Central European Science Journals
www.cesj.com

Central European Journal of Mathematics
Central European Science Journals
CEJM 3(4) 2005 766–793

K–theory from the point of view of C^*–algebras and Fredholm representations

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Received 3 January 2005; accepted 1 July 2005

Abstract: These notes represent the subject of five lectures which were delivered as a mini-course during the VI conference in Krynica, Poland, “Geometry and Topology of Manifolds”, May, 2-8, 2004.

Keywords: Vector bundle, K-theory, C-algebras, Fredholm operators, Fredholm representations*
MSC (2000): 19L, 19K, 19J25, 55N15

1 Introduction

In the second half of the last century, research commenced and developed in what is now called "non-commutative geometry". As a matter of fact, this term concentrates on a circle of problems and tools which originally was based on the quite simple idea of re-formulating topological properties of spaces and continuous mappings in terms of appropriate algebras of continuous functions.

This idea looks very old (it goes back to the theorem of I.M. Gelfand and M.A. Naimark (see, for example [1]) on the one-to-one correspondence between the category of compact topological spaces and the category of commutative unital C^*-algebras), and was developed by different authors both in the commutative and in the non-commutative cases. The first to clearly proclaim it as an action program was Alain Connes in his book "non-commutative geometry" [2].

The idea, along with commutative C^*-algebras (which can be interpreted as algebras of continuous functions on the spaces of maximal ideals), to also consider non-commutative

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algebras as functions on a non-existing ”non-commutative” space was so fruitful that it allowed the joining together of a variety of methods and conceptions from different areas such as topology, differential geometry, functional analysis, representation theory, asymptotic methods in analysis and resulted in mutual enrichment by new properties and theorems.

One of classical problems of smooth topology, which consists in the description of topological and homotopy properties of characteristic classes of smooth and piecewise-linear manifolds, has been almost completely and exclusively shaped by the different methods of functional analysis that were brought to bear on it. Vice versa, attempts to formulate and to solve classical topological problems have led to the enrichment of the methods of functional analysis. It is typical that solutions of particular problems of a new area lead to the discovery of new horizons in the development of mathematical methods and new properties of classical mathematical objects.

The following notes should not be considered as a complete exposition of the subject of non-commutative geometry. The lectures were devoted to topics of interest to the author and are indicative of his point of view in the subject. Consequently, the contents of the lectures were distributed as follows:

(1) Topological K–theory as a cohomology theory. Bott periodicity. Relation between the real, the complex and the quaternionic K–theories.
(2) Elliptic operators as the homology K–theory, Atiyah homology K–theory as an ancestor of KK–theory.
(3) C^*–algebras, Hilbert C^*-modules and Fredholm operators. Homotopical point of view.
(4) Higher signature, C^*–signature of non-simply connected manifolds.

2 Some historical remarks on the formation of non-commutative geometry

2.1 From Poincare duality to the Hirzebruch formula.

The Pontryagin characteristic classes, though not homotopy invariants, are nevertheless closely connected with the problem of the description of smooth structures of given homotopy type. Therefore, the problem of finding all homotopy invariant Pontryagin characteristic classes was a very actual one. However, in reality, another problem turned out to be more natural. It is clear that Pontryagin classes are invariants of smooth structures on a manifold. For the purpose of the classification of smooth structures, the most suitable objects are not the smooth structures but the so-called inner homology of manifolds or, using contemporary terms, bordisms of manifolds.

Already L.S. Pontryagin [3] conjectured that inner homology could be described in terms of some algebraic expression of Pontryagin classes, the so-called Pontryagin numbers. He established that the Pontryagin numbers are at least invariants of inner homology [4, Theorem 3]. W. Browder and S.P. Novikov were the first to prove that only the Pontryagin number which coincides with the signature of an oriented manifold is homotopy invariant. This fact was established by means of surgery theory developed in [5], [6].

The formula that asserts the coincidence of the signature with a Pontryagin number is known now as the Hirzebruch formula [7], though its special case was obtained by V.A. Rokhlin [8] a year before. Investigations of the Poincare duality and the Hirzebruch formula have a long history, which is partly related to the development of non-commutative geometry. Here we shall describe only some aspects that were typical of the Moscow school of topology.

The start of that history should be located in the famous manuscript of Poincare in 1895 [9], where Poincare duality was formulated. Although the complete statement and its full proof were presented much later, one can, without reservations, regard Poincare as the founder of the theory.

After that was required the discovery of homology groups (E. Noether, 1925) and cohomology groups (J.W. Alexander, A.N. Kolmogorov, 1934). The most essential was, probably, the discovery of characteristic classes (E.L. Stiefel, Y. Whitney (1935); L. Pontryagin (1947); S.S. Chern (1948)).

