# On Compact and Fredholm Operators over C*-algebras and a New Topology in the Space of Compact Operators *†

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## Abstract

It is well-known that bounded operators in Hilbert $C^*$-modules over $C^*$-algebras may not be adjointable and the same is true for compact operators. So, there are two analogs for classical compact operators in Hilbert C*-modules: adjointable compact operators and *all* compact operators, i.e. not necessarily having an adjoint ones.

Classical Fredholm operators are those that are invertible modulo compact operators. When the notion of a Fredholm operator in a Hilbert $C^*$-module was developed in [6], the first analog was used: Fredholm operators were defined as operators that are invertible modulo adjointable compact operators.

In this paper we use the second analog and develop a more general version of Fredholm operators over $C^*$-algebras. Such operators are defined as bounded operators that are invertible modulo the ideal of *all* compact operators. The main property of this new class is that a Fredholm operator still has a decomposition into a direct sum of an isomorphism and a finitely generated operator.

The special case of Fredholm operators (in sense of [6]) over the commutative $C^*$-algebra $C(K)$ of continuous functions on a compact topological space $K$ was also considered in [2]. In order to describe general Fredholm operators (invertible modulo *all* compact operators over $C(K)$) we construct a new IM-topology on the space of compact operators on a Hilbert space such that continuous families of compact operators generate the ideal of *all* compact operators over $C^*$-algebra.

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# 1 Introduction

In the paper [2] M. Atiyah and G. Segal have considered families of Fredholm operators parameterized by points of a compact space $K$ which are continuous in a topology weaker than the uniform topology, i.e. the norm topology in the space of bounded operators $B(H)$ in a Banach space $H$.

Therefore, it is interesting to ascertain whether the conditions, characterized families of Fredholm operators, from the paper [2] precisely describe the families of Fredholm operators which forms a Fredholm operator over the $C^*$–algebra $\mathcal{A} = C(K)$ of all continuous functions on $K$.

It is well-known that bounded operators in Hilbert $C^*$-modules over $C^*$-algebras may not be adjointable and the same is true for compact operators. So, there are two analogs for classical compact operators in Hilbert C*-modules: adjointable compact operators and *all* compact operators, i.e. not necessarily having an adjoint ones.

Classical Fredholm operators are those that are invertible modulo compact operators. When the notion of a Fredholm operator in a Hilbert $C^*$-module was developed in [6], the first analog was used: Fredholm operators were defined as bounded operators that are invertible modulo adjointable compact operators.

Compact adjointable operators form an ideal only in the algebra of adjointable bounded operators but may not form an ideal in the algebra of *all* bounded operators.

Since the main class of operators considered in the paper [6] is the class of pseudodifferential operators, for which the adjoint operator automatically is a bounded one, existence of the adjoint operator was not the actual question for the main goals of that paper.

Here we follow the paper by M.Atiyah and G.Segal ([2]), where arguments are given that show necessity to consider bounded operators over an algebra of continuous functions, which may not be adjointable.

Therefore in this paper we use the second analog of the notion of a compact operator and develop a more general version of Fredholm operators over $C^*$-algebras. Such operators are defined as bounded operators that are invertible modulo the ideal of *all* compact operators. The main property of this new class is that a Fredholm operator still has a (Noether) decomposition into a direct sum of an isomorphism and a finitely generated operator.

Such a decomposition allows to get a well-defined homotopy invariant of the index type of a Fredholm operator, i.e. allows to prove that the index of a Fredholm operator as an element of corresponding $K$–group does not depend of the choice of a (Noether) docomposition.

The special case of Fredholm operators (in sense of [6]) over the commutative $C^*$-algebra $C(K)$ of continuous functions on a compact topological space $K$ was also considered in [2] since a continuous family of compact operators with respect to uniform topology generates an adjointable compact operator over $C(K)$.

Earlier in the paper ([4]) a topology in the space of the Fredholm operators was constructed such that the maps from the space $K$ that are continuous in this topology generate Fredholm operators over the algebra $C(K)$, but description

of general compact operators over $C(K)$ was unknown. Therefore in addition to ([4]) in order to describe general Fredholm operators (invertible modulo *all* compact operators over $C(K)$) we construct a new IM-topology on the space of compact operators on a Hilbert space such that continuous families of compact operators generate the ideal of *all* compact operators over $C^*$-algebra.

## 2  A Notion of Compact Operator over $C^*$–algebra

Let $A$ be a unital $C^*$–algebra. We shall consider so called Hilbert $C^*$–modules over the algebra $A$. The simplest Hilbert modules are the free finitely generated $A$–modules

$$A^n = \underbrace{A \oplus A \oplus \cdots \oplus A}_{n \text{ times}}$$

and the $A$-module $l_2(A) = A^\omega$. All such modules have a convenient description. Any element $x$ of a module $A^\alpha$, $\alpha \in [1..\omega]$ is a sequence $x = \{x_1, x_2, \ldots, x_k, \ldots\}$, $x_k \in A$, $1 \le k < 1 + \alpha$, such that the sum

$$\langle x, x \rangle = \sum_{k=1}^{\alpha} x_k x_k^* \in A \tag{1}$$

converges in the algebra $A$. It is clear that if $\alpha < \omega$ then the sum (1) automatically converges. The elements $e^k \in A^\alpha$, $e_j^k = \delta_j^k$ form a free basis in the module $A^\alpha$ both for finite $\alpha$ and for infinite $\alpha$, in the sense that any element $x \in A^\alpha$ can be represented as a converged sum

$$x = \sum_{k=1}^{\alpha} x_k e^k. \tag{2}$$

In general, a Hilbert $C^*$–module $M$ is a Banach space. We say that $C^*$–module $M$ is a finitely generated $C^*$–module if $M$ is a finitely generated $C^*$–module in the algebraic sense. In other words, there exists a free $C^*$–module $A^n$, $n < \omega$, and an algebraic epimorphism

$$f : A^n \longrightarrow M \longrightarrow 0. \tag{3}$$

It is easily verified that the epimorphism $f$ is a bounded map. Indeed, if $x \in A^n$,

$x = \{x_k\}$ then

$$\|f(x)\|^2 = \|\langle f(x), f(x)\rangle\| = \left\|\left\langle \left(\sum_k x_k f(e^k)\right), \left(\sum_j x_j f(e^j)\right)\right\rangle\right\| =$$

$$= \|\sum_{k,j} \left(x_k \langle f(e^k), f(e^j)\rangle x_j^*\right)\| \le \|\sum_{k,j} \|\langle f(e^k), f(e^j)\rangle\| \cdot \left(x_k x_j^*\right)\| \le \tag{4}$$

$$\le \sum_{k,j} \|\langle f(e^k), f(e^j)\rangle\| \cdot \|x_k x_j^*\| \le \sum_{k,j} \|\langle f(e^k), f(e^j)\rangle\| \cdot \|x_k\| \cdot \|x_j^*\| \le$$

$$\le \sum_{k,j} \|\langle f(e^k), f(e^j)\rangle\| \cdot \|x\|^2 \le n^2 C \|x\|^2,$$

where

$$C = \max_{k,j} \|\langle f(e^k), f(e^j)\rangle\|. \tag{5}$$

A Hilbert $C^*$–module is called a projective finitely generated $C^*$–module if it is isomorphic to a direct summand of a finite free $C^*$–module $L_n(A) = A^n$.

