ORTHOGONALITY-PRESERVING, C^*-CONFORMAL AND CONFORMAL MODULE MAPPINGS ON HILBERT C^*-MODULES

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ABSTRACT. We investigate orthonormality-preserving, C^*-conformal and conformal module mappings on full Hilbert C^*-modules to obtain their general structure. Orthogonality-preserving bounded module maps T act as a multiplication by an element λ of the center of the multiplier algebra of the C^*-algebra of coefficients combined with an isometric module operator as long as some polar decomposition conditions for the specific element λ are fulfilled inside that multiplier algebra. Generally, T always fulfils the equality $\langle T(x), T(y) \rangle = |\lambda|^2 \langle x, y \rangle$ for any elements x, y of the Hilbert C^*-module. At the contrary, C^*-conformal and conformal bounded module maps are shown to be only the positive real multiples of isometric module operators.

The set of all orthogonality-preserving bounded linear mappings on Hilbert spaces is fairly easy to describe, and it coincides with the set of all conformal linear mappings there: a linear map T between two Hilbert spaces H_1 and H_2 is orthogonality-preserving if and only if T is the scalar multiple of an isometry V with $V^*V = \mathrm{id}_{H_1}$. Furthermore, the set of all orthogonality-preserving mappings $\{\lambda \cdot V : \lambda \in \mathbb{C}, V^*V = \mathrm{id}_{H_1}\}$ corresponds to the set of all those maps which transfer tight frames of H_1 into tight frames of (norm-closed) subspaces $V(H_1)$ of H_2, cf. [10].

The latter fact transfers to the more general situation of standard tight frames of Hilbert C^*-modules in case the image submodule is an orthogonal summand of the target Hilbert C^*-module, cf. [7, Prop. 5.10]. Also, module isometries of Hilbert C^*-modules are always induced by module unitary operators between them, [13], [11, Prop. 2.3]. However, in case of a non-trivial center of the multiplier algebra of the C^*-algebra of coefficients the property of a bounded module map to be merely orthogonality-preserving might not infer the property of that map to be (C^*-)conformal or even isometric. So the goal of the present note is to derive the structure of arbitrary orthogonality-preserving, C^*-conformal or conformal bounded module mappings on Hilbert C^*-modules over (non-)unital C^*-algebras without any further assumption.

Partial solutions can be found in a publication by D. Ilišević and A. Turnšek for C^*-algebras A of coefficients which admit a faithful $*$-representation π on some Hilbert space H such that $K(H) \subseteq \pi(A) \subseteq B(H)$, cf. [11, Thm. 3.1]. Orthogonality-preserving mappings have been mentioned also in a paper by J. Chmieliński, D. Ilišević, M. S. Moslehian, Gh. Sadeghi, [6, Th. 2.2]. In two working drafts [14, 15] by Chi-Wai Leung, Chi-Keung

Ng and Ngai-Ching Wong found by a Google search in May 2009 we obtained further partial results on orthogonality-preserving linear mappings on Hilbert C^*-modules.

Orthogonality-preserving bounded linear mappings between C^*-algebras have been considered by J. Schweizer in his Habilitation Thesis in 1996, [19, Prop. 4.5-4.8]. His results are of interest in application to the linking C^*-algebras of Hilbert C^*-modules.

A bounded module map T on a Hilbert C^*-module $\mathcal{M}$ is said to be *orthogonality-preserving* if $\langle T(x), T(y) \rangle = 0$ in case $\langle x, y \rangle = 0$ for certain $x, y \in \mathcal{M}$. In particular, for two Hilbert C^*-modules $\mathcal{M}$, $\mathcal{N}$ over some C^*-algebra A a bounded module map $T : \mathcal{M} \to \mathcal{N}$ is orthogonality-preserving if and only if the validity of the inequality $\langle x, x \rangle \leq \langle x + ay, x + ay \rangle$ for some $x, y \in \mathcal{M}$ and any $a \in A$ forces the validity of the inequality $\langle T(x), T(x) \rangle \leq \langle T(x) + aT(y), T(x) + aT(y) \rangle$ for any $a \in A$, cf. [11, Cor. 2.2]. So the property of a bounded module map to be orthogonality preserving has a geometrical meaning considering pairwise orthogonal one-dimensional C^*-submodules and their orthogonality in a geometric sense.

Orthogonality of elements of Hilbert C^*-modules with respect to their C^*-valued inner products is different from the classical James-Birkhoff orthogonality defined with respect to the norm derived from the C^*-valued inner products, in general. Nevertheless, the results are similar in both situations, and the roots of both these problem fields coincide for the particular situation of Hilbert spaces. For results in this parallel direction the reader might consult publications by A. Koldobsky [12], by A. Turnšek [21], by J. Chmieliński [4, 5], and by A. Blanco and A. Turnšek [3], among others.

Further resorting to C^*-*conformal* or *conformal* mappings on Hilbert C^*-modules, i.e. bounded module maps preserving either a generalized C^*-valued angle $\langle x, y \rangle / \|x\| \|y\|$ for any x, y of the Hilbert C^*-module or its normed value, we consider a particular situation of orthogonality-preserving mappings. Surprisingly, both these sets of orthogonality-preserving and of $(C^*$-$)$conformal mappings are found to be different in case of a non-trivial center of the multiplier algebra of the underlying C^*-algebra of coefficients.

The content of the present paper is organized as follows: In the following section we investigate the general structure of orthogonality-preserving bounded module mappings on Hilbert C^*-modules. The results are formulated in Theorem 1.3 and Theorem 1.4. In the last section we characterize C^*-conformal and conformal bounded module mappings on Hilbert C^*-modules, see Theorem 2.1 and Theorem 2.3.