The Hirzebruch formula is an excellent example of the application of categorical method as a basic tool in algebraic and differential topology. Indeed, Poincare seems always to indicate when he had proved the coincidence of the Betti numbers of manifolds which are equidistant from the ends. But after the introduction of the notion of homology groups, Poincare duality began to be expressed a little differently: as the equality of the ranks of the corresponding homology groups. At that time it was not significant what type of homology groups were employed, whether with integer or with rational coefficients, since the rank of an integer homology group coincides with the dimension of the homology group over rational coefficients. But the notion of homology groups allowed *to enrich* ⇒ **to expand** Poincare duality by consideration of the homology groups over finite fields. Taking into account torsions of the homology groups, one obtained isomorphisms of some homology groups, but not in the same dimensions where the Betti numbers coincide. This apparent inconsistency was understood after the discovery of the cohomology groups and their duality to the homology groups. Thus finally, Poincare duality became sound as an isomorphism between the homology groups and the cohomology groups

$$H_k(M;Z) = H^{n-k}(M;Z). \tag{1}$$

The crucial understanding here is that the Poincare duality is not an abstract isomorphism of groups, but the isomorphism generated by a natural operation in the category of manifolds. For instance, in a special case of middle dimension for even-dimensional manifolds ($\dim M = n = 2m$) with rational coefficients, the condition (1) becomes trivial since

$$H^m(M;Q) = \mathbf{Hom}\,(H_m(M;Q), Q) \equiv H_m(M;Q). \tag{2}$$

But in the equation (2), the isomorphism between the homology groups and the cohomology groups is not chosen at will. Poincare duality says that there is the definite homomorphism generated by the intersection of the fundamental cycle $[M]$

$$\cap_{[M]} : H^{n-k}(M;Q) \longrightarrow H_k(M;Q).$$

This means that the manifold M gives rise to a non-degenerate quadratic form which has an additional invariant — the signature of the quadratic form. The signature plays a crucial role in many problems of differential topology.

2.2 Homotopy invariants of non-simply connected manifolds.

This collection of problems is devoted to finding the most complete system of invariants of smooth manifolds. In a natural way the smooth structure generates on a manifold a system of so-called characteristic classes, which take values in the cohomology groups with a different system of coefficients. Characteristic classes not only have natural descriptions and representations in differential geometric terms, but their properties also allow us to classify the structures of smooth manifolds in practically an exhaustive way modulo a finite number of possibilities. Consequently, the theory of characteristic classes is a most essential tool for the study of geometrical and topological properties of manifolds.

However, the system of characteristic classes is in some sense an over-determined system of data. More precisely, this means that for some characteristic classes their dependence on the choice of smooth structure is inessential. Therefore, one of the problems was to find out to what extent one or other characteristic class is invariant with respect to an equivalence relation on manifolds. The best known topological equivalence relations between manifolds are piece-linear homeomorphisms, continuous homeomorphisms, homotopy equivalences and bordisms. For such kinds of relations one can formulate a problem: which characteristic classes are: a) combinatorially invariant, b) topologically invariant, c) homotopy invariant. The last relation (bordism) gives a trivial description of the invariance of characteristic classes: only characteristic numbers are invariant with respect to bordisms.

Let us now restrict our considerations to rational Pontryagin classes. S.P. Novikov has proved (1965) that all rational Pontryagin classes are topologically invariant. In the case of homotopy invariance, at the present time the problem is very far from being solved. On the other hand, the problem of homotopy invariance of characteristic classes seems to be quite important on account of the fact that the homotopy type of manifolds seems to be more accessible to classification in comparison with its topological type. Moreover, existing methods of classification of smooth structures on a manifold can reduce this problem to a description of its homotopy type and its homology invariants.

Thus, the problem of homotopy invariance of characteristic classes seemed to be one of the essential problems in differential topology. In particular, the problem of homotopy invariance of rational Pontryagin classes happened to be the most interesting (and probably the most difficult) from the point of view of mutual relations. For example, the importance of the problem is confirmed by that fact that the classification of smooth structures on a manifold by means the Morse surgeries demands a description of all homotopy invariant rational Pontryagin classes.

In the case of simply connected manifolds, the problem was solved by Browder and Novikov who have proved that only signature is a homotopy invariant rational Pontryagin number. For non-simply connected manifolds, the problem of the description of all homotopy invariant rational Pontryagin classes which are responsible for obstructions to surgeries of normal mappings to homotopy equivalence, turned out to be more difficult. The difficulties are connected with the essential role that the structure of the fundamental group of the manifold plays here. This circumstance is as interesting as the fact that the description and identification of fundamental groups in finite terms is impossible. In some simple cases when the fundamental group is, for instance, free abelian, the problem could be solved directly in terms of differential geometric tools.

In the general case, it turned out that the problem can be reduced to the one that the so-called higher signatures are homotopy invariant. The accurate formulation of this problem is known as the Novikov conjecture. A positive solution of the Novikov conjecture may permit, at least partly, the avoidance of algorithmic difficulties of description and the recognition of fundamental groups in the problem of the classification of smooth structures on non-simply connected manifolds.

The Novikov conjecture says that any characteristic number of kind $\operatorname{sign}_x(M) = \langle L(M)f^*(x), [M]\rangle$ is a homotopy invariant of the manifold M , where $L(M)$ is the full Hirzebruch class, $x \in H^*(B\pi; Q)$ is an arbitrary rational cohomology class of the classifying space of the fundamental group $\pi = \pi_1(M)$ of the manifold M, $f : M \longrightarrow B\pi$ is the isomorphism of fundamental groups induced by the natural mapping. The numbers $\operatorname{sign}_x(M)$ are called higher signatures of the manifold M to indicate that when $x = 1$ the number $\operatorname{sign}_1(M)$ coincides with the classical signature of the manifold M.