**Theorem 1** [7](Theorem 1.1, p. 69.) *Let $M$ — be a finitely generated Hilbert $A$–module. Then $M$ is a projective $A$–module, i.e. $M$ is isomorphic to a direct summand of a finite free $A$–module $L_n(A)$.*

So, we can give the following definition

**Definition 1** *Let* **End** *$(l_2(A))$ be a Banach algebra of all bounded $A$–operators of a Hilbert $A$–module $l_2(A)$. An $A$–operator $K : l_2(A) \longrightarrow l_2(A)$ is called a finitely generated $A$–operator if it can be represented as a composition of bounded $A$–operators $f_1$ and $f_2$:*

$$K : l_2(A) \xrightarrow{f_1} M \xrightarrow{f_2} l_2(A),$$

*where $M$ — is a finitely generated Hilbert $C^*$–module. The set $\mathcal{FG}(A) \subset$* **End** *$(l_2(A))$ of all finitely generated $A$–operators forms a two side ideal. By definition, an $A$–operator $K$ is called a compact if it belongs to the closure $\mathcal{K}(l_2(A)) = \overline{\mathcal{FG}(A)} \subset$* **End** *$(l_2(A))$, which also forms two side ideal.*

In general, the set $\mathcal{FG}(A) \subset$ **End** $(l_2(A))$ is not closed subset. For example, in classical case, when $\mathcal{A} = \mathbf{C}$, the set $\mathcal{FG}(A)$ consists of all finite dimensional operators, while not all compact operators are finite dimensional.

**Lemma 1** *The ideal $\mathcal{K}(l_2(A))$ is a proper ideal.*

**Proof.** It is sufficient to prove that the identity operator $\mathbf{id} \in$ **End** $(l_2(A))$ does not belong to $\mathcal{K}(l_2(A))$. Or, to prove that the distance (in the sense of the operator norm) between this operator and the set $\mathcal{FG}(A)$ is a positive number. In other words, it is sufficient to prove that any finitely generated $A$–operator is not invertible. Indeed, if a finitely generated $A$–operator $K$ :

$l_2(A) \xrightarrow{f_1} M \xrightarrow{f_2} l_2(A)$ is an invertible $A$–operator then that means that the $A$–operator $f_2$ is an epimorphism. Since $C^*$–module $M$ is a finitely generated $C^*$–module then there exists an epimorphism $p : L_n(A) \longrightarrow M$. Then the $A$–operator $f_2 \circ p : L_n(A) \longrightarrow l_2(A)$ is an epimorphism. But this is impossible. ∎

Let $l_2(A) = (L_n(A))^\perp \oplus L_n(A)$ be an orthogonal decomposition which is given by a pair of projectors

$$p_n, q_n : l_2(A) \longrightarrow l_2(A), p_n + q_n = \mathbf{id}, \mathbf{Im}\, p_n = L_n(A). \tag{6}$$

Any $A$–operator $f : l_2(A) \longrightarrow l_2(A)$ forms a matrix composed from the bounded operators

$$f = \begin{pmatrix} q_n f q_n & q_n f p_n \\ p_n f q_n & p_n f p_n \end{pmatrix} : (L_n(A))^\perp \oplus L_n(A) \longrightarrow (L_n(A))^\perp \oplus L_n(A). \tag{7}$$

**Theorem 2** *A bounded $A$–operator $K : l_2(A) \longrightarrow l_2(A)$ is a compact $A$-operator iff for any $\varepsilon > 0$ there exists a number $N$ such that for any $m > N$ we have*

$$\|q_m K\| \leq \varepsilon. \tag{8}$$

**Proof.** Let us assume that the property (8) holds. Let $K_m = p_m K$. Since

$$K_m : l_2(A) \xrightarrow{f_1 = p_m K} L_m(A) \xrightarrow{f_2 = i} l_2(A) \tag{9}$$

then the operator $K_m$ is a finitely generated $A$–operator, i.e. $K_m \in \mathcal{FG}(A)$. Since for any $\varepsilon > 0$ there exists a natural number $N$ such that for any $m > N$

$$\|K - K_m\| = \|K - p_m K\| = \|q_m K\| \leq \varepsilon,$$

then $K \in \overline{\mathcal{FG}(A)}$, i.e. the operator $K$ is a compact $A$-operator.

Inverse, Let $K$ be a compact $A$–operator. It follows from the definition 1 that there exists a finitely generated $A$–operator $K' \in \mathcal{FG}(A)$ such that

$$\|K - K'\| \leq \frac{\varepsilon}{2}. \tag{10}$$

The finitely generated $A$–operator $K'$ can be represented as a composition

$$K' : l_2(A) \xrightarrow{f_1} M \xrightarrow{f_2} l_2(A), \tag{11}$$

in which, without loss of generality, we can assume that $M = L_n(A)$ with the basis $e_1, e_2, \ldots, e_n$. In the other words, the operator $f_1$ can be described as linear combination of bounded functionals

$$f_1(x) = \sum_{j=1}^{n} e_j \varphi^j(x), \quad \|\varphi_j\| \leq C. \tag{12}$$

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Correspondingly, the operator $f_2$ is given by a set of vectors $y_j = f_2(e_j) \in l_2(A)$. Thus, the operator $K'$ can be represented by the formula

$$K'(x) = \sum_{j=1}^{n} y_j \varphi^j(x). \tag{13}$$

Under the formula (10) the operator $K'$ has the following matrix form:

$$K' = \begin{pmatrix} q_m K' q_m & q_m K' p_m \\ p_m K' q_m & p_m K' p_m \end{pmatrix} : (L_m(A))^{\perp} \oplus L_m(A) \longrightarrow (L_m(A))^{\perp} \oplus L_m(A). \tag{14}$$

We have:

$$q_m K'(x) = \sum_{j=1}^{n} q_m(y_j) \varphi^j(x). \tag{15}$$

Then

$$\|q_m K'(x)\| \leq \| \sum_{j=1}^{n} q_m(y_j) \varphi^j(x) \| \leq$$
$$\leq \sum_{j=1}^{n} \|q_m(y_j)\| \cdot \|\varphi^j\| \cdot \|x\|. \tag{16}$$

Since the number of vectors $y_j$ is finite then there exists a number $N$ such that for any $m > N$ $\|q_m(y_j)\| \leq \frac{\varepsilon}{2nC}$. Then for any $m > N$ we have

$$\|q_m K'(x)\| \leq \frac{\varepsilon}{2}\|x\|, \tag{17}$$

i.e. $\|q_m K'\| \leq \frac{\varepsilon}{2}$. Taking in account the inequality (10) we obtain the desired inequality

$$\|q_m K\| \leq \varepsilon. \tag{18}$$

∎

**Corollary 1** *Let $K : l_2(A) \longrightarrow l_2(A)$ be a compact A-operator. Then for any $\varepsilon > 0$ there exists a number $N$ such that for any $m > N$ we have*

$$\|q_m K q_m\| \leq \varepsilon. \tag{19}$$

**Proof.** We are interested in the operator $q_m K' q_m$ from the formula (14).
We have:

$$q_m K' q_m(x) = \sum_{j=1}^{n} q_m(y_j) \varphi^j(q_m(x)). \tag{20}$$

Then

$$\|q_m K' q_m(x)\| \leq \| \sum_{j=1}^{n} q_m(y_j) \varphi^j(q_m(x)) \| \leq$$
$$\leq \sum_{j=1}^{n} \|q_m(y_j)\| \cdot \|\varphi^j\| \cdot \|q_m\| \cdot \|x\|. \tag{21}$$

Since the number of vectors $y_j$ is finite then there exists a number $N$ such that for any $m > N$ $\|q_m(y_j)\| \leq \frac{\varepsilon}{2nC}$. Then for any $m > N$ we have

$$\|q_m K' q_m(x)\| \leq \frac{\varepsilon}{2}\|x\|, \tag{22}$$

i.e. $\|q_m K' q_m\| \leq \frac{\varepsilon}{2}$. Taking in account the inequality (10) we obtain the desired inequality

$$\|q_m K q_m\| \leq \varepsilon. \tag{23}$$

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# 3    Fredholm Operators over C*-algebra

**Definition 2** *A bounded $A$–operator $F : l_2(A) \longrightarrow l_2(A)$ is called a Fredholm $A$–operator if there exists a bounded $A$–operator $G : l_2(A) \longrightarrow l_2(A)$ such that*

$$\mathbf{id} - FG \in \mathcal{K}(l_2(A)), \quad \mathbf{id} - GF \in \mathcal{K}(l_2(A)). \tag{24}$$