Since we rely only on the very basics of $*$-representation and duality theory of C^*-algebras and of Hilbert C^*-module theory, respectively, we refer the reader to the monographs by M. Takesaki [20] and by V. M. Manuilov and E. V. Troitsky [16], or to other relevant monographical publications for basic facts and methods of both these theories.

1. Orthogonality-preserving mappings

The set of all orthogonality-preserving bounded linear mappings on Hilbert spaces is fairly easy to describe. For a given Hilbert space H it consists of all scalar multiples of isometries V, where an isometry is a map $V : H \to H$ such that $V^*V = \mathrm{id}_H$. Any bounded linear orthogonality-preserving map T induces a bounded linear map $T^*T : H \to H$. For a non-zero element $x \in H$ set $T^*T(x) = \lambda_x x + z$ with $z \in \{x\}^\perp$ and $\lambda_x \in \mathbb{C}$. Then the

given relation $\langle x, z \rangle = 0$ induces the equality

$$0 = \langle T(x), T(z) \rangle = \langle T^*T(x), z \rangle = \langle \lambda_x x + z, z \rangle = \langle z, z \rangle.$$

Therefore, $z = 0$ by the non-degeneratedness of the inner product, and $\lambda_x \geq 0$ by the positivity of T^*T. Furthermore, for two orthogonal elements $x, y \in H$ one has the equality

$$\lambda_{x+y}(x + y) = T^*T(x + y) = \lambda_x x + \lambda_y y$$

which induces the equality $\lambda_{x+y}\langle x, x \rangle = \lambda_x \langle x, x \rangle$ after scalar multiplication by $x \in H$. Since the element $\langle x, x \rangle$ is invertible in $\mathbb{C}$ we can conclude that the orthogonality-preserving operator T induces an operator T^*T which acts as a positive scalar multiple $\lambda \cdot \mathrm{id}_H$ of the identity operator on any orthonormal basis of the Hilbert space H. So $T^*T = \lambda \cdot \mathrm{id}_H$ on the Hilbert space H by linear continuation. The polar decomposition of T inside the von Neumann algebra $B(H)$ of all bounded linear operators on H gives us the equality $T = \sqrt{\lambda}V$ for an isometry $V : H \to H$, i.e. with $V^*V = \mathrm{id}_H$. The positive number $\sqrt{\lambda}$ can be replaced by an arbitrary complex number of the same modulus multiplying by a unitary $u \in \mathbb{C}$. In this case the isometry V has to be replaced by the isometry u^*V to yield another decomposition of T in a more general form.

As a natural generalization of the described situation one may change the algebra of coefficients to arbitrary C^*-algebras A and the Hilbert spaces to C^*-valued inner product A-modules, the (pre-)Hilbert C^*-modules. Hilbert C^*-modules are an often used tool in the study of locally compact quantum groups and their representations, in noncommutative geometry, in KK-theory, and in the study of completely positive maps between C^*-algebras, among other research fields.

To be more precise, a (left) *pre-Hilbert C^*-module* over a (not necessarily unital) C^*-algebra A is a left A-module $\mathcal{M}$ equipped with an A-valued inner product $\langle \cdot, \cdot \rangle : \mathcal{M} \times \mathcal{M} \to A$, which is A-linear in the first variable and has the properties $\langle x, y \rangle = \langle y, x \rangle^*$, $\langle x, x \rangle \geq 0$ with equality if and only if $x = 0$. We always suppose that the linear structures of A and $\mathcal{M}$ are compatible. A pre-Hilbert A-module $\mathcal{M}$ is called a *Hilbert A-module* if $\mathcal{M}$ is a Banach space with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$.

Consider bounded module orthogonality-preserving maps T on Hilbert C^*-modules $\mathcal{M}$. For several reasons we cannot repeat the simple arguments given for Hilbert spaces in the situation of an arbitrary Hilbert C^*-module, in general. First of all, the bounded module operator T might not admit a bounded module operator T^* as its adjoint operator, i.e. satisfying the equality $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$ for any $x, y \in \mathcal{M}$. Secondly, orthogonal complements of subsets of a Hilbert C^*-module might not be orthogonal direct summands of it. Last but not least, Hilbert C^*-modules might not admit analogs (in a wide sense) of orthogonal bases. So the understanding of the nature of bounded module orthogonality-preserving operators on Hilbert C^*-modules involves both more global and other kinds of localisation arguments.

Example 1.1. Let A be the C^*-algebra of continuous functions on the unit interval $[0, 1]$ equipped with the usual Borel topology. Let $I = C_0((0, 1])$ be the C^*-subalgebra of all continuous functions on $[0, 1]$ vanishing at zero. I is a norm-closed two-sided ideal of A. Let $\mathcal{M}_1 = A \oplus A$ be the Hilbert A-module that consists of two copies of A, equipped with the standard A-valued inner product on it. Consider the multiplication T_1 of both parts

of $\mathcal{M}_1$ by the function $a(t) \in A$, $a(t) := t$ for any $t \in [0,1]$. Obviously, the map T_1 is bounded, A-linear, injective and orthogonality-preserving. However, its range is even not norm-closed in $\mathcal{M}_1$.

Let $\mathcal{M}_2 = I \oplus l_2(A)$ be the orthogonal direct sum of a proper ideal I of A and of the standard countably generated Hilbert A-module $l_2(A)$. Consider the shift operator $T_2 : \mathcal{M}_2 \to \mathcal{M}_2$ defined by the formula $T_2((i, a_1, a_2, ...)) = (0, i, a_1, a_2, ...)$ for $a_k \in A$, $i \in I$. It is an isometric A-linear embedding of $\mathcal{M}_2$ into itself and, hence, orthogonality-preserving, however T_2 is not adjointable.