The situation with non-simply connected manifolds turns out to be quite different from the case of simply connected manifolds in spite of the fact that C. T. C. Wall had constructed a non-simply connected analogue of Morse surgeries. The obstructions to such kinds of surgeries does not have an effective description. One way to avoid this difficulty is to find out which rational characteristic classes for non-simply connected manifolds are homotopy invariant. Here we should define more accurately what we mean by characteristic classes for non-simply connected manifolds. As was mentioned above, we should consider only such invariants for a non-simply connected manifold as a) can be expressed in terms of the cohomology of the manifold and b) are invariants of non-simply connected bordisms. In other words, each non-simply connected manifold M with fundamental group $\pi = \pi_1(M)$ has a natural continuous map $f_M : M \longrightarrow B\pi$ which induces an isomorphism of fundamental groups

$$(f_M)_* : \pi = \pi_1(M) \xrightarrow{\approx} \pi_1(B\pi) = \pi. \tag{3}$$

Then the bordism of a non-simply connected manifold M is the singular bordism $[M, f_M] \in \Omega(B\pi)$ of the space $B\pi$. Hence, from the point of view of bordism theory, a rational characteristic number for the singular bordism $[M, f_M]$ is a number of the following form

$$\alpha([M, f_M]) = \langle \mathcal{P}\left(p_1(M), \ldots, p_n(M); f_M^*(x_1), \ldots, f_M^*(x_k)\right), [M]\rangle\,, \tag{4}$$

where $p_j(M)$ are the Pontryagin classes of the manifold M, $x_j \in H^*(B\pi; \mathbf{Q})$ are arbitrary cohomology classes. Following the classical paper of R.Thom ([14]) one can obtain the result that the characteristic numbers of the type (4) form a complete system of invariants of the group $\Omega_*(B\pi) \otimes \mathbf{Q}$. Using the methods developed by C. T. C. Wall ([13]) one can prove that only higher signatures of the form

$$\text{sign}_x(M) = \langle L(M) f_M^*(x); [M] \rangle \tag{5}$$

may be homotopy invariant rational characteristic numbers for a non-simply connected manifold $[M]$.

3 Topological K–theory

3.1 Locally trivial bundles, their structure groups, principal bundles

Definition 3.1. Let E and B be two topological spaces with a continuous map

$$p : E \longrightarrow B.$$

The map p is said to define a *locally trivial bundle* if there is a topological space F such that for any point $x \in B$ there is a neighborhood $U \ni x$ for which the inverse image $p^{-1}(U)$ is homeomorphic to the Cartesian product $U \times F$. Moreover, it is required that the homeomorphism φ preserves fibers. This means in algebraic terms that the diagram

$$\begin{array}{ccccc} U \times F & \xrightarrow{\varphi} & p^{-1}(U) & \subset & E \\ \downarrow \pi & & \downarrow p & & \downarrow p \\ U & = & U & \subset & B \end{array}$$

is commutative where

$$\pi : U \times F \longrightarrow U, \quad \pi(x, f) = x$$

is the projection onto the first factor. The space E is called *total space of the bundle* or *the fiberspace*, the space B is called *the base of the bundle*, the space F is called *the fiber of the bundle* and the mapping p is called *the projection.*

One can give an equivalent definition using the so-called *transition functions:*

Definition 3.2. Let B and F be two topological spaces and $\{U_\alpha\}$ be a covering of the space B by a family of open sets. The system of homeomorphisms which form the commutative diagram

$$\begin{array}{ccc} (U_\alpha \cap U_\beta) \times F & \xrightarrow{\varphi_{\alpha\beta}} & (U_\alpha \cap U_\beta) \times F \\ \downarrow & & \downarrow \\ (U_\alpha \cap U_\beta) & = & (U_\alpha \cap U_\beta \end{array} \tag{6}$$

and satisfy the relations

$$\begin{aligned} \varphi_{\alpha\gamma}\varphi_{\gamma\beta}\varphi_{\beta\alpha} = \mathbf{id}, & \text{ for any three indices } \alpha, \beta, \gamma \\ & \text{ on the intersection } (U_\alpha \cap U_\beta \cap U_\gamma) \times F \\ \varphi_{\alpha\alpha} = \mathbf{id} & \text{ for each } \alpha. \end{aligned} \tag{7}$$

By analogy with the terminology for smooth manifolds, the open sets U_α are called *charts*, the family $\{U_\alpha\}$ is called *the atlas of charts*, the homeomorphisms

$$\begin{array}{ccccc} U_\alpha \times F & \xrightarrow{\varphi_\alpha} & p^{-1}(U_\alpha) & \subset & E \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ U_\alpha & = & U_\alpha & \subset & B \end{array} \tag{8}$$

are called *the coordinate homeomorphisms* and the $\varphi_{\alpha\beta}$ are called *the transition functions* or *the sewing functions*.