**Definition 3** *We say that a bounded $A$–operator $F : l_2'(A) \longrightarrow l_2''(A)$ admits an inner (Noether) decomposition if there is a decomposition of the preimage and the image*

$$l_2'(A) = M_1 \oplus N_1, \quad l_2''(A) = M_2 \oplus N_2, \tag{25}$$

*where $C^*$–modules $N_1$ and $N_2$ are finitely generated Hilbert $C^*$–modules, and if $F$ has the following matrix form*

$$F = \begin{pmatrix} F_1 & F_2 \\ 0 & F_4 \end{pmatrix} : M_1 \oplus N_1 \longrightarrow M_2 \oplus N_2, \tag{26}$$

*where $F_1 : M_1 \longrightarrow M_2$ is an isomorphism.*

**Definition 4** *We put by definition index $F = [N_2] - [N_1] \in K(A)$.*

**Definition 5** *We say that a bounded $A$–operator $F : l_2'(A) \longrightarrow l_2''(A)$ admits an external (Noether) decomposition if there exist finitely generated $C^*$–modules $X_1$ and $X_2$ and bounded $A$–operators $E_2, E_3$ such that the matrix operator*

$$F_0 = \begin{pmatrix} F & E_2 \\ E_3 & 0 \end{pmatrix} : l_2'(A) \oplus X_1 \longrightarrow l_2''(A) \oplus X_2 \tag{27}$$

*is an invertible operator.*

**Definition 6** *We put by definition index $F = [X_1] - [X_2] \in K(A)$.*

**Theorem 3** *A bounded $A$–operator $F : l_2'(A) \longrightarrow l_2''(A)$ admits an external (Noether) decomposition iff it admits an inner (Noether) decomposition.*

**Proof.** If we have an inner (Noether) decomposition (26) then we can construct an external decomposition by an $A$–operator $F_0$ which has the following matrix form

$$F_0 = \begin{pmatrix} F_1 & F_2 & 0 \\ 0 & F_4 & \mathbf{id} \\ 0 & \mathbf{id} & 0 \end{pmatrix} : M_1 \oplus N_1 \oplus N_2 \longrightarrow M_2 \oplus N_2 \oplus N_1. \qquad (28)$$

It is obvious that the operator $F_0$ is an invertible $A$–operator.

Now, let an external decomposition (27) is given. Then the operator $E_3 : l_2'(A) \longrightarrow X_2$ is an epimorphism. Since the module $X_2$ is a projective $C^*$–module then there exists a decomposition

$$l_2'(A) = M_1 \oplus N_1, \quad M_1 = \mathbf{Ker}\, E_3, \quad E_3' = (E_3)_{|N_1} : N_1 \approx X_2. \qquad (29)$$

Analogously, let the inverted operator $G_0 = F_0^{-1}$ has the following matrix form

$$G_0 = \begin{pmatrix} G & G_2 \\ G_3 & G_4 \end{pmatrix} : l_2''(A) \oplus X_2 \longrightarrow l_2'(A) \oplus X_1. \qquad (30)$$

The condition $G_0 = F_0^{-1}$ can be rewrited as $F_0 G_0 = \mathbf{id}_{(l_2''(A) \oplus X_2)}$, $G_0 F_0 = \mathbf{id}_{(l_2'(A) \oplus X_1)}$, which have the following matrix forms

$$F_0 G_0 = \begin{pmatrix} F & E_2 \\ E_3 & 0 \end{pmatrix} \begin{pmatrix} G & G_2 \\ G_3 & G_4 \end{pmatrix} = \begin{pmatrix} \mathbf{id} & 0 \\ 0 & \mathbf{id} \end{pmatrix}, \qquad (31)$$

$$G_0 F_0 = \begin{pmatrix} G & G_2 \\ G_3 & G_4 \end{pmatrix} \begin{pmatrix} F & E_2 \\ E_3 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{id} & 0 \\ 0 & \mathbf{id} \end{pmatrix}. \qquad (32)$$

The conditions (31), (32) can be rewrited as

$$\begin{aligned}
\mathbf{id} &= FG + E_2 G_3 &:& \quad l_2''(A) \longrightarrow l_2''(A); \\
0 &= FG_2 + E_2 G_4 &:& \quad X_2 \longrightarrow l_2''(A); \\
0 &= E_3 G &:& \quad l_2''(A) \longrightarrow X_2; \\
\mathbf{id} &= E_3 G_2 &:& \quad X_2 \longrightarrow X_2; \\
\mathbf{id} &= GF + G_2 E_3 &:& \quad l_2'(A) \longrightarrow l_2'(A); \\
0 &= GE_2 &:& \quad X_1 \longrightarrow l_2'(A); \\
0 &= G_3 F + G_4 E_3 &:& \quad l_2'(A) \longrightarrow X_1; \\
\mathbf{id} &= G_3 E_2 &:& \quad X_1 \longrightarrow X_1.
\end{aligned} \qquad (33)$$

In particular, the operator $G_3 : l_2''(A) \longrightarrow X_1$ is also an epimorphism. Hence, there exists a decomposition

$$l_2''(A) = M_2 \oplus N_2, \quad M_2 = \mathbf{Ker}\, G_3, \quad G_3' = (G_3)_{|N_2} : N_2 \approx X_1. \qquad (34)$$

Then the operator $F_0$ has the following matrix form

$$F_0 = \begin{pmatrix} F_1 & F_2 & * \\ 0 & F_4 & * \\ 0 & E_3' & 0 \end{pmatrix} : M_1 \oplus N_1 \oplus X_1 \longrightarrow M_2 \oplus N_2 \oplus X_2. \qquad (35)$$

Indeed, if $x \in M_1$ then $E_3(x) = 0$. Hence, $G_3 F(x) = 0$, i.e. $F(x) \in$ **Ker** $G_3 = M_2$, and $F_0(x) \in M_2$. If $y \in M_2$ then $G_3(y) = 0$, and $E_3 G(y) = 0$, i.e. $G(y) \in M_1$. Moreover, if $x \in M_1$ then $x = GF(x)$, and for $y \in M_2$ we have $y = FG(y)$. Hence, the operator $F_1$ is an invertible $A$–operator. Since the operators $E'_3$ and $G'_3$ are invertible $A$–operators then the modules $N_1$ and $N_2$ are finitely generated Hilbert $C^*$–modules. ∎

**Corollary 2** *The index constructed by inner or external decomposition does not depend on the method of decomposition.*

**Theorem 4** *Let $K : l_2(A) \longrightarrow l_2(A)$ — be a compact operator in the sense of definition 1. Then the operator $\mathbf{id} + K$ admits an inner (Noether) decomposition.*

**Proof.** Under the formula (7) any operator $f : l_2(A) \longrightarrow l_2(A)$ has the following matrix form:

$$f = \begin{pmatrix} q_n f q_n & q_n f p_n \\ p_n f q_n & p_n f p_n \end{pmatrix} : (L_n(A))^\perp \oplus L_n(A) \longrightarrow (L_n(A))^\perp \oplus L_n(A). \quad (36)$$

Due to the corollary 1 we can find a natural number $N$ such that for any $m > N$

$$\|q_m K q_m\| < 1. \quad (37)$$

The operator $F = \mathbf{id} + K$ can be represented in the following matrix form

$$F = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix} : (L_m(A))^\perp \oplus L_m(A) \longrightarrow (L_m(A))^\perp \oplus L_m(A), \quad (38)$$

where the operator $F_1$ has the form $F_1 = \mathbf{id} + q_m K q_m$, and hence, is an invertible $A$-operator. The invertibility of the operator $F_1$ allows to represent the matrix (38) in the following form

$$\begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix} =$$

$$= \begin{pmatrix} \mathbf{id} & 0 \\ F_3 F_1^{-1} 0 & \mathbf{id} \end{pmatrix} \cdot \begin{pmatrix} F_1 & 0 \\ 0 & F_4 - F_3 F_1^{-1} F_2 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{id} & F_1^{-1} F_2 \\ 0 & \mathbf{id} \end{pmatrix}, \quad (39)$$

This proves the theorem. ∎

**Theorem 5** *Any Fredholm operator in the sense of definition 2 admits both the inner and external (Noether) decomposition.*

