathbf{Id} \otimes \rho_{g'(a)}(h)) \circ A^a[1].$$

The last equation means that the operator should $A^a[1]$ permute these representations, or equivalently, such an operator exists only when the representations ρ and $\rho_{g'(a)}$ are equivalent. Recalling the commutation rule (7), we see that this is the case we are been considering.

Thus, if the representations ρ and ρ_g are equivalent, we have an (inverse) splitting operator $C(g)$, satisfying the equation

$$\rho_g(h) = \rho\left(g^{-1}hg\right) = C(g)\rho(h)C^{-1}(g). \tag{20}$$

for every $g \in G$. The operator $C(g)$ is defined up to multiplication by a scalar operator $\mu_g \in SS^1 \subset \mathbf{C}^1$.

So

$$A^a[1] \circ (\mathbf{Id} \otimes \rho(h)) = (\mathbf{Id} \otimes C(g'(a)) \circ \rho(h) \circ C^{-1}(g'(a))) \circ A^a[1],$$

or

$$(\mathbf{Id} \otimes C^{-1}(g'(a))) \circ A^a[1] \circ (\mathbf{Id} \otimes \rho(h)) = (\mathbf{Id} \otimes \rho(h)) \circ (\mathbf{Id} \otimes C^{-1}(g'(a))) \circ A^a[1],$$

Then, by the Schur's lemma,

$$(\mathbf{Id} \otimes C^{-1}(g'(a))) \circ A^a[1] = B^a[1] \otimes \mathbf{Id},$$

i.e.

$$A^a[1] = B^a[1] \otimes C(g'(a)),$$

Using the formula (19), we obtain

$$A^a[g] = (\mathbf{Id} \otimes \rho(u(ga)u^{-1}(a)))(B^a[1] \otimes C(g'(a)))(\mathbf{Id} \otimes \rho(u^{-1}(g))),$$

i.e.

$$A^a[g] = B^a[1] \otimes (\rho(u(ga)u^{-1}(a)) \circ C(g'(a)) \circ \rho(u^{-1}(g))). \tag{21}$$

This means, that by defining the matrix $B^a[1]$, it is possible to obtain all the operators $A^a[g]$ satisfying the equation (19).

It remains to verify the commutation rule (15), i.e. in the formula

$$(\mathbf{Id} \otimes \rho(u(g_1 ga)u^{-1}(ga)))A^a[g] = A^a[g_1 g](\mathbf{Id} \otimes \rho(u(g_1 g)u^{-1}(g)))$$

we substitute the expression (21):

$$(\mathbf{Id} \otimes \rho(u(g_1 ga)u^{-1}(ga))) \circ (B^a[1] \otimes (\rho(u(ga)u^{-1}(a)) \circ C(g'(a)) \circ \rho(u^{-1}(g)))) =$$

$$= (B^a[1] \otimes (\rho(u(g_1 ga)u^{-1}(a)) \circ C(g'(a)) \circ \rho(u^{-1}(g_1 g)))) \circ (\mathbf{Id} \otimes \rho(u(g_1 g)u^{-1}(g)))$$

that is

$$B^a[1] \otimes \rho(u(g_1 ga)u^{-1}(ga))) \circ (\rho(u(ga)u^{-1}(a)) \circ C(g'(a)) \circ \rho(u^{-1}(g)))) =$$

$$= B^a[1] \otimes (\rho(u(g_1 ga)u^{-1}(a)) \circ C(g'(a)) \circ \rho(u^{-1}(g_1 g)))) \circ (\rho(u(g_1 g)u^{-1}(g)))$$