Two systems of the transition functions $\varphi_{\beta\alpha}$, and $\varphi'_{\beta\alpha}$ define isomorphic locally trivial bundles iff there exist fiber-preserving homeomorphisms

$$\begin{array}{ccc} U_\alpha \times F & \xrightarrow{h_\alpha} & U_\alpha \times F \\ \Big\downarrow & & \Big\downarrow \\ U_\alpha & = & U_\alpha \end{array}$$

such that

$$\varphi_{\beta\alpha} = h_\beta^{-1}\varphi'_{\beta\alpha}h_\alpha. \tag{9}$$

Let **Homeo** (F) be the group of all homeomorphisms of the fiber F. Each fiberwise homeomorphism

$$\varphi : U \times F \longrightarrow U \times F, \tag{10}$$

defines a map

$$\overline{\varphi} : U \longrightarrow \mathbf{Homeo}\ (F), \tag{11}$$

So instead of $\varphi_{\alpha\beta}$ a family of functions

$$\overline{\varphi}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathbf{Homeo}\ (F),$$

can be defined on the intersection $U_\alpha \cap U_\beta$ and having values in the group **Homeo** (F).

The condition (7) means that

$$\begin{gathered} \overline{\varphi}_{\alpha\alpha}(x) = \mathbf{id}, \\ \overline{\varphi}_{\alpha\gamma}(x)\overline{\varphi}_{\gamma\beta}(x)\overline{\varphi}_{\beta\alpha}(x) = \mathbf{id}. \\ x \in U_\alpha \cap U_\beta \cap U_\gamma. \end{gathered}$$

and we say that the cochain $\{\overline{\varphi}_{\alpha\beta}\}$ is a *cocycle*.

The condition (9) means that there is a zero-dimensional cochain $h_\alpha : U_\alpha \longrightarrow \mathbf{Homeo}\,(F)$ such that

$$\overline{\varphi}_{\beta\alpha}(x) = h_\beta^{-1}(x)\overline{\varphi}'_{\beta\alpha}(x)h_\alpha(x),\ x \in U_\alpha \cap U_\beta.$$

Using the language of homological algebra the condition (9) means that cocycles $\{\overline{\varphi}_{\beta\alpha}\}$ and $\{\overline{\varphi}'_{\beta\alpha}\}$ are cohomologous. Thus the family of locally trivial bundles with fiber F and base B is in one-to-one correspondence with the one-dimensional cohomology of the space B with coefficients in the sheaf of germs of continuous **Homeo** (F)–valued functions for the given open covering $\{U_\alpha\}$. Despite obtaining a simple description of the family of locally trivial bundles in terms of homological algebra, it is ineffective since there is no simple method for calculating cohomologies of this kind.

Nevertheless, this representation of the transition functions as a cocycle turns out to be very useful because of the situation described below.

First of all, notice that using the new interpretation, a locally trivial bundle is determined by the base B, the atlas $\{U_\alpha\}$ and the functions $\{\varphi_{\alpha\beta}\}$ taking values in the group $G = \mathbf{Homeo}\,(F)$. The fiber F itself does not directly take part in the description of the bundle. Hence, one can at first describe a locally trivial bundle as a family of functions $\{\varphi_{\alpha\beta}\}$ with values in some topological group G, and thereafter construct the total space of the bundle with fiber F by additionally defining an action of the group G on the space F, that is, defining a continuous homomorphism of the group G into the group **Homeo** (F).

Secondly, the notion of locally trivial bundle can be generalized and the structure of bundle made richer by requiring that both the transition functions $\overline{\varphi}_{\alpha\beta}$ and the functions h_α not be arbitrary but take values in some subgroup of the homeomorphism group **Homeo** (F).

Thirdly, sometimes information about locally trivial bundle may be obtained by substituting some other fiber F' for the fiber F but using the 'same' transition functions. Thus, we come to a new definition of a locally trivial bundle with additional structure — the group where the transition functions take their values, the so-called *the structure group.*

Definition 3.3. A locally trivial bundle with the structure group G is called a principal G–bundle if $F = G$ and the action of the group G on F is defined by the left translations.

Theorem 3.4. *Let $p : E \longrightarrow B$ be a principal G–bundle. Then there is a right action of the group G on the total space E such that:*

1) the right action of the group G is fiberwise,

2) the homeomorphism φ_α^{-1} transforms the right action of the group G on the total space into right translations on the second factor.

Using the transition functions it is very easy to define the *inverse image* of the bundle. Namely, let $p : E \longrightarrow B$ be a locally trivial bundle with structure group G and the collection of transition functions $\varphi_{\alpha\beta} : U_{\alpha\beta} \longrightarrow G$ and let $f : B' \longrightarrow B$ be a continuous mapping. Then the inverse image $f^*(p : E \longrightarrow B)$ is defined as a collection of charts $U'_\alpha = f^{-1}(U_\alpha)$ and a collection of transition functions $\varphi'_{\alpha\beta}(x) = \varphi_{\alpha\beta}(f(x))$.

Another geometric definition of the inverse bundle arises from the diagram

$$\begin{array}{ccccc} E' = f^*(E) & \subset & E \times B' & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ B' & = & B' & \xrightarrow{f} & B \end{array} \tag{12}$$

where E' consists of points $(e, b') \in E \times B'$ such that $f(b') = p(e)$. The map $\widehat{f} : f^*(E) \longrightarrow E$ is canonically defined by the map f.