**Proof.** Let operators $F : l'_2(A) \longrightarrow l''_2(A)$, $G : l''_2(A) \longrightarrow l'_2(A)$ are chosen such that

$$K' = \mathbf{id} - FG \in \mathcal{K}(l_2(A)), \quad K'' = \mathbf{id} - GF \in \mathcal{K}(l_2(A)). \quad (40)$$

In accordance with the theorem 4 there exist decompositions

$$\begin{aligned} l'_2(A) = M_1 \oplus N_1, & \quad M_1 = \mathbf{Im}\, p_1, & \quad N_1 = \mathbf{Im}\,(\mathbf{id} - p_1), \\ l'_2(A) = M_2 \oplus N_2, & \quad M_2 = \mathbf{Im}\, p_2, & \quad N_2 = \mathbf{Im}\,(\mathbf{id} - p_2), \end{aligned} \quad (41)$$

such that the modules $N_1$ and $N_2$ are finitely generated $C^*$–modules, and the matrix of the operator $\mathbf{id} - K''$ has a diagonal form

$$\mathbf{id} - K'' = GF = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} : M_1 \oplus N_1 \xrightarrow{F} l_2''(A) \xrightarrow{G} M_2 \oplus N_2, \qquad (42)$$

where the operator $K_1$ is invertible. Let us consider the operator

$$P : l_2''(A) \longrightarrow l_2''(A), \quad P(x) = FK_1^{-1}p_2G(x). \qquad (43)$$

We have

$$PP(x) = FK_1^{-1}p_2G \cdot FK_1^{-1}p_2G(x) = FK_1^{-1}p_2K_1K_1^{-1}p_2G(x) = $$
$$= FK_1^{-1}p_2p_2G(x) = FK_1^{-1}p_2G(x) = P(x), \qquad (44)$$

i.e. the operator $P$ is a projector. This means that the module $l_2''(A)$ can be decomposed in direct sum

$$l_2''(A) = \mathbf{Im}\, P \oplus \mathbf{Ker}\, P = M_3 \oplus N_3, \qquad (45)$$

and in the decomposition (45) the operator $F$ has the following matrix form

$$F = \begin{pmatrix} F_1 & * \\ 0 & F_4 \end{pmatrix} : M_1 \oplus N_2 \longrightarrow M_3 \oplus N_3, \qquad (46)$$

where the operator $F_1$ is an isomorphism.

Now it is necessary to prove that the module $N_3$ is a finitely generated $C^*$–module. For, by the theorem 4 the operator $K' = \mathbf{id} - FG$ in the decompositions

$$\begin{aligned} l_2''(A) &= M_4 \oplus N_4, \\ l_2''(A) &= M_5 \oplus N_5 \end{aligned} \qquad (47)$$

has the following matrix form

$$\mathbf{id} - K' = FG : \begin{pmatrix} K_3 & 0 \\ 0 & K_4 \end{pmatrix} : M_4 \oplus N_4 \longrightarrow M_5 \oplus N_5, \qquad (48)$$

where the operator $K_3$ is an isomorphism. In particular, the operator

$$A = \begin{pmatrix} K_3 & 0 & 0 \\ 0 & K_4 & \mathbf{id} \end{pmatrix} : M_4 \oplus N_4 \oplus N_5 \longrightarrow M_5 \oplus N_5 \qquad (49)$$

is an epimorphism. It is convenient to represent this operator in the following matrix form

$$A = \begin{pmatrix} FG & , & a \end{pmatrix} : l_2''(A) \oplus N_5 \longrightarrow l_2''(A). \qquad (50)$$

We can represent the operator $A$ as composition

$$A = \begin{pmatrix} F & , & a \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & \mathbf{id} \end{pmatrix} : l_2''(A) \oplus N_5 \longrightarrow l_2'(A) \oplus N_5 \longrightarrow l_2''(A). \qquad (51)$$

Hence, the operator

$$B = \begin{pmatrix} F & , & a \end{pmatrix} : l_2'(A) \oplus N_5 \longrightarrow l_2''(A) \tag{52}$$

also is an epimorphism.

By (46) the operator $B$ has the following matrix form

$$B = \begin{pmatrix} F_1 & * & a_1 \\ 0 & F_4 & a_2 \end{pmatrix} : M_1 \oplus N_2 \oplus N_5 \longrightarrow M_3 \oplus N_3. \tag{53}$$

Hence the operator

$$D = \begin{pmatrix} F_4 & , & a_2 \end{pmatrix} : N_2 \oplus N_5 \longrightarrow N_3. \tag{54}$$

is an epimorphism. This means that the module $N_3$ is a finitely generated $C^*$–module. ∎

**Corollary 3** *Let $K : l_2(A) \longrightarrow l_2(A)$ — be a compact operator in the sense of definition 1 and $F : l_2(A) \longrightarrow l_2(A)$ be a Fredholm $A$–operator in the sense of definition 2. Then the operator $F + K$ is a Fredholm $A$–operator in the sense of definition 2 and admits an inner (Noether) decomposition.*

# 4 Fredholm and Compact Operators over Commutative $C^*$–algebras

Let $\mathcal{A} = C(X)$ be an algebra of continuous functions on a compact Hausdorff space $X$. We can identify the Hilbert $\mathcal{A}$–module $l_2(\mathcal{A})$ with the set $[X, H]$ of all continuous maps from the space $X$ into the Hilbert space $H$. Denote the set of all maps from the space $X$ into the space of bounded linear operators $\mathcal{B}(H)$, continuous in the strong topology, by $[X, \mathcal{B}(H)^s]$. We denote by $\mathcal{E}$ the map

$$\mathcal{E} : [X, \mathcal{B}(H)^s] \rightarrow End_\mathcal{A}(l_2(\mathcal{A})) \tag{55}$$

defined by the formula

$$(\mathcal{E}(T)(\varphi))(x) = T(x)\varphi(x), \tag{56}$$

where $T \in [X, \mathcal{B}(H)^s]$ is fixed, $x \in X$, and $\varphi \in l_2(\mathcal{A}) = [X, H]$ are arbitrary, and denote by $\mathcal{D}$ the map

$$\mathcal{D} : End_\mathcal{A}(l_2(\mathcal{A})) \rightarrow [X, \mathcal{B}(H)^s] \tag{57}$$

defined by the formula

$$(\mathcal{D}(B)(x))(a) = (B\varphi)(x), \tag{58}$$

where $B \in End_\mathcal{A}(l_2(\mathcal{A}))$ and the map $\varphi \in [X, H]$ is chosen so that $\varphi(x) \equiv a$, $a \in H$.

It was shown in the paper [3] that the definitions (55)–(58) are correct, the maps $\mathcal{E}$ and $\mathcal{D}$ are $\mathcal{A}$-isomorphisms, and

$$\mathcal{E}\mathcal{D} = \mathbf{id}_{End_{\mathcal{A}}(l_2(\mathcal{A}))} \quad \mathcal{D}\mathcal{E} = \mathbf{id}_{[X,\mathcal{B}(H)^s]}. \tag{59}$$

Further, it was shown ibidem that each invertible $\mathcal{A}$–operator $S \in GL(l_2(\mathcal{A}))$ under the map $\mathcal{D}$ can be represented as a family of invertible operators $S_x \in GL(H)^s$ continuous in the strong topology such that $\sup_{x \in X} \|S_x^{-1}\| < \infty$, and, conversely, each family of invertible operators $S_x \in GL(H)^s$ continuous in the strong topology such that $\sup_{x \in X} \|S_x^{-1}\| < \infty$ is mapped by $\mathcal{E}$ into an invertible $\mathcal{A}$–operator.

In the paper [4] the F-topology was introduced in the space of Fredholm operators $\mathcal{F}(H)$.