To formulate the result on orthogonality-preserving mappings we need a construction by W. L. Paschke ([17]): for any Hilbert A-module $\mathcal{M}$ over any C^*-algebra A one can extend $\mathcal{M}$ canonically to a Hilbert A^{**}-module $\mathcal{M}^{\#}$ over the bidual Banach space and von Neumann algebra A^{**} of A [17, Th. 3.2, Prop. 3.8, §4]. For this aim the A^{**}-valued pre-inner product can be defined by the formula

$$[a \otimes x, b \otimes y] = a\langle x, y\rangle b^*,$$

for elementary tensors of $A^{**} \otimes M$, where $a, b \in A^{**}$, $x, y \in M$. The quotient module of $A^{**} \otimes M$ by the set of all isotropic vectors is denoted by $\mathcal{M}^{\#}$. It can be canonically completed to a self-dual Hilbert A^{**}-module $\mathcal{N}$ which is isometrically algebraically isomorphic to the A^{**}-dual A^{**}-module of $\mathcal{M}^{\#}$. $\mathcal{N}$ is a dual Banach space itself, (cf. [17, Thm. 3.2, Prop. 3.8, §4].) Every A-linear bounded map $T : \mathcal{M} \to \mathcal{M}$ can be continued to a unique A^{**}-linear map $T : \mathcal{M}^{\#} \to \mathcal{M}^{\#}$ preserving the operator norm and obeying the canonical embedding $\pi'(\mathcal{M})$ of $\mathcal{M}$ into $\mathcal{M}^{\#}$. Similarly, T can be further extended to the self-dual Hilbert A^{**}-module $\mathcal{N}$. The extension is such that the isometrically algebraically embedded copy $\pi'(\mathcal{M})$ of $\mathcal{M}$ in $\mathcal{N}$ is a w*-dense A-submodule of $\mathcal{N}$, and that A-valued inner product values of elements of $\mathcal{M}$ embedded in $\mathcal{N}$ are preserved with respect to the A^{**}-valued inner product on $\mathcal{N}$ and to the canonical isometric embedding π of A into its bidual Banach space A^{**}. Any bounded A-linear operator T on $\mathcal{M}$ extends to a unique bounded A^{**}-linear operator on $\mathcal{N}$ preserving the operator norm, cf. [17, Prop. 3.6, Cor. 3.7, §4]. The extension of bounded A-linear operators from $\mathcal{M}$ to $\mathcal{N}$ is continuous with respect to the w*-topology on $\mathcal{N}$.

Example 1.2. Let A be the C^*-algebra of all continuous functions on the unit interval, i.e. $A = C([0,1])$. In case we consider A as a Hilbert C^*-module over itself and an orthogonality-preserving map T_0 defined by the multiplication by the function $a(t) = t \cdot (\sin(1/t) + \mathbf{i}\cos(1/t))$ we obtain that the operator T_0 cannot be written as the combination of a multiplication by a positive element of A and of an isometric module operator U_0 on $\mathcal{M} = A$. The reason for this phenomenon is the lack of a polar decomposition of $a(t)$ inside A. Only a lift to the bidual von Neumann algebra A^{**} of A restores the simple description of the continued operator T_0 as the combination of a multiplication by a positive element (of the center) of A and an isometric module operator on $\mathcal{M}^{\#} = \mathcal{N} = A^{**}$. The unitary part of $a(t)$ is a so-called local multiplier of $C([0,1])$, i.e. a multiplier of $C_0((0,1))$. But it is not a multiplier of $C([0,1])$ itself. We shall show that this example is a very canonical one.

We are going to demonstrate the following fact on the nature of orthogonality-preserving bounded module mappings on Hilbert C^*-modules. Without loss of generality, one may

assume that the range of the A-valued inner product on $\mathcal{M}$ in A is norm-dense in A. Such Hilbert C^*-modules are called *full* Hilbert C^*-modules. Otherwise A has to be replaced by the range of the A-valued inner product which is always a two-sided norm-closed $*$-ideal of A. The sets of all adjointable bounded module operators and of all bounded module operators on $\mathcal{M}$, resp., are invariant with respect to such changes of sets of coefficients of Hilbert C^*-modules, cf. [17].

Theorem 1.3. *Let A be a C^*-algebra, $\mathcal{M}$ be a full Hilbert A-module and $\mathcal{M}^{\#}$ be its canonical A^{**}-extension. Any orthogonality-preserving bounded A-linear operator T on $\mathcal{M}$ is of the form $T = \lambda V$, where $V : \mathcal{M}^{\#} \to \mathcal{M}^{\#}$ is an isometric A-linear embedding and λ is a positive element of the centre $Z(M(A))$ of the multiplier algebra $M(A)$ of A. If any element $\lambda' \in Z(M(A))$ with $|\lambda'| = \lambda$ admits a polar decomposition inside $Z(M(A))$ then the operator V preserves $\pi'(\mathcal{M}) \subset \mathcal{M}^{\#}$. So $T = \lambda \cdot V$ on $\mathcal{M}$.*

In [11, Thm. 3.1] D. Ilišević and A. Turnšek proved Theorem 1.3 for the particular case if for some Hilbert space H the C^*-algebra A admits an isometric representation π on H with the property $K(H) \subset \pi(A) \subset B(H)$. In this situation $Z(M(A)) = \mathbb{C}$.