Note that this identity does not depend on the particular matrix $B^a[1]$, thus, this means that we only need to verify the identity for arbitrary a, g and g_1:

$$\rho(u(g_1 ga)u^{-1}(ga))) \circ (\rho(u(ga)u^{-1}(a)) \circ C(g'(a)) \circ \rho(u^{-1}(g)))) =$$

$$= (\rho(u(g_1 ga)u^{-1}(a)) \circ C(g'(a)) \circ \rho(u^{-1}(g_1 g)))) \circ (\rho(u(g_1 g)u^{-1}(g))),$$

which is obvious, after the natural simplifications

$$\rho(u(g_1 ga)u^{-1}(a)) \circ C(g'(a)) \circ \rho(u^{-1}(g)))) =$$

$$= (\rho(u(g_1 ga)u^{-1}(a)) \circ C(g'(a)) \circ \rho(u^{-1}(g))),$$

So, it follows, that for every element $[a] \in G_0$ there exist an element $(A^a, a) \in \mathrm{Aut}_G\,(G_0 \times (F \otimes V))$. This means that the homomorphism

$$\mathrm{Aut}_G\,(G_0 \times (F \otimes V)) \xrightarrow{pr} G_0$$

is in fact an epimorphism, and the lemma is proved. ∎

It is clear that there is an equivalence between G-vector bundles with fiber $G_0 \times (F \otimes V)$ over a (compact) base X, where G acts trivially over the base and canonically over the fiber, and homotopy classes of mappings from X to the space $B\mathrm{Aut}_G\,(G_0 \times (F \otimes V))$.

Lets denote by $\mathrm{Vect}_G(M, \rho)$ the category of G-equivariant vector bundles $\xi = \xi_0 \otimes V$ with base M, where the action of the group G is quasi-free over the base with finite normal stationary subgroup $H < G$, the group H acts trivially over the bundle ξ_0 and V denotes the trivial bundle with fiber V and with fiberwise action of the group H given by an irreducible linear representation ρ. Here we need to require for the representations $\rho_g(h) = \rho(g^{-1}hg)$ to be equivalent for every $g \in G$, in the other case, in view of the commutation rule, this category may be void.

This is a category because, in fact, we are just taking vector bundles over the space M, then applying tensor product by the fixed bundle V and defining some action of the group G over the resulting spaces. The inclusion $GL(F) \hookrightarrow \mathrm{Aut}_G\,(G_0 \times (F \otimes V))$ from lemma 2 ensures that the identities are included.

Denote by $\mathrm{Bundle}(X, L)$ the category of principal L-bundles over the base X.

Theorem 2 *There is a monomorphism*

$$\mathrm{Vect}_G(M, \rho) \longrightarrow \mathrm{Bundle}(M/G_0, \mathrm{Aut}_G\,(G_0 \times (F \otimes V))). \tag{22}$$

Proof. By corollary 3, every element $\xi \in \mathrm{Vect}_G(M, \rho)$ is defined by transition functions

$$\Psi_{\alpha\beta}:\ (U_\alpha \cap [g_{\alpha\beta}]U_\beta) \to \mathrm{Aut}_G\,(G_0 \times (F \otimes V))$$

where by construction, when $[g] \neq [g_{\alpha\beta}]$, we have $U_\alpha \cap [g]U_\beta = \emptyset$ and if $[g] \neq 1$, then $U_\alpha \cap [g]U_\alpha = \emptyset$ and $U_\beta \cap [g]U_\beta = \emptyset$. This means that the sets U_α and U_β project homeomorphically to open sets under the natural projection $M \to M/G_0$. So, these transition functions are well-defined over an atlas of the quotient space M/G_0 and they form a G-bundle with fiber $G_0 \times (F \otimes V)$ over this quotient space.

By the same arguments, it is obvious that every G-equivariant map

$$h_\alpha : O_\alpha \times (F \otimes V) \to O_\alpha \times (F \otimes V) \tag{23}$$

can be interpreted as a map

$$h_\alpha : U_\alpha \times (G_0 \times (F \otimes V)) \to U_\alpha \times (G_0 \times (F \otimes V)) \tag{24}$$

by means of the homeomorphism $O_\alpha \approx U_\alpha \times G_0$, where the set U_α can be thought as an open set of the space M/G_0. Equivalently,

$$h_\alpha : U_\alpha \to \mathrm{Aut}_G\,(G_0 \times (F \otimes V)) \tag{25}$$

where U_α is homeomorphic to an open set of the space M/G_0. Therefore, the map (22) is well defined.