**Definition 7** [4] *The following sets form a subbase of the F-topology*

$$U_{\varepsilon,a_1,\ldots,a_n,A} = \{B \in \mathcal{F}(H) \mid \|(B-A)a_i\| < \varepsilon \ \ \forall \ i = 1,\ldots,n\},$$

$$U_{\varepsilon,V,A} = \{B \in \mathcal{F}(H) \mid \exists R \in GL(H), \ R(V) \subset V, \ such \ that \ \|RB - A\| < \varepsilon\}.$$

*Here $V$ denotes a finite dimensional subspace of the Hilbert space $H$ and $a_1, \ldots, a_n \in H$.*

Let $f : [0,1] \rightarrow \mathcal{F}(H)^F$ be any continuous map in the $F$–topology. Then, *index $f(x) = const$.* On the other hand, there exists a map $f : [0,1] \rightarrow \mathcal{F}(H)^s$ continuous in the strong topology such that *index $f(0) \neq$ index $f(1)$* (see [4]), so the $F$–topology is strictly stronger than the strong topology in the space of Fredholm operators.

Let $F \in End_{\mathcal{A}}(l_2(\mathcal{A}))$ be any Fredholm $\mathcal{A}$–operator. Then for any $x \in X$ $(\mathcal{D}(F))(x) \in \mathcal{F}(H)$ and the map $\mathcal{D}(F) : X \rightarrow \mathcal{F}(H)^s$ is continuous in the strong topology. It was shown in [4] that the map

$$\mathcal{D}(F) : X \rightarrow \mathcal{F}(H)^F \xrightarrow{\ \mathbf{id}\ } \mathcal{F}(H)^s \subset \mathcal{B}(H)^s$$

is continuous in the $F$–topology and vice versa if a map $f : X \rightarrow \mathcal{F}(H)^F$ is continuous in the $F$–topology then the $\mathcal{A}$–operator $\mathcal{E}(f)$ is a Fredholm $\mathcal{A}$–operator. Thus, the map

$$\mathcal{D}|_{\mathcal{F}(l_2(\mathcal{A}))} : \mathcal{F}(l_2(\mathcal{A})) \rightarrow [X, \mathcal{F}(H)^F], \tag{60}$$

where $\mathcal{F}(l_2(\mathcal{A}))$ is the space of Fredholm operators over the algebra $\mathcal{A}$ and $[X, \mathcal{F}(H)^F]$ is the set of continuous maps from the space $X$ into the space of Fredholm operators $\mathcal{F}(H)^F$ with the F-topology, is an isomorphism.

Denote by $\mathcal{U}(H)^s$ the space of unitary operators in $H$ with the strong topology. Due to the formula

$$U(x)^{-1} - U(x_0)^{-1} = U(x)^{-1} \left(U(x_0) - U(x)\right) U(x_0)^{-1}, \tag{61}$$

we can assert that if a map $U : X \rightarrow \mathcal{U}(H)^s$ is continuous in the strong topology then the map $U^{-1} : X \rightarrow \mathcal{U}(H)^s$ is also continuous in the strong topology.

**Theorem 6** *Let $X$ be a compact Hausdorff space and maps $U : X \to \mathcal{U}(H)^s$, $F : X \to \mathcal{F}(H)^F$ are continuous in the strong topology and the $F$-topology, respectively. Then the map $UFU^{-1} : X \to \mathcal{F}(H)^s \subset \mathcal{B}(H)^s$, given by the formula $UFU^{-1}(x) = U(x)F(x)U^{-1}(x)$, is continuous in the $F$-topology:*

$$UFU^{-1} : X \to \mathcal{F}(H)^F \xrightarrow{\ \mathbf{id}\ } \mathcal{F}(H)^s \subset \mathcal{B}(H)^s.$$

**Proof.** Since the $\mathcal{A}$–operators $\mathcal{E}(U)$, $\mathcal{E}\left(U^{-1}\right)$ are unitary $\mathcal{A}$–operators, and $\mathcal{E}(F)$ is a Fredholm $\mathcal{A}$–operator then the operator $\mathcal{E}\left(UFU^{-1}\right) = \mathcal{E}(U)\mathcal{E}(F)\mathcal{E}\left(U^{-1}\right)$ is a Fredholm $\mathcal{A}$–operator. Hence, by the isomorphism (60) the map $UFU^{-1} = \mathcal{D}\mathcal{E}\left(UFU^{-1}\right)$ is continuous in the $F$-topology. ∎

Let us consider the set of compact $\mathcal{A}$–operators $\mathcal{K}(l_2(\mathcal{A}))$. In the paper [6] has been considered the following class of compact operators $\mathcal{K}^*(l_2(\mathcal{A}))$. By definition (see [6]) an $\mathcal{A}$–operator $K : l_2(A) \to l_2(A)$ belongs to the set $\mathcal{K}^*(l_2(\mathcal{A}))$ iff

$$\lim_{n \to \infty} \|Kq_n\| = 0, \tag{62}$$

where the operator $q_n$ is defined by the formula (6). It was shown in ([8], Prop. 2.2.1.) that the set $\mathcal{K}^*(l_2(\mathcal{A}))$ coincides with the closure of the set of linear combinations of elementary operators $\theta_{x,y}(z) := x < y, z >$, where $x, y, z \in l_2(\mathcal{A})$. Hence, any $K \in \mathcal{K}^*(l_2(\mathcal{A}))$ automatically admits the adjoint operator. On the other hand, our notion of compact $\mathcal{A}$–operator does not demand existing of adjoint operator unlike that was assumed in many papers on $KK$–theory and so we shall distinguish the set $\mathcal{K}(l_2(\mathcal{A}))$ of all compact $\mathcal{A}$–operators and the subset $\mathcal{K}^*(l_2(\mathcal{A})) \subset \mathcal{K}(l_2(\mathcal{A}))$ of compact $\mathcal{A}$–operators which admit adjoint operator.

**Theorem 7** [4] *A compact $\mathcal{A}$–operator $K$ admits adjoint operator, i.e. $K \in \mathcal{K}^*(l_2(\mathcal{A}))$, iff the map*

$$\mathcal{D}(K) : X \to \mathcal{K}(H)^u \xrightarrow{\ \mathbf{id}\ } \mathcal{K}(H)^s \subset \mathcal{B}(H)^s$$

*is continuous in the uniform topology.*

The following example shows that there exists a self-adjoint family of compact operators continuous in the strong topology such that the corresponding $\mathcal{A}$–operator does not belong to the set $\mathcal{K}(l_2(\mathcal{A}))$.

**Example.** Let $X = \{0\} \cup \bigcup_{i=1}^{\infty}\{\frac{1}{i}\} \subset \mathbf{R}$. We define the map $K : X \to \mathcal{K}(H)$ by the following formula

$$K\left(\frac{1}{i}\right)(\xi) = -\xi_i e_i, \qquad K(0) = 0, \tag{63}$$

where $\xi = (\xi_1, \xi_2, \ldots)$ is an element of the standard Hilbert space with an orthonormal basis $\{e_i\}_{i \in \mathbf{N}}$. Then $K\left(\frac{1}{i}\right)(\xi) \to 0$ as $i \to \infty$ for every $\xi \in H$. But $\mathcal{E}(K) \notin \mathcal{K}(l_2(\mathcal{A}))$. Indeed, if we suppose the contrary then by the theorem 4 the

operator $\mathbf{id} + \mathcal{E}(K)$ is a Fredholm $\mathcal{A}$–operator. Due to the isomorphism (60) the map $\mathcal{D}(\mathbf{id} + \mathcal{E}(K)) = \mathbf{id} + K : X \to \mathcal{F}(H)^F$ is continuous in the $F$-topology. But for any invertible operator $S$ we have

$$\left\| S\left(\mathbf{id} + K\left(\frac{1}{i}\right)\right) - (\mathbf{id} + K(0)) \right\| \geq \left\| \left(S\left(\mathbf{id} + K\left(\frac{1}{i}\right)\right) - \mathbf{id}\right)(e_i) \right\| = 1.$$

That means that the map $\mathbf{id} + K : X \to \mathcal{F}(H)^F$ is not continuous in the $F$-topology. ∎

Thus, the example poses the problem of finding a topology in the space of compact operators $\mathcal{K}(H)$ such that any family continuous in this topology forms a compact $\mathcal{A}$–operator, and vice versa, any compact $\mathcal{A}$–operator maps by the map $\mathcal{D}$ to a family of compact operators continuous in the sought topology.