Proof. We want to make use of the canonical nondegenerate isometric $*$-representation π of a C^*-algebra A in its bidual Banach space and von Neumann algebra A^{**} of A, as well as of its extension $\pi' : \mathcal{M} \to \mathcal{M}^{\#} \to \mathcal{N}$ and of its operator extension. That is, we switch from the triple $\{A, \mathcal{M}, T\}$ to the triple $\{A^{**}, \mathcal{M}^{\#} \subseteq \mathcal{N}, T\}$.

We have to demonstrate that for orthogonality-preserving bounded A-linear mappings T on $\mathcal{M}$ the respective extended bounded A^{**}-linear operator on $\mathcal{N}$ is still orthogonality-preserving for $\mathcal{N}$. Consider two non-zero elements $x, y \in \mathcal{N}$ such that $\langle x, y \rangle = 0$. The bi-orthogonal complements of the sets $\{x\}$ and $\{y\}$ are pairwise orthogonal Hilbert A^{**}-submodules of $\mathcal{N}$ which are orthogonal direct summands of $\mathcal{N}$ because of their self-duality, cf. [17]. By the construction of $\mathcal{N}$ from $\mathcal{M}$, the sets $\{x\}^{\perp\perp} \cap \mathcal{M}$ and $\{y\}^{\perp\perp} \cap \mathcal{M}$ are w*-dense in the sets $\{x\}^{\perp\perp}$ and $\{y\}^{\perp\perp}$, respectively. Since the extension of T is continuous with respect to the w*-topology on $\mathcal{N}$ and since $\langle T(z), T(s) \rangle = 0$ for any $z \in \{x\}^{\perp\perp} \cap \mathcal{M}$, $s \in \{y\}^{\perp\perp} \cap \mathcal{M}$ we conclude $\langle T(x), T(y) \rangle = 0$.

Next, we want to consider only *discrete* W*-algebras, i.e. W*-algebras for which the supremum of all minimal projections contained in them equals their identity. (We prefer to use the word discrete instead of atomic.) To connect to the general C^*-case we make use of a theorem by Ch. A. Akemann stating that the $*$-homomorphism of a C^*-algebra A into the discrete part of its bidual von Neumann algebra A^{**} which arises as the composition of the canonical embedding π of A into A^{**} followed by the projection ρ to the discrete part of A^{**} is an injective $*$-homomorphism, [1, p. 278] and [2, p. I]. The injective $*$-homomorphism ρ is partially implemented by a central projection $p \in Z(A^{**})$ in such a way that A^{**} multiplied by p gives the discrete part of A^{**}. Applying this approach to our situation we reduce the problem further by investigating the triple $\{pA^{**}, p\mathcal{N}, pT\}$ instead of the triple $\{A^{**}, \mathcal{N}, T\}$, where we rely on the injectivity of the algebraic embeddings $\rho \circ \pi : A \to pA^{**}$ and $\rho' \circ \pi' : \mathcal{M} \to p\mathcal{N}$. The latter map is injective since $\langle x, x \rangle \neq 0$ forces $\langle px, px \rangle = p\langle x, x \rangle = \rho \circ \pi(\langle x, x \rangle) \neq 0$. Obviously, the bounded pA^{**}-linear operator pT is orthogonality-preserving for the self-dual Hilbert pA^{**}-module $p\mathcal{N}$ because the orthogonal

projection of $\mathcal{N}$ onto $p\mathcal{N}$ and the operator T commute, and both they are orthogonality-preserving.

In the sequel we have to consider the multiplier algebra $M(A)$ and the left multiplier algebra $LM(A)$ of the C^*-algebra A. By [18] every non-degenerate injective $*$-representation of A in a von Neumann algebra B extends to an injective $*$-representation of the multiplier algebra $M(A)$ in B and to an isometric algebraic representation of the left multiplier algebra $LM(A)$ of A preserving the strict and the left strict topologies on $M(A)$ and on $LM(A)$, respectively. In particular, the injective $*$-representation $\rho \circ \phi$ extends to $M(A)$ and to $LM(A)$ in such a way that

$$
\begin{aligned}
\rho \circ \phi(M(A)) &= \{b \in pA^{**} : b\rho \circ \phi(a) \in A, \ \rho \circ \phi(a)b \in A \text{ for every } a \in A\}, \\
\rho \circ \phi(LM(A)) &= \{b \in pA^{**} : b\rho \circ \phi(a) \in A \text{ for every } a \in A\}.
\end{aligned}
$$

Since $Z(LM(A)) = Z(M(A))$ for the multiplier algebra of A of every C^*-algebra A, we have the description

$$
\rho \circ \phi(Z(M(A))) = \{b \in pA^{**} : b\rho \circ \phi(a) = \rho \circ \phi(a)b \in A \text{ for every } a \in A\}.
$$

Since the von Neumann algebra pA^{**} is discrete the identity p can be represented as the w*-sum of a maximal set of pairwise orthogonal atomic projections $\{q_\alpha : \alpha \in I\}$ of the centre $Z(pA^{**})$ of pA^{**}. Note, that $\sum_{\alpha \in I} q_\alpha = p$. Select a single atomic projection $q_\alpha \in Z(pA^{**})$ of this collection and consider the part $\{q_\alpha pA^{**}, q_\alpha p\mathcal{N}, q_\alpha pT\}$ of the problem for every single $\alpha \in I$.

By [11, Thm. 3.1] the operator $q_\alpha pT$ can be described as a non-negative constant λ_{q_α} multiplied by an isometry V_{q_α} on the Hilbert $q_\alpha pA^{**}$-module $q_\alpha p\mathcal{N}$, where the isometry V_{q_α} preserves the $q_\alpha pA$-submodule $q_\alpha p\mathcal{M}$ inside $q_\alpha p\mathcal{N}$ since the operator $q_\alpha pT$ preserves it, and multiplication by a positive number does not change this fact. In case $\lambda_{q_\alpha} = 0$ we set simply $V_{q_\alpha} = 0$.