Conversely, if we start from mappings of the form (25) where the sets U_α are open in M/G_0, by refining the atlas, if it is necessary, we can always think that the inverse image of the open sets U_α under the quotient map $M \to M/G_0$ are homeomorphic to the product $U_\alpha \times G_0$ and then obtain mappings of the form (23). Therefore, the map (22) is a monomorphism. ∎

Of course, the map (22) its not in general an epimorphism, since, when we define the category $\mathrm{Vect}_G(M, \rho)$, we are automatically fixing a bundle $M \to M/G_0$, or equivalently, a homotopy class in $[M/G_0, BG_0]$.

Theorem 3 *If the space X is compact, then*

$$\mathrm{Bundle}(X, \mathrm{Aut}_G\,(G_0 \times (F \otimes V))) \approx \bigsqcup_{M \in \mathrm{Bundle}(X, G_0)} \mathrm{Vect}_G(M, \rho). \tag{26}$$

Proof. By theorem 5, there is an inclusion

$$\bigcup_{M \in \mathrm{Bundle}(X, G_0)} \mathrm{Vect}_G(M, \rho) \hookrightarrow \mathrm{Bundle}(X, \mathrm{Aut}_G\,(G_0 \times (F \otimes V))). \tag{27}$$

Now we will construct an inverse to the map (27), so the fact that the last union is disjoint will follow. Let

$$\Psi_{\alpha\beta} : \ (U_\alpha \cap U_\beta) \to \mathrm{Aut}_G\,(G_0 \times (F \otimes V))$$

be the transition functions of a bundle $\xi \in \mathrm{Bundle}(X, \mathrm{Aut}_G\,(G_0 \times (F \otimes V)))$. By lemma 2, there is a continuous projection of groups $pr : \mathrm{Aut}_G\,(G_0 \times (F \otimes V)) \to G_0$. So, by composition with pr we obtain a bundle with the discrete fiber G_0, and it is well known that G_0 acts fiberwise and freely over the total space M of this bundle and that $M/G_0 = X$.

Also, we can assume that we have chosen an atlas such that there is a homeomorphism

$$M \approx \bigcup_\alpha (U_\alpha \times G_0) \approx \bigcup_\alpha \left(\bigsqcup_{[g] \in G_0} [g] U_\alpha \right)$$

where the intersections are defined by the rule

$$[1] U_\alpha \cap [g_{\alpha\beta}] U_\beta \approx U_\alpha \cap U_\beta$$

where $[g_{\alpha\beta}] = pr \circ \Psi_{\alpha\beta}$.

On the other hand, we have

$$\xi \approx \bigcup_\alpha (U_\alpha \times (G_0 \times (F \otimes V)))$$

where $U_\alpha \times (G_0 \times (F \otimes V))$ intersects $U_\beta \times (G_0 \times (F \otimes V))$ on the points $(x, g, f \otimes v) = (x, \Psi_{\alpha\beta}([g], f \otimes v)) = (x, [g_{\alpha\beta} g], A_{\alpha\beta}[g](f \otimes v))$ where $x \in U_\alpha \cap U_\beta$ and, once again, we are using lemma 2 for the description of the operators $\Psi_{\alpha\beta}$.

Taking into account the homeomorphism

$$U_\alpha \times G_0 \approx \bigsqcup_{[g] \in G_0} [g] U_\alpha$$

we can rewrite

$$([g]x, f \otimes v) = ([g g_{\alpha\beta}]x, A_{\alpha\beta}[g](f \otimes v))$$

.

Therefore, the projection

$$(U_\alpha \times G_0) \times (F \otimes V) \to U_\alpha \times G_0$$

extends to a well-defined and continuous projection

$$\xi \to M.$$

It is clear by the preceding formulas, that this projection will be G-equivariant, if G acts canonically over the fibers and in by left translations on G_0 under the quotient map $G \to G/H = G_0$. So, we have $\xi \in \mathrm{Vect}_G(M, \rho)$.

To end the proof, we make the remark that, by the theory of principal G_0-bundles, the construction of the space M is up to equivariant homeomorphism. This means that the inverse to (27) is well defined. ∎

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