We define a new $IM$-topology in the space of compact operators $\mathcal{K}(H)$ in the following way. Let

$$U_{\varepsilon, a_1, \ldots, a_n, K} = \{B \in \mathcal{K}(H) \mid \|(B - K)a_i\| < \varepsilon \ \ \forall \ i = 1, \ldots, n\},$$

$$U_{\varepsilon, n, S, K} = \{B \in \mathcal{K}(H) \mid \exists R \in GL(H), \ such \ that$$

$$\|R(S + Q_n B) - (S + Q_n K)\| < \varepsilon\},$$

where $\varepsilon > 0$, $S \in GL(H)$, and $Q_n : H = (L_n)^\perp \oplus L_n \longrightarrow (L_n)^\perp \subset H$ is the orthogonal projection along the subspace $L_n$ spanned by the first $n$ orthonormal basis vectors $e_1, \ldots, e_n$.

**Definition 8** *As a subbase of the $IM$-topology we take the following sets*

$$U_{\varepsilon, a_1, \ldots, a_n, K} \quad and \quad U_{\varepsilon, S, K} := \bigcap_{n=0}^{\infty} U_{\varepsilon, n, S, K}.$$

**Remark 1** *It follows from the definition of the $IM$–topology that the identity map*

$$\mathcal{K}(H)^{IM} \xrightarrow{\mathbf{id}} \mathcal{K}(H)^s \subset \mathcal{B}(H)^s$$

*from the space of compact operators with the $IM$–topology to the same space with the strong topology is continuous. Since all sets $U_{\varepsilon, a_1, \ldots, a_n, K}$ and $U_{\varepsilon, S, K}$ contain the ball $B(K, \varepsilon) = \{Z \in \mathcal{K}(H) \mid \|Z - K\| < \varepsilon\}$, then the map*

$$\mathcal{K}(H)^u \xrightarrow{\mathbf{id}} \mathcal{K}(H)^{IM}$$

*from the space of compact operators with the norm topology to the same space with the $IM$–topology is continuous.*

**Theorem 8** *An $\mathcal{A}$–operator $K$ is a compact operator, i.e. $K \in \mathcal{K}(l_2(\mathcal{A}))$, iff the map*

$$\mathcal{D}(K) : X \to \mathcal{K}(H)^{IM} \xrightarrow{\mathbf{id}} \mathcal{K}(H)^s \subset \mathcal{B}(H)^s$$

*is continuous in the $IM$-topology.*

**Proof.** Let $K \in \mathcal{K}(l_2(\mathcal{A}))$. We have to prove that the map $\mathcal{D}(K)$ is a continuous map from $X$ to $\mathcal{K}(H)$ with the $IM$–topology. For, it is sufficient to show that for any $\varepsilon > 0$, for any $S \in GL(H)$, and for any $x_0 \in X$ there exists a neighbourhood $U_{x_0} \subset X$, $x_0 \in U_{x_0}$ such that for any $n \geq 0$

$$\mathcal{D}(K)(U_{x_0}) \subset U_{\varepsilon, n, S, \mathcal{D}(K)(x_0)}. \tag{64}$$

Let $l_2(\mathcal{A}) = L_{n,\mathcal{A}}^\perp \oplus L_{n,\mathcal{A}}$ be an orthogonal decomposition which is given by a pair of projectors $p_n$, $q_n$, $p_n + q_n = \mathbf{id}$, $\mathbf{Im}\, p_n = L_{n,\mathcal{A}}$, $\mathbf{Im}\, q_n = L_{n,\mathcal{A}}^\perp$, $\mathcal{D}(q_n)(x) \equiv Q_n$.

Let $s : X \to GL(H)$ be a constant map, $s(x) \equiv S \in GL(H)$, and $\hat{S} = \mathcal{E}(s) \in GL(\mathcal{A})$, i.e. $\mathcal{D}(\hat{S})(x) \equiv S \in GL(H)$.

Let us choose $n \in \mathbf{N}$ such that for all $m > n$ the $\mathcal{A}$–operator $G_m = \hat{S} + q_m K$ is invertible, i.e.

$$G_m = \hat{S} + q_m K \in GL(l_2(\mathcal{A})). \tag{65}$$

Then for any $x \in X$ we have

$$\mathbf{id} = \mathcal{D}(G_m^{-1} G_m)(x) = \mathcal{D}(G_m^{-1})(x)(S + Q_m \mathcal{D}(K)(x)).$$

If we put $R := \mathcal{D}(G_m)(x_0)\mathcal{D}(G_m^{-1})(x) \in GL(H)$ then for any $x \in X$ and $m > n$ we have

$$\|\mathcal{D}(G_m)(x_0)\mathcal{D}(G_m^{-1})(x)\,(S + Q_m \mathcal{D}(K)(x)) - (S + Q_m \mathcal{D}(K)(x_0))\,\| = 0.$$

The last equality means that $\forall x \in X$, $\forall m > n$, and $\forall \varepsilon > 0$

$$\mathcal{D}(K)(x) \in U_{\varepsilon, m, S, \mathcal{D}(K)(x_0)}.$$

Since $K$ is a compact $\mathcal{A}$–operator then by the corollary 3 for any Fredholm $\mathcal{A}$–operator $F$ the operator $F + q_l K$ is a Fredholm $\mathcal{A}$–operator. In particular, the operator $\hat{S} + q_l K$ is a Fredholm $\mathcal{A}$–operator for all $l \geq 0$. By the isomorphism (60), the maps $\mathcal{D}(\hat{S} + q_l K) : X \to \mathcal{F}(H)^F$, $\mathcal{D}(\hat{S} + q_l K)(x) = S + Q_l \mathcal{D}(K)(x)$, $l = 1, \ldots, n$, are continuous in the $F$-topology. Hence, for any fixed finite dimensional subspace $V_l \subset H$, $l = 1, \ldots, n$, there exists a neighbourhood $U_{x_0}^l \subset X$ such that for any $x \in U_{x_0}^l$

$$\mathcal{D}(\hat{S} + q_l K)(x) \in U_{\varepsilon, V_l, S + Q_l \mathcal{D}(K)(x_0)},$$

where $U_{\varepsilon, V_l, S + Q_l \mathcal{D}(K)(x_0)}$ is an open set in $F$-topology. This means that $\exists R \in GL(H)$, $R(V_l) \subset V_l$, such that

$$\|R\,(S + Q_l \mathcal{D}(K)(x)) - (S + Q_l \mathcal{D}(K)(x_0))\,\| < \varepsilon,$$

i.e.

$$\mathcal{D}(K)(x) \in U_{\varepsilon, l, S, \mathcal{D}(K)(x_0)}.$$

Then, if we put $U_{x_0} := \bigcap_{l=1}^n U_{x_0}^l$ we obtain the necessary condition (64). This proves that the map $\mathcal{D}(K) : X \to \mathcal{K}(H)^{IM} \xrightarrow{\mathbf{id}} \mathcal{K}(H)^s \subset \mathcal{B}(H)^s$ is continuous in the $IM$-topology.

To prove the inverse assertion of the theorem let us suppose the contrary. This means that there exists a continuous map $C : X \to \mathcal{K}(H)^{IM}$ such that the operator $\mathcal{E}(C)$ is not a compact $\mathcal{A}$-operator, i.e. $\mathcal{E}(C) \notin \mathcal{K}(l_2(\mathcal{A}))$. Due to the theorem 2 there exist a number $c_1 > 0$ and an increasing sequence of natural numbers $\{n_i\}_{i \in \mathbf{N}}$ such that

$$\|q_{n_i}\mathcal{E}(C)\| > c_1. \tag{66}$$

Since

$$\|q_{n_i}\mathcal{E}(C)\| = \sup_{x \in X} \|Q_{n_i}C(x)\|, \tag{67}$$

then there exist an element $x_i \in X$ and a vector $v^i \in H$, $\|v^i\| = 1$, such that

$$\left\|Q_{n_i}C(x_i)(v^i)\right\| > c_1. \tag{68}$$

Let

$$c_0 := \frac{c_1}{2} \quad and \quad \varepsilon_0 := \min\left(c_0, \frac{c_0^2}{\|\mathcal{E}(C)\|}\right). \tag{69}$$

Since the map $C$ is continuous in the $IM$–topology, which is stronger than the strong topology, then we can conclude that $\|\mathcal{E}(C)\| < \infty$ and $\varepsilon_0 > 0$.