We have to show the existence of global operators on the Hilbert pA^{**}-module $p\mathcal{N}$ build as w*-limits of nets of finite sums with pairwise distinct summands of the sets $\{\lambda_{q_\alpha} q_\alpha : \alpha \in I\}$ and $\{q_\alpha V_{q_\alpha} : \alpha \in I\}$, respectively. Additionally, we have to establish key properties of them. First, note that the collection of all finite sums with pairwise distinct summands of $\{\lambda_{q_\alpha} q_\alpha : \alpha \in I\}$ form an increasingly directed net of positive elements of the centre of the operator algebra $\mathrm{End}_{pA^{**}}(p\mathcal{N})$, which is $*$-isomorphic to the von Neumann algebra $Z(pA^{**})$. This net is bounded by $\|pT\| \cdot \mathrm{id}_{p\mathcal{N}}$ since the operator pT admits an adjoint operator on the self-dual Hilbert pA^{**}-module $p\mathcal{N}$ by [17, Prop. 3.4] and since for any finite subset I_0 of I the inequality

$$
0 \leq \sum_{\alpha \in I_0} \lambda_{q_\alpha}^2 \cdot \mathrm{id}_{q_\alpha p\mathcal{N}} = \sum_{\alpha \in I_0} q_\alpha pT^*T \leq pT^*T \leq \|pT\|^2 \cdot \mathrm{id}_{p\mathcal{N}}
$$

holds in the operator algebra $\mathrm{End}_{pA^{**}}(p\mathcal{N})$, the centre of which is $*$-isomorphic to $Z(pA^{**})$. Therefore, the supremum of this increasingly directed bounded net of positive elements exists as an element of the centre of the operator algebra $\mathrm{End}_{pA^{**}}(p\mathcal{N})$, which is $*$-isomorphic to the von Neumann algebra $Z(pA^{**})$. We denote the supremum of this net by λ_p. By construction and by the w*-continuity of transfers to suprema of increasingly directed

bounded nets of self-adjoint elements of von Neumann algebras we have the equality

$$\lambda_p = \mathrm{w}^* - \lim_{I_0 \subseteq I} \sum_{\alpha \in I_0} \lambda_{q_\alpha} \cdot q_\alpha \in Z(pA^{**}) \equiv Z(\mathrm{End}_{pA^{**}}(p\mathcal{N}))$$

where I_0 runs over the partially ordered net of all finite subsets of I. Since $\langle q_\alpha p T^* T(z), z \rangle = \lambda_{q_\alpha}^2 q_\alpha \langle z, z \rangle$ for any $z \in q_\alpha \mathcal{N}$ and for any $\alpha \in I$, we arrive at the equality

$$\langle p T^* T(z), z \rangle = \lambda_p^2 \cdot p \langle z, z \rangle$$

for any $z \in p\mathcal{N}$ and for the constructed positive $\lambda_p \in Z(pA^{**}) \equiv Z(\mathrm{End}_{pA^{**}}(p\mathcal{N}))$. Consequently, the operator pT can be written as $pT = \lambda_p V_p$ for some isometric pA^{**}-linear map $V_p \in \mathrm{End}_{pA^{**}}(p\mathcal{N})$, cf. [11, Proposition 2.3].

Consider the operator pT on $p\mathcal{N}$. Since the formula

(1) $$\langle pT(x), pT(x) \rangle = \lambda_p^2 \langle x, x \rangle \in \rho \circ \pi(A)$$

holds for any $x \in \rho' \circ \pi'(\mathcal{M}) \subseteq p\mathcal{N}$ and since the range of the A-valued inner product on $\mathcal{M}$ is supposed to be the entire C^*-algebra A, the right side of this equality and the multiplier theory of C^*-algebras forces $\lambda_p^2 \in LM(pA) \cap Z(pA^{**}) = Z(M(\rho \circ \pi(A))) = \rho \circ \pi(Z(M(A)))$, [18]. Taking the square root of λ_p^2 in a C^*-algebraical sense is an operation which results in a (unique) positive element of the C^*-algebra itself. So we arrive at $\lambda_p \in \rho \circ \pi(Z(M(A)))$ as the square root of $\lambda_p^2 \geq 0$. In particular, the operator $\lambda_p \cdot \mathrm{id}_{p\mathcal{N}}$ preserves the $\rho \circ \pi(A)$-submodule $\rho' \circ \pi'(\mathcal{M})$.

As a consequence, we can lift the bounded pA^{**}-linear orthogonality-preserving operator pT on $p\mathcal{N}$ back to $\mathcal{M}^\#$ since A^{**} allows polar decomposition for any element, the embedding $\rho \circ \pi : A \to pA^{**}$ and the module and operator mappings, induced by $\rho \circ \pi$ and by Paschke's embedding were isometrically and algebraically, just by multiplying with or, resp., acting by p in the second step. So we obtain a decomposition $T = \lambda V$ of $T \in \mathrm{End}_A(\mathcal{M})$ with a positive function $\lambda \in Z(M(A)) \equiv Z(\mathrm{End}_A(\mathcal{M}))$ derived from λ_p, and with an isometric A-linear embedding $V \in \mathrm{End}_A(\mathcal{M}^\#)$, V derived from V_p.

In case any element $\lambda' \in Z(M(A)))$ with $|\lambda'| = \lambda$ admits a polar decomposition inside $Z(M(A)))$ then the operator V preserves $\pi'(\mathcal{M}) \subset \mathcal{M}^\#$. So $T = \lambda \cdot V$ on $\mathcal{M}$.