Theorem 3.5. *Let*

$$\psi : E' \longrightarrow E \tag{13}$$

be a continuous map of total spaces for principal G–bundles over bases B' and B. The map (13) is generated by a continuous map $f : B' \longrightarrow B$ if and only if the map ψ is equvivariant (with respect to right actions of the structure group G on the total spaces).

3.2 Homotopy properties, classifying spaces

Theorem 3.6. *The inverse images with respect to homotopic mappings are isomorphic bundles.*

Therefore, the category of all bundles with structure group G, $Bndls_G(B)$ forms a homotopy functor from the category of CW-spaces to the category of sets.

Definition 3.7. A principal bundle $p : E_G \longrightarrow B_G$ is called a classifying bundle iff for any CW–space B there is a one-to-one correspondence

$$Bndls_G(B) \approx [B, B_G] \tag{14}$$

generated by the map

$$\begin{gathered} \varphi : [B, B_G] \longrightarrow Bndls_G(B), \\ \varphi(f) = f^*(p : E_G \longrightarrow B_G). \end{gathered} \tag{15}$$

Theorem 3.8. *The principal G–bundle,*

$$p_G : E_G \longrightarrow B_G \tag{16}$$

is a classifying bundle if all homotopy groups of the total space E_G are trivial:

$$\pi_i(E_G) = 0, \ 0 \le i < \infty. \tag{17}$$

3.3 Characteristic classes

Definition 3.9. A mapping $\alpha : Bndls_G(B)\longrightarrow H^*(B)$ is called a characteristic class if the following diagram is commutative

$$\begin{array}{ccc} Bndls_G(B) & \xrightarrow{\alpha} & H^*(B) \\ \downarrow f^* & & \downarrow f^* \\ Bndls_G(B') & \xrightarrow{\alpha} & H^*(B') \end{array} \tag{18}$$

for any continuous mapping $f : B'\longrightarrow B$, that is, α is a natural transformation of functors.

Theorem 3.10. *The family of all characteristic classes is in one-to-one correspondence with the cohomology $H^*(B_G)$ by the assignment*

$$\begin{gathered} \alpha(x)(\xi) = f^*(x) \quad \textit{for} \\ x \in H^*(B_G), \quad f : B\longrightarrow B_G, \quad \xi = f^*(p : E_G\longrightarrow B_G). \end{gathered} \tag{19}$$

3.4 Vector bundles, K–theory, Bott periodicity

Members of the special (and very important) class of locally trivial bundles are called (real) vector bundles with structure groups $GL(n, \mathbf{R})$ and fiber $\mathbf{R}^n$. The structure group can be reduced to the subgroup $O(n)$. If the structure group $O(1)$ can be reduced to the subgroup $G = SO(1)$ then the vector bundle is trivial and is denoted by $\overline{\mathbf{1}}$.

Similar versions arise for other structure groups:

1) Complex vector bundles with the structure group $GL(n, \mathbf{C})$ and fiber $\mathbf{C}^n$.

2) Quaternionic vector bundles with structure group $GL(n, \mathbf{K})$ and fiber $\mathbf{K}^n$, where $\mathbf{K}$ is the (non-commutative) field of quaternions.

All of them admit useful algebraic operations:

1. Direct sum, $\xi = \xi_1 \oplus \xi_2$,
2. tensor product,$\xi = \xi_1 \otimes \xi_2$,
3. other tensor operations, **HOM** (ξ_1, ξ_2),$\Lambda_k(\xi)$. etc.

Let ξ be a vector bundle over the field $F = \mathbf{R}, \mathbf{C}, \mathbf{K}$. Let $\Gamma(\xi)$ be the space of all sections ξ of the bundle. Then $\Gamma(\overline{\mathbf{1}}) = C(B)$ — the ring of continuous functions with values in F.

Theorem 3.11. *The space $\Gamma(\xi)$ has a natural structure of (left) $C(B)$–module by fiberwise multiplication. If B is a compact space, then $\Gamma(\xi)$ is a finitely generated projective $C(B)$–module. Conversely, each finitely generated projective $C(B)$–module can be presented as a space $\Gamma(\xi)$ for a vector bundle ξ.*

The property of compactness of B is essential for $\Gamma(\xi)$ to be a finitely generated module.

Definition 3.12. Let $K(X)$ denotes the abelian group where the generators are (isomorphism classes of) vector bundles over the base X subject to the following relations:

$$[\xi] + [\eta] - [\xi \oplus \eta] = 0 \tag{20}$$

for vector bundles ξ and η, and where $[\xi]$ denotes the element of the group $K(X)$ defined by the vector bundle ξ.

The group defined in Definition 3.12 is called *the Grothendieck group* of the category of all vector bundles over the base X.

To avoid confusion, the group generated by all real vector bundles will be denoted by $K_O(X)$, the group generated by all complex vector bundles will be denoted by $K_U(X)$ and the group generated by all quaternionic vector bundles will be denoted by $K_{Sp}(X)$.

Let $K^0(X, x_0)$ denote the kernel of the homomorphism $K(X)\longrightarrow K(x_0)$:

$$K^0(X, x_0) = \mathbf{Ker}\ (K(X)\longrightarrow K(x_0)).$$

Elements of the subring $K^0(X, x_0)$ are represented by differences $[\xi]-[\eta]$ for which $\dim \xi = \dim \eta$. The elements of the ring $K(X)$ are called *virtual* bundles and elements of the ring $K^0(X, x_0)$ are *virtual* bundles of trivial dimension over the point x_0.