We assert that we can choose from the sequence $\{x_i\}_{i \in \mathbf{N}}$ a subsequence $\{y_i\}_{i \in \mathbf{N}}$, an increasing subsequence of natural numbers $\{r_i\}_{i \in \mathbf{N}}$, and an orthonormal sequence of vectors $w_i \in H$, $i \in \mathbf{N}$, such that

$$\|Q_{r_i}C(y_i)(w_i)\| > c_0 \tag{70}$$

and

$$\|Q_{r_i}C(y_j)(w_j)\| < \frac{\varepsilon_0}{2^{i+2}} \quad for\ all \quad j < i. \tag{71}$$

We shall prove our assertion by mathematical induction. Let us put $y_1 := x_1$ and $w_1 := v^1$. Let us suppose that $k$ points $y_1, \ldots, y_k \in X$ and $k$ orthonormal vectors $w_1, \ldots, w_k \in H$ have already been chosen such that the conditions (70) and (71) hold. We consider $2k$ functions $\varphi_i : X \to H$, $\varphi_i(x) \equiv w_i$, and $\mathcal{E}(C)\varphi_i$, $(\mathcal{E}(C)\varphi_i)(x) = C(x)w_i$, $i = 1, \ldots, k$, as elements of $\mathcal{A}$–module $l_2(\mathcal{A}) = [X, H]$. For any $\varepsilon > 0$ there exists a natural number $N(\varepsilon)$ such that for any natural number $l > N(\varepsilon)$ the inequality

$$\|q_l\mathcal{E}(C)\varphi_j\| < \varepsilon \tag{72}$$

holds for $j = 1, \ldots, k$. Let $n_m > N\left(\frac{\varepsilon_0}{\sqrt{k}2^{k+3}}\right)$ and $H_1 := span < w_1, \ldots, w_k >$. Let $v^m = v_1^m + v_2^m$, $v_1^m \in H_1$, $v_2^m \in (H_1)^\perp$, be a representation of the vector $v^m$ in accordance with the decomposition $H = H_1 \oplus (H_1)^\perp$. We have

$$c_1 < \|(Q_{n_m}C(x_m))(v^m)\| = \|(Q_{n_m}C(x_m))(v_1^m + v_2^m)\| \leq$$
$$\frac{\sqrt{k}\varepsilon_0}{\sqrt{k}2^{k+3}}\|v_1^m\| + \|(Q_{n_m}C(x_m))(v_2^m)\|. \tag{73}$$

If we put $y_{k+1} := x_m$, $w_{k+1} := \frac{v_2^m}{\|v_2^m\|}$, and $r_{k+1} := n_m$, then by (73) we obtain the inequality (70) for $i = k+1$:

$$\left\|\left(Q_{r_{k+1}}C(y_{k+1})\right)(w_{k+1})\right\| \geq \left\|\left(Q_{n_m}C(x_m)\right)(v_2^m)\right\| > c_1 - \frac{\varepsilon_0}{2^{k+3}} > c_0. \quad (74)$$

The inequality (71) for $i = k+1$ follows from the inequality (72):

$$\left\|\left(Q_{r_{k+1}}C(y_j)\right)(w_j)\right\| \leq \|q_{n_m}\mathcal{E}(C)\varphi_j\| \leq \frac{\varepsilon_0}{\sqrt{k}2^{k+3}}, \quad j \leq k.$$

Let $a = (a_1, a_2, \ldots) \in H$, $\sum_{i=1}^{\infty} a_i \bar{a}_i = 1$. Let us estimate the norm of the element $\sum_{i=s}^{t} a_i \cdot (Q_{r_i}C(y_i))(w_i) \in H$. We have

$$\left\|\sum_{i=s}^{t} a_i \cdot (Q_{r_i}C(y_i))(w_i)\right\|^2 = \left(\sum_{i=s}^{t} a_i \cdot (Q_{r_i}C(y_i))(w_i), \sum_{i=s}^{t} a_i \cdot (Q_{r_i}C(y_i))(w_i)\right) =$$

$$\sum_{i=s}^{t} |a_i|^2 \left(Q_{r_i}C(y_i)(w_i), Q_{r_i}C(y_i)(w_i)\right)$$

$$+2 \sum_{s \leq i < j \leq t} \mathbf{Re}\left(a_i \cdot (Q_{r_i}C(y_i))(w_i), a_j \cdot (Q_{r_j}C(y_j))(w_j)\right) \leq$$

$$\|\mathcal{E}(C)\|^2 \sum_{i=s}^{t} |a_i|^2 + 2 \sum_{s \leq i < j \leq t} \mathbf{Re}\left(a_i \cdot (Q_{r_j}C(y_i))(w_i), a_j \cdot (Q_{r_j}C(y_j))(w_j)\right) \leq$$

$$\|\mathcal{E}(C)\|^2 \sum_{i=s}^{t} |a_i|^2 + 2\|\mathcal{E}(C)\| \sum_{i=s}^{t-1} \sum_{j=i+1}^{t} \frac{\varepsilon_0}{2^{j+2}} =$$

$$\|\mathcal{E}(C)\|^2 \sum_{i=s}^{t} |a_i|^2 + 2\|\mathcal{E}(C)\|\varepsilon_0 \left(\frac{1}{2^{s+1}} - \frac{t-s+2}{2^{t+2}}\right), (75)$$

and

$$\left\|\sum_{i=1}^{t} a_i \cdot (Q_{r_i}C(y_i))(w_i)\right\|^2 \geq$$

$$\left|c_0^2 \sum_{i=1}^{t} |a_i|^2 - 2 \sum_{1 \leq i < j \leq t} \mathbf{Re}\left(a_i \cdot (Q_{r_j}C(y_i))(w_i), a_j \cdot (Q_{r_j}C(y_j))(w_j)\right)\right| \geq$$

$$\left|c_0^2 \sum_{i=1}^{t} |a_i|^2 - 2\|\mathcal{E}(C)\|\frac{\varepsilon_0}{4}\right| \geq \frac{c_0^2}{2}. \quad (76)$$

Let us choose from the sequence $\{y_i\}_{i \in \mathbf{N}}$ a subsequence $\{y_{l(i)}\}$ such that both closures of the subspaces $W$ and $C(W)$ spanned by the vectors $\{w_{l(i)}\}_{i \in \mathbf{N}}$ and $\{Q_{r_{l(i)}}C(y_{l(i)})(w_{l(i)})\}_{i \in \mathbf{N}}$, respectively, have infinite codimension. Due to the inequalities (75) and (76) the map

$$S|_W : W \to C(W), \qquad S(w_{l(i)}) = Q_{r_{l(i)}}C(y_{l(i)})(w_{l(i)}),$$

is bounded and bijective. Hence, by theorem III.11 of [9] the map $S|_W$ is an isomorphism, and we can extend it to an invertible operator $S : H \to H$ by choosing any isomorphism between orthogonal complements of the spaces $W$ and $C(W)$.

Since $X$ is a compact space then there exists a point $x_0 \in X$ such that for any open neighbourhood $U_{x_0}$ of the point $x_0$ there are infinitely many members of the subsequence $\{y_{l(i)}\}_{i \in \mathbf{N}}$ lying in $U_{x_0}$. For any open neighbourhood $U_{\frac{1}{d}, -S, C(x_0)}$ in the $IM$–topology of the operator $C(x_0)$, $d \in \mathbf{N}$, we denote by $U_{x_0}^d \subset X$ the open neighbourhood

$$U_{x_0}^d := C^{-1}\left( U_{\frac{1}{d}, -S, C(x_0)} \right)$$

of the point $x_0 \in X$.