For completeness just note, that the adjointability of V goes lost on this last step of the proof in case T has not been adjointable on $\mathcal{M}$ in the very beginning. $\square$

Theorem 1.4. *Let A be a C^*-algebra and $\mathcal{M}$ be a Hilbert A-module. Any orthogonality-preserving bounded A-linear operator T on $\mathcal{M}$ fulfils the equality*

$$\langle T(x), T(y) \rangle = \kappa \langle x, y \rangle$$

for a certain T-specific positive element $\kappa \in Z(M(A))$ and for any $x, y \in \mathcal{M}$.

Proof. We have only to remark that the values of the A-valued inner product on $\mathcal{M}$ do not change if $\mathcal{M}$ is canonically embedded into $\mathcal{M}^\#$ or $\mathcal{N}$. Then the obtained formula works in the bidual situation, cf. (1). $\square$

Problem 1.5. We conjecture that any orthogonality-preserving map T on Hilbert A-modules $\mathcal{M}$ over C^*-algebras A are of the form $T = \lambda V$ for some element $\lambda \in Z((M(A))$ and some A-linear isometry $V : \mathcal{M} \to \mathcal{M}$. To solve this problem one has possibly to

solve the problem of general polar decomposition of arbitrary elements of (commutative) C^*-algebras inside corresponding local multiplier algebras or in similarly derived algebras.

Corollary 1.6. *Let A be a C^*-algebra and $\mathcal{M}$ be a Hilbert A-module. Let T be an orthogonality-preserving bounded A-linear operator on $\mathcal{M}$ of the form $T = \lambda V$, where $V : \mathcal{M} \to \mathcal{M}$ is an isometric adjointable bounded A-linear embedding and λ is an element of the centre $Z(M(A))$ of the multiplier algebra $M(A)$ of A. Then the following conditions are equivalent:*

(i) T is adjointable.
(ii) V is adjointable.
(iii) The graph of the isometric embedding V is a direct orthogonal summand of the Hilbert A-module $\mathcal{M} \oplus \mathcal{M}$.
(iv) The range $\mathrm{Im}(V)$ of V is a direct orthogonal summand of $\mathcal{M}$.

Proof. Note, that a multiplication operator by an element $\lambda \in Z(M(A))$ is always adjointable. So, if T is supposed to be adjointable, then the operator V has to be adjointable, and vice versa. By [8, Cor. 3.2] the bounded operator V is adjointable if and only if its graph is a direct orthogonal summand of the Hilbert A-module $\mathcal{M} \oplus \mathcal{M}$. Moreover, since the range of the isometric A-linear embedding V is always closed, adjointability of V forces V to admit a bounded A-linear generalized inverse operator on $\mathcal{M}$, cf. [9, Prop. 3.5]. The kernel of this inverse to V mapping serves as the orthogonal complement of $\mathrm{Im}(V)$, and $\mathcal{M} = \mathrm{Im}(V) \oplus \mathrm{Im}(V)^\perp$ as an orthogonal direct sum by [9, Th. 3.1]. Conversely, if the range $\mathrm{Im}(V)$ of V is a direct orthogonal summand of $\mathcal{M}$, then there exists an orthogonal projection of $\mathcal{M}$ onto this range and, therefore, V is adjointable. $\square$

2. C^*-CONFORMAL AND CONFORMAL MAPPINGS

We want to describe generalized C^*-conformal mappings on Hilbert C^*-modules. A full characterization of such maps involves isometries as for the orthogonality-preserving case since we resort to a particular case of the latter.

Let $\mathcal{M}$ be a Hilbert module over a C^*-algebra A. An injective bounded module map T on $\mathcal{M}$ is said to be C^*-*conformal* if the identity

$$\text{(2)} \qquad \frac{\langle Tx, Ty \rangle}{\|Tx\|\|Ty\|} = \frac{\langle x, y \rangle}{\|x\|\|y\|}$$

holds for all non-zero vectors $x, y \in \mathcal{M}$. It is said to be *conformal* if the identity

$$\text{(3)} \qquad \frac{\|\langle Tx, Ty \rangle\|}{\|Tx\|\|Ty\|} = \frac{\|\langle x, y \rangle\|}{\|x\|\|y\|}$$

holds for all non-zero vectors $x, y \in \mathcal{M}$.

Theorem 2.1. *Let $\mathcal{M}$ be a Hilbert A-module over a C^*-algebra A and T be an injective bounded module map. The following conditions are equivalent:*

(i) T is C^*-conformal;
(ii) $T = \lambda U$ for some non-zero positive $\lambda \in \mathbb{R}$ and for some isometrical module operator U on $\mathcal{M}$.

Proof. The condition (ii) implies condition (i) because the condition $\|Ux\| = \|x\|$ for all $x \in M$ implies the condition $\langle Ux, Uy \rangle = \langle x, y \rangle$ for all $x, y \in M$ by [11, Proposition 2.3]. So we have only to verify the implication (i)$\rightarrow$(ii).