Now consider a pair (X, Y) of the cellular spaces, $Y \subset X$. Denote by $K^0(X, Y)$ the ring

$$K^0(X, Y) = K^0(X/Y, [Y]) = \ker(K(X/Y)\longrightarrow K([Y]))$$

where X/Y is the quotient space where the subspace Y is collapsed to a point $[Y]$. For any negative integer $-n$, let

$$K^{-n}(X, Y) = K^0(S^n X, S^n Y)$$

where $S^n(X)$ denotes the n–times suspension of the space X:

$$S^n X = (S^n \times X)/(S^n \vee X).$$

Theorem 3.13. *The pair (X, Y) induces an exact sequence*

$$\begin{array}{l}
K^0(Y, x_0)\longleftarrow K^0(X, x_0)\longleftarrow K^0(X, Y)\longleftarrow \\
\longleftarrow K^{-1}(Y, x_0)\longleftarrow K^{-1}(X, x_0)\longleftarrow K^{-1}(X, Y)\longleftarrow \\
\longleftarrow K^{-2}(Y, x_0)\longleftarrow K^{-2}(X, x_0)\longleftarrow K^{-2}(X, Y)\longleftarrow \dots \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\dots \longleftarrow K^{-n}(Y, x_0)\longleftarrow K^{-n}(X, x_0)\longleftarrow K^{-n}(X, Y)\longleftarrow \dots \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{array} \tag{21}$$

Consider a complex n–dimensional vector bundle ξ over the base X and let $p : E\longrightarrow X$ be the projection of the total space E onto the base X. Consider the space E, as a new base space, and a complex of vector bundles

$$0\longrightarrow \Lambda^0\eta \xrightarrow{\varphi_0} \Lambda^1\eta \xrightarrow{\varphi_1} \Lambda^2\eta \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_{n-1}} \Lambda^n\eta \longrightarrow 0, \tag{22}$$

where $\eta = p^*\xi$ is the inverse image of the bundle ξ, $\Lambda^k\eta$ is the k-skew power of the vector bundle η and the homomorphism

$$\varphi_k : \Lambda^k\eta \longrightarrow \Lambda^{k+1}\eta$$

is defined as exterior multiplication by the vector $y \in E,\ y \in \xi_x,\ x = p(y)$. It is known that if the vector $y \in \xi_x$ is non-zero, $y \neq 0$, then the complex (22) is exact. Consider the subspace $D(\xi) \subset E$ consisting of all vectors $y \in E$ such that $|y| \leq 1$ with respect to a fixed Hermitian structure on the vector bundle ξ. Then the subspace $S(\xi) \subset D(\xi)$ of all unit vectors gives the pair $(D\xi), S(\xi))$ for which the complex (22) is exact on $S(\xi)$. Denote the element defined by (22) by $\beta(\xi) \in K^0(D(\xi), S(\xi))$. Then, one has the homomorphism given by multiplication by the element $\beta(\xi)$

$$\beta : K(X) \longrightarrow K^0\left(D(\xi), S(\xi)\right). \tag{23}$$

The homomorphism (23) is an isomorphism called *the Bott homomorphism.*

In particular the Bott element $\beta \in K^0(\mathbf{S}^2, s_0) = K^{-2}(\mathbf{S}^0, s_0) = \mathbf{Z}$ generates a homomorphism $\widetilde{h}$:

$$K^{-n}(X,Y) \xrightarrow{\otimes\beta} K^{-(n+2)}(X,Y) \tag{24}$$

which is called *the Bott periodicity isomorphism* and hence forms a periodic cohomology K–theory.

3.5 Relations between complex, symplectic and real bundles

Let G be a compact Lie group. A *G–space* X is a topological space X with continuous action of the group G on it. The map $f : X \longrightarrow Y$ is said to be *equivariant* if

$$f(gx) = gf(x), g \in G.$$

Similarly, if f is a locally trivial bundle and also equivariant then f is called an *equivariant* locally trivial bundle. An equivariant vector bundle is defined similarly. The theory of equivariant vector bundles is very similar to the classical theory. In particular, equivariant vector bundles admit the operations of direct sum and tensor product. In certain simple cases the description of equivariant vector bundles is reduced to the description of the usual vector bundles.

The category of G– equivariant vector bundles is good place to give consistent descriptions of three different structures on vector bundles — complex, real and symplectic.

Consider the group $G = \mathbf{Z}_2$ and a complex vector bundle ξ over the G–space X. This means that the group G acts on the space X. Let E be the total space of the bundle ξ and let

$$p : E \longrightarrow X$$

be the projection in the definition of the vector bundle ξ. Let G act on the total space E as a fiberwise operator which is linear over the real numbers and anti-complex over complex numbers, that is, if $\tau \in G = \mathbf{Z}_2$ is the generator then

$$\tau(\lambda x) = \overline{\lambda}\tau(x),\ \lambda \in \mathbf{C},\ x \in E. \tag{25}$$

A vector bundle ξ with the action of the group G satisfying the condition (25) is called a *KR–bundle* . The operator τ is called the anti-complex involution. The corresponding Grothendieck group of KR–bundles is denoted by $KR(X)$.