Let $z_d := y_{l(i_d)} \in U_{x_0}^d \cap \{y_{l(i)}\}_{i \in \mathbf{N}}$ which exists by the choice of the point $x_0 \in X$. Then there exists a sequence of invertible operators $G_d \in GL(H)$ such that

$$\frac{1}{d} > \left\| G_d \left( -S + Q_{r_{l(i_d)}} C(z_d) \right) - \left( -S + Q_{r_{l(i_d)}} C(x_0) \right) \right\| \geq$$
$$\left\| \left( G_d \left( -S + Q_{r_{l(i_d)}} C(z_d) \right) - \left( -S + Q_{r_{l(i_d)}} C(x_0) \right) \right) (w_{l(i_d)}) \right\| =$$
$$\left\| \left( -S + Q_{r_{l(i_d)}} C(x_0) \right) (w_{l(i_d)}) \right\|. \qquad (77)$$

Hence, for sufficiently large $d$

$$\| C(x_0)(w_{l(i_d)}) \| \geq \frac{1}{2} \| S(w_{l(i_d)}) \| \geq \frac{c_0}{2} \qquad (78)$$

But the inequality (78) contradicts the condition that $C(x_0)$ is a compact operator. ∎

**Corollary 4** *The uniform topology is stronger than the $IM$–topology.*

**Proof.** Let $X = \{0\} \cup \bigcup_{i=1}^{\infty} \{\frac{1}{i}\} \subset \mathbf{R}$ and $\mathcal{A} = C(X)$. We shall construct an $\mathcal{A}$–operator $K \in \mathcal{K}(l_2(\mathcal{A}))$ such that the map $\mathcal{D}(K) : X \to \mathcal{K}(H)^{IM} \xrightarrow{\mathbf{id}} \mathcal{K}(H)^s \subset \mathcal{B}(H)^s$ is not continuous in the uniform topology but by the theorem 8 is continuous in the $IM$–topology.

Let us define continuous functions $\varphi_i : X \to \mathbf{C}$, $i \in \mathbf{N}$, by the following rule

$$\varphi_i \left( \frac{1}{i} \right) = 1, \quad \varphi_i \left( \frac{1}{j} \right) = 0, \quad for \quad j \neq i, \quad \varphi_i(0) = 0.$$

We define the $\mathcal{A}$–operator $K : l_2(\mathcal{A}) \to l_2(\mathcal{A})$ by the following formula

$$K(\xi) = \left( \sum_{i=1}^{\infty} \varphi_i \xi_i, 0, 0, \ldots \right), \qquad (79)$$

where $\xi = (\xi_1, \xi_2, \ldots)$ is an element of the $l_2(\mathcal{A})$, $\xi_i \in \mathcal{A}$, $i = 1 \ldots, \infty$. Then $K \in \mathcal{K}(l_2(\mathcal{A}))$ and hence the map $\mathcal{D}(K) : X \to \mathcal{K}(H)^{IM} \xrightarrow{\mathbf{id}} \mathcal{K}(H)^s \subset \mathcal{B}(H)^s$ is continuous in the $IM$–topology.

By definition (79) of the $\mathcal{A}$–operator $K$ we have

$$\mathcal{D}(K)(0) = 0 \in \mathcal{B}(H).$$

But

$$\left\| \mathcal{D}(K) \left( \frac{1}{i} \right) \right\| \geq \left\| \mathcal{D}(K) \left( \frac{1}{i} \right) (e_i) \right\| = 1,$$

where $e_i \in H$, $i = 1, \ldots, \infty$ is the standard basis of $H$. That means that the map $\mathcal{D}(K) : X \to \mathcal{K}(H)^{IM} \xrightarrow{\ \mathbf{id}\ } \mathcal{K}(H)^s \subset \mathcal{B}(H)^s$ is not continuous in the uniform topology. ∎

Now, let us discuss the representing space for **K**-theory introduced in the paper [2]. In this paper M. Atiyah and G. Segal have considered locally trivial bundles $P \to X$ whose fibers $P_x = \mathbf{P}(H)$ are the projective space of a separable infinite dimensional complex Hilbert space $H$ and structural group is the projective unitary group $\mathcal{PU}(H)^{c.o}$, with the compact-open topology. With the aim to define a twisted **K**-theory they need to replace fiber $P_x$ by a representing space for **K**-theory such that the structural group $\mathcal{PU}(H)^{c.o}$ acts continuously on it by conjugation (ibid. sect.3, p.12). It is well known (see [1] and [5]) that the space $\mathcal{F}(H)^u$ of Fredholm operators in $H$ with the uniform topology is a representing space for **K**-theory. Unfortunately, the unitary group $\mathcal{U}(H)^{c.o}$ (and $\mathcal{PU}(H)^{c.o}$), with the compact-open topology, does not act continuously on $\mathcal{F}(H)^u$ by conjugation. To surmount this obstacle M. Atiyah and G. Segal have suggested (ibid.) to use as a representing space for **K**-theory the following set

$$Fred'(H) = \{ (A, B) \in \mathcal{F}(H) \times \mathcal{F}(H) \mid AB - I \in \mathcal{K}(H) \quad and \quad BA - I \in \mathcal{K}(H) \}$$

with the topology induced by the embedding

$$Fred'(H) \hookrightarrow \mathcal{B}(H)^{c.o} \times \mathcal{B}(H)^{c.o} \times \mathcal{K}(H)^u \times \mathcal{K}(H)^u$$

$$(A, B) \to (A, B, AB - I, BA - I),$$

where $\mathcal{B}(H)^{c.o}$ is the space of bounded operators in $H$, with the compact-open topology, and $\mathcal{K}(H)^u$ is the space of compact operators in $H$, with the uniform topology. Let $X$ be a compact space. In this case, by Banach-Steinhaus theorem (see [9] Theorem III.9) the continuous maps $X \to \mathcal{B}(H)$ are the same for the compact-open and for the strong operator topologies. Then any continuous map $f : X \to Fred'(H)$ of compact space $X$ into $Fred'(H)$ can be considered as a pair of continuous maps in the strong operator topology

$$A^f : X \to \mathcal{F}(H)^s, \tag{80}$$

$$B^f : X \to \mathcal{F}(H)^s \tag{81}$$

such that for any $x \in X$ the operators $A^f(x)B^f(x) - I$ and $B^f(x)A^f(x) - I$ are compact and the maps

$$A^f B^f - I : X \to \mathcal{K}(H)^u, \tag{82}$$

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$$B^f A^f - I : X \to \mathcal{K}(H)^u \tag{83}$$

are continuous in the uniform topology.

Now, we can relax the conditions (82) and (83) to strictly extend the set of admitted maps $A^f$, in the following way. The maps

$$A^f B^f - I : X \to \mathcal{K}(H)^{IM}, \tag{84}$$

$$B^f A^f - I : X \to \mathcal{K}(H)^{IM} \tag{85}$$

are continuous in the $IM$-topology. Indeed, by the theorem 8 we have

$$\mathcal{E}(A^f B^f - I), \quad \mathcal{E}(B^f A^f - I) \in \mathcal{K}(l_2(\mathcal{A})), \tag{86}$$

By the theorems 4 and 5 the $\mathcal{A}$–operators $\mathcal{E}(A^f)$ and $\mathcal{E}(B^f)$ are Fredholm $\mathcal{A}$–operators. Due to the isomorphism (60) the maps

$$A^f : X \to \mathcal{F}(H)^F, \tag{87}$$

$$B^f : X \to \mathcal{F}(H)^F \tag{88}$$

are continuous in the $F$-topology. Hence, the class of continuous maps $X \to \mathcal{F}(H)^F$ is strictly wider than the class of continuous maps $A^f : X \to \mathcal{F}(H)^s$ for which the conditions (80), (81), (82), (83) hold. Moreover, it was shown in the paper [4] that the space $\mathcal{F}(H)^F$ of Fredholm operators with the $F$-topology is a representing space for **K**-theory. Taking in account the theorem 6 we conclude that the space $\mathcal{F}(H)^F$ can be taken as a representing space for **K**-theory in the construction of the twisted **K**-theory.

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