Assume an injective bounded module map T on $\mathcal{M}$ to be C^*-conformal. We can rewrite (2) in the following equivalent form:

$$(4) \qquad \langle Tx, Ty \rangle = \langle x, y \rangle \frac{\|Tx\|\|Ty\|}{\|x\|\|y\|}, \quad x, y \neq 0.$$

Consider the left part of this equality as a new A-valued inner product on $\mathcal{M}$. Consequently, the right part of (4) has to satisfy all the conditions of a C^*-valued inner product, too. In particular, the right part of (4) has to be additive in the second variable, what exactly means

$$\langle x, y_1 + y_2 \rangle \frac{\|Tx\|\|T(y_1 + y_2)\|}{\|x\|\|y_1 + y_2\|} = \langle x, y_1 \rangle \frac{\|Tx\|\|Ty_1\|}{\|x\|\|y_1\|} + \langle x, y_2 \rangle \frac{\|Tx\|\|Ty_2\|}{\|x\|\|y_2\|}$$

for all non-zero $x, y_1, y_2 \in M$. Therefore,

$$(y_1 + y_2) \frac{\|T(y_1 + y_2)\|}{\|y_1 + y_2\|} = y_1 \frac{\|Ty_1\|}{\|y_1\|} + y_2 \frac{\|Ty_2\|}{\|y_2\|},$$

by the arbitrarity of $x \in \mathcal{M}$, which can be rewritten as

$$(5) \qquad y_1 \left(\frac{\|T(y_1 + y_2)\|}{\|y_1 + y_2\|} - \frac{\|Ty_1\|}{\|y_1\|} \right) + y_2 \left(\frac{\|T(y_1 + y_2)\|}{\|y_1 + y_2\|} - \frac{\|Ty_2\|}{\|y_2\|} \right) = 0$$

for all non-zero $y_1, y_2 \in M$. In case the elements y_1 and y_2 are not complex multiples of each other both the complex numbers inside the brackets have to equal to zero. So we arrive at

$$(6) \qquad \frac{\|T(y_1)\|}{\|y_1\|} = \frac{\|T(y_2)\|}{\|y_2\|}$$

for any $y_1, y_2 \in \mathcal{M}$ which are not complex multiples of one another. Now, if the elements would be non-trivial complex multiples of each other both the coefficients would have to be equal, what again forces equality (6).

Let us denote the positive real number $\frac{\|Tx\|}{\|x\|}$ by t. Then the equality (6) provides

$$\left\| \left(\frac{1}{t} T \right)(z) \right\| = \|z\|,$$

which means $U = \frac{1}{t}T$ is an isometrical operator. The proof is complete. $\qquad \square$

Example 2.2. Let $A = C_0((0,1]) = \mathcal{M}$ and T be a C^*-conformal mapping on $\mathcal{M}$. Our aim is to demonstrate that $T = tU$ for some non-zero positive $t \in \mathbb{R}$ and for some isometrical module operator U on $\mathcal{M}$. To begin with, let us recall that the Banach algebra $\mathrm{End}_A(M)$ of all bounded module maps on $\mathcal{M}$ is isomorphic to the algebra $LM(A)$ of left multipliers of A under the given circumstances. Moreover, $LM(A) = C_b((0,1])$, the C^*-algebra of all bounded continuous functions on $(0,1]$. So any A-linear bounded operator on $\mathcal{M}$ is just a multiplication by a certain function of $C_b((0,1])$. In particular,

$$T(g) = f_T \cdot g, \quad g \in A,$$

for some $f_T \in C_b((0,1])$. Let us denote by x_0 the point of $(0,1]$, where the function $|f_T|$ achieves its supremum, i.e. $|f_T(x_0)| = \|f_T\|$, and set $t := \|f_T\|$. We claim that the operator $\frac{1}{t}T$ is an isometry, what exactly means

(7)
$$\frac{|f_T(x)|}{\|f_T\|} = 1$$

for all $x \in (0,1]$. Indeed, consider any point $x \neq x_0$. Let $\theta_x \in C_0((0,1])$ be an Urysohn function for x, i.e. $0 \leq \theta_x \leq 1$, $\theta_x(x) = 1$ and $\theta_x = 0$ outside of some neighborhood of x, and let θ_{x_0} be a Urysohn function for x_0. Moreover, we can assume that the supports of θ_x and θ_{x_0} do not intersect each other. Now the condition (2) written for T and for coinciding vectors $x = y = \theta_x + \theta_{x_0}$ yields the equality

$$\frac{|f_T|^2(\theta_x + \theta_{x_0})^2}{\|f_T(\theta_x + \theta_{x_0})\|^2} = \frac{(\theta_x + \theta_{x_0})^2}{\|\theta_x + \theta_{x_0}\|^2},$$

which implies

$$\frac{|f_T|^2(\theta_x + \theta_{x_0})^2}{\|f_T\|^2} = (\theta_x + \theta_{x_0})^2.$$

This equality at point x takes the form (7) for any $x \in (0,1]$.

Theorem 2.3. *Let $\mathcal{M}$ be a Hilbert A-module over a C^*-algebra A and T be an injective bounded module map. The following conditions are equivalent:*

  (i) T is conformal;
  (ii) $T = \lambda U$ for some non-zero positive $\lambda \in \mathbb{R}$ and for some isometrical module operator U on $\mathcal{M}$.

Proof. As in the proof of the theorem on orthogonality-preserving mappings we switch from the setting $\{A, \mathcal{M}, T\}$ to its faithful isometric representation in $\{pA^{**}, p\mathcal{M}^{\#} \subseteq p\mathcal{N}, T\}$, where $p \in A^{**}$ is the central projection of A^{**} mapping A^{**} to its discrete part.

First, consider a minimal projections $e \in pA^{**}$. Then the equality (3) gives

$$\frac{\|\langle ex, ey\rangle\|}{\|ex\|\|ey\|} = \frac{\|\langle T(ex), T(ey)\rangle\|}{\|T(ex)\|\|T(ey)\|}$$

for any $x, y \in p\mathcal{M}^{\#}$. Since $\{e\mathcal{M}^{\#}, \langle .,.\rangle\}$ becomes a Hilbert space after factorization by the set $\{x \in p\mathcal{M}^{\#} : e\langle x, x\rangle e = 0\}$, the map T acts as a positive scalar multiple of a linear isometry on $e\mathcal{M}^{\#}$, i.e. $eT = \lambda_e U_e$.