Below we describe some of the relations with classical real and complex vector bundles.

Proposition 3.14. *Suppose that the G–space X has the form $X = Y \times \mathbf{Z}_2$ and the involution τ transposes the second factor. Then the group $KR(X)$ is naturally isomorphic to the group $K_U(Y)$ and this isomorphism coincides with restriction of a vector bundle to the summand $Y \times \{1\}$, $1 \in G = \mathbf{Z}_2$, ignoring the involution τ.*

Proposition 3.15. *Suppose the involution τ on X is trivial. Then*

$$KR(X) \approx K_O(X). \tag{26}$$

The isomorphism (26) associates to any KR–bundle the fixed points of the involution τ.

Proposition 3.16. *The operation of forgetting the involution induces a homomorphism*

$$KR(X) \longrightarrow K_U(X)$$

and when the involution is trivial on the base X this homomorphism coincides with complexification

$$c : K_O(X) \longrightarrow K_U(X).$$

Moreover, the proof of Bott periodicity can be extended word by word to KR–theory:

Theorem 3.17. *There is an element $\beta \in KR(D^{1,1}, S^{1,1}) = KR^{-1,-1}(\mathbf{pt})$ such that the homomorphism given by multiplication by β*

$$\beta : KR^{p,q}(X, Y) \longrightarrow KR^{p-1,q-1}(X.Y) \tag{27}$$

is an isomorphism.

It turns out that this scheme can be modified so that it includes another type of K–theory – that of quaternionic vector bundles.

Let K be the (non-commutative) field of quaternions. As for real or complex vector bundles, we can consider locally trivial vector bundles with fiber K^n and structure group $\mathbf{GL}(n, K)$, the so called quaternionic vector bundles. Each quaternionic vector bundle can be considered as a complex vector bundle $p : E \longrightarrow X$ with additional structure defined by a fiberwise anti-complex linear operator J such that

$$J^2 = -1, IJ + JI = 0,$$

where I is fiberwise multiplication by the imaginary unit.

More generally, let J be a fiberwise anti-complex linear operator which acts on a complex vector bundles ξ and satisfies

$$J^4 = 1, IJ + JI = 0. \tag{28}$$

Then, the vector bundle ξ can be split into two summands $\xi = \xi_1 \oplus \xi_2$ both invariant under the action of J, that is, $J = J_1 \oplus J_2$ such that

$$J_1^2 = 1,\ J_2^2 = -1. \tag{29}$$

Hence, the vector bundle ξ_1 is the complexification of a real vector bundle and ξ_2 is a quaternionic vector bundle.

Consider a similar situation over a base X with an involution τ such that the operator (28) commutes with τ. Such a vector bundle will be called a *KRS–bundle* .

Lemma 3.18. *A KRS–bundle ξ is split into an equivariant direct sum $\xi = \xi_1 \oplus \xi_2$ such that $J^2 = 1$ on ξ_1 and $J^2 = -1$ on ξ_2.*

Lemma 3.18 shows that the Grothendieck group $KRS(X)$ generated by KRS–bundles has a $\mathbf{Z}_2$–grading, that is,

$$KRS(X) = KRS_0(X) \oplus KRS_1(X).$$

It is clear that $KRS_0(X) = KR(X)$. In the case when the involution τ acts trivially, $KRS_1(X) = K_Q(X)$, that is,

$$KRS(X) = K_O(X) \oplus K_Q(X)$$

where $K_Q(X)$ is the group generated by quaternionic bundles.

q	-8	-7	-6	-5	-4	-3	-2	-1	0
K_O	$\mathbf{Z}$ $\alpha = \gamma^2$	0	0	0	$\mathbf{Z}$ $u\gamma$	0	$\mathbf{Z}_2$ h^2	$\mathbf{Z}_2$ h	$\mathbf{Z}$ $u^2 = 4$
K_Q	$\mathbf{Z}$ $u\gamma^2$	0	$\mathbf{Z}_2$ $h^2\gamma$	$\mathbf{Z}_2$ $h\gamma$	$\mathbf{Z}$ γ	0	0	0	$\mathbf{Z}$ u

Fig. 1 A list of the groups K_O and K_Q.

4 Elliptic operators as the homology K–theory, Atiyah homology K–theory as an ancestor of KK–theory

4.1 Homology K–theory. Algebraic categorical setting

A naive point of view of homology theory is that the homology groups dual to the cohomology groups $h^*(X)$ should be considered as

$$h_*(X) \stackrel{def}{=} \mathbf{Hom}\ (h^*(X), \mathbf{Z}). \tag{30}$$

This naive definition is not good since it gives a non-exact functor. A more appropriate definition is the following.

Consider a natural transformation of functors

$$\alpha_Y : h^*(X \times Y) \longrightarrow h^*(Y) \tag{31}$$

which is the homomorphism of $h^*(Y)$–modules and for a continuous mapping $f : Y' \longrightarrow Y$ gives the commutative diagram

$$\begin{array}{ccc} h^*(X \times Y) & \xrightarrow{\alpha} & h^*(Y) \\ \Big\downarrow (\mathbf{id} \times f)^* & & \Big\downarrow f^* \\ h^*(X \times Y') & \xrightarrow{\alpha} & h^*(Y') \end{array} \tag{32}$$

Let be the family of all natural transformations of the type (31, 32). The functor $h^*(X)$ defines a homology theory.

4.2 PDO

Consider a linear differential operator A which acts on the space of smooth functions of n real variables:

$$A : C^\infty(\mathbf{R}^n) \longrightarrow C^\infty(\mathbf{R}^n).$$

and is presented as a finite linear combination of partial derivatives

$$A = \sum_{|\alpha| \le m} a_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}. \tag{33}$$

Put

$$a(x, \xi) = \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha i^{|\alpha|}.$$

The function $a(x, \xi)$ is called the *symbol of a differential operator* A. The operator A can be reconstructed from its symbol as

$$A = a\left(x, \frac{1}{i}\frac{\partial}{\partial x}\right).$$

Since the symbol is a polynomial with respect to the variables ξ, it can be split into homogeneous summands

$$a(x, \xi) = a_m(x, \xi) + a_{m-1}(x, \xi) + \cdots + a_0(x, \xi).$$

The highest term $a_m(x, x)$ is called the *principal symbol* of the operator A while whole symbol is sometimes called the *full symbol*.

Proposition 4.1. *Let $y = y(x)$ be a smooth change of variables. Then, in the new coordinate system the operator B defined by the formula*

$$(Bu)(y) = (Au\,(y(x)))_{x=x(y)}$$

is again a differential operator of order m for which the principal symbol is

$$b_m(y,\eta) = a_m\left(x(y), \eta\frac{\partial y(x(y))}{\partial x}\right). \tag{34}$$

The formula (34) shows that the variables ξ change as a tensor of valency $(0,1)$, that is, as components of a cotangent vector.

The concept of a differential operator exists on an arbitrary smooth manifold M. The concept of a whole symbol is not well defined but the principal symbol can be defined as a function on the total space of the cotangent bundle T^*M. It is clear that the differential operator A does not depend on the principal symbol alone but only up to the addition of an operator of smaller order.

The notion of a differential operator can be generalized in various directions. First of all, notice that

$$(Au)(x) = \mathbf{F}_{\xi\longrightarrow x}\left(a(x,\xi)\left(\mathbf{F}_{x\longrightarrow\xi}(u)(\xi)\right)\right), \tag{35}$$

where $\mathbf{F}$ is the Fourier transform.

Hence, we can enlarge the family of symbols to include some functions which are not polynomials.

Then the operator A defined by formula (35) with non-polynomial symbol is called a *pseudodifferential operator of order* m (more exactly, not greater than m). The pseudodifferential operator A acts on the Schwartz space $\mathbf{S}$.

This definition of a pseudodifferential operator can be extended to the Schwartz space of functions on an arbitrary compact manifold M. Let $\{U_\alpha\}$ be an atlas of charts with a local coordinate system x_α. Let $\{\varphi_\alpha\}$ be a partition of unity subordinate to the atlas of charts, that is,

$$0 \leq \varphi_\alpha(x) \leq 1,\ \sum_\alpha \varphi_\alpha(x) \equiv 1,\ \mathbf{supp}\ \varphi_\alpha \subset U_\alpha.$$

Let $\psi_\alpha(x)$ be functions such that

$$\mathbf{supp}\ \psi_\alpha \subset U_\alpha,\ \varphi_\alpha(x)\psi_\alpha(x) \equiv \varphi_\alpha(x).$$

Define an operator A by the formula

$$A(u)(x) = \sum_\alpha \psi_\alpha(x) A_\alpha\left(\varphi_\alpha(x)u(x)\right), \tag{36}$$

where A_α is a pseudodifferential operator on the chart U_α (which is diffeomorphic to $\mathbf{R}^n$) with principal symbol $a_\alpha(x_\alpha,\xi_\alpha) = a(x,\xi)$.

In general, the operator A depends on the choice of functions ψ_α, φ_α and the local coordinate system x_α, uniquely up to the addition of a pseudodifferential operator of order strictly less than m.

The next useful generalization consists of a change from functions on the manifold M to smooth sections of vector bundles.

The crucial property of the definition (36) is the following

Proposition 4.2. *Let*

$$a : \pi^*(\xi_1)\longrightarrow\pi^*(\xi_2),\ b : \pi^*(\xi_a)\longrightarrow\pi^*(\xi_3)$$

be two symbols of orders m_1, m_2. *Let* $c = ba$ *be the composition of the symbols. Then the operator*

$$b(D)a(D) - c(D) : \Gamma^\infty(\xi_1)\longrightarrow\Gamma^\infty(\xi_3)$$

is a pseudodifferential operator of order $m_1 + m_2 - 1$.

Proposition 4.2 leads to a way of solving equations of the form

$$Au = f \tag{37}$$

for certain pseudodifferential operators A. To find a solution of (37), it suffices to construct a left inverse operator B, that is, $BA = 1$. Usually, this is not possible, but a weaker condition can be realized.

Condition 4.3. $a(x, \xi)$ is invertible for sufficiently large $|\xi| \geq C$.

The pseudodifferential operator $A = a(D)$ is called an *elliptic* if