Secondly, every two minimal projections $e, f \in pA^{**}$ with the same minimal central support projection $q \in pZ(A^{**})$ are connected by a (unique) partial isometry $u \in pA^{**}$ such that $u^*u = f$ and $uu^* = e$. Arguments analogous to those given at [11, p. 303] show

$$
\begin{aligned}
\lambda_e^2 \cdot e\langle x, x\rangle e &= ufu^*\langle T(x), T(x)\rangle ufu^* \\
&= uf\langle T(u^*x)t(u^*x)\rangle fu^* \\
&= u\lambda_f^2 f\langle u^*x, u^*x\rangle f \\
&= \lambda_f^2 \cdot e\langle x, x\rangle e.
\end{aligned}
$$

Therefore, $qT = \lambda_q U$ for some positive $\lambda_q \in \mathbb{R}$, for a qA-linear isometric mapping $U : q\mathcal{M}^{\#} \to q\mathcal{M}^{\#}$ and for any minimal central projection $q \in pA^{**}$.

Thirdly, suppose e, f are two minimal central projections of pA^{**} that are orthogonal. For any $x, y \in p\mathcal{M}^{\#}$ consider the supposed equality

$$\frac{\|\langle (e+f)x, (e+f)y \rangle\|}{\|(e+f)x\|\|(e+f)y\|} = \frac{\|\langle T((e+f)x), T((e+f)y) \rangle\|}{\|T((e+f)x)\|\|T((e+f)y)\|}.$$

Since T is a bounded module mapping which acts on $ep\mathcal{M}^{\#}$ like $\lambda_e \cdot \mathrm{id}$ and on $fp\mathcal{M}^{\#}$ like $\lambda_f \cdot \mathrm{id}$ we arrive at the equality

$$\frac{\|\langle (e+f)x, (e+f)y \rangle\|}{\|(e+f)x\|\|(e+f)y\|} = \frac{\|\langle (\lambda_e e + \lambda_f f)x, (\lambda_e e + \lambda_f f)y \rangle\|}{\|(\lambda_e e + \lambda_f f)x\|\|(\lambda_e e + \lambda_f f)y\|}.$$

Involving the properties of e, f to be central and orthogonal to each other and exploiting modular linear properties of the pA^{**}-valued inner product we transform the equality further to

$$\frac{\|\langle ex, ey \rangle + \langle fx, fy \rangle\|}{\|\langle ex, ex \rangle + \langle fx, fx \rangle\|^{1/2} \cdot \|\langle ey, ey \rangle + \langle fy, fy \rangle\|^{1/2}} =$$
$$= \frac{\|\lambda_e^2 \langle ex, ey \rangle + \lambda_f^2 \langle fx, fy \rangle\|}{\|\lambda_e^2 \langle ex, ex \rangle + \lambda_f^2 \langle fx, fx \rangle\|^{1/2} \cdot \|\lambda_e^2 \langle ey, ey \rangle + \lambda_f^2 \langle fy, fy \rangle\|^{1/2}}.$$

Since e, f are pairwise orthogonal central projections we can transform the equality further to

$$\frac{\sup\{\|\langle ex, ey \rangle\|, \|\langle fx, fy \rangle\|\}}{\sup\{\|\langle ex, ex \rangle\|^{1/2}, \|\langle fx, fx \rangle\|^{1/2}\} \cdot \sup\{\|\langle ey, ey \rangle\|^{1/2}, \|\langle fy, fy \rangle\|^{1/2}\}} =$$
$$= \frac{\sup\{\|\lambda_e^2 \langle ex, ey \rangle\|, \|\lambda_f^2 \langle fx, fy \rangle\|\}}{\sup\{\|\lambda_e^2 \langle ex, ex \rangle\|^{1/2}, \|\lambda_f^2 \langle fx, fx \rangle\|^{1/2}\} \cdot \sup\{\|\lambda_e^2 \langle ey, ey \rangle\|^{1/2}, \|\lambda_f^2 \langle fy, fy \rangle\|^{1/2}\}}.$$

By the w*-density of $p\mathcal{M}^{\#}$ in $p\mathcal{N}$ we can distinguish a finite number of cases at which of the central parts the respective six suprema may be admitted, at epA^{**} or at fpA^{**}. For this aim we may assume, in particular, that x, y belong to $p\mathcal{N}$ to have a larger set for these elements to be selected specifically. Most interesting are the cases when (i) both $(e+f)x$ and $(\lambda_e e + \lambda_f f)x$ admit their norm at the e-part, (ii) both $(e+f)y$ and $(\lambda_e e + \lambda_f f)y$ admit their norm at the f-part, and (iii) both $\langle (e+f)x, (e+f)y \rangle$ and $\langle (\lambda_e e + \lambda_f f)x, (\lambda_e e + \lambda_f f)y \rangle$ admit their norm (either) at the e-part (or at the f-part). In these cases the equality above gives $\lambda_e = \lambda_f$. (All the other cases either give the same result or do not give any new information on the interrelation of λ_e and λ_f.)

Finally, if for any central minimal projection $f \in pA^{**}$ the operator T acts on $fp\mathcal{N}$ as λU for a certain (fixed) positive constant λ and a certain module-linear isometry U then T acts on $p\mathcal{N}$ in the same way. Consequently, T acts on $\mathcal{M}$ in the same manner since U preserves $p\mathcal{M}^{\#}$ inside $p\mathcal{N}$. □

Remark 2.4. Obviously, the C^*-conformity of a bounded module map follows from the conformity of it, but the converse is not obvious, even it is true for Hilbert C^*-modules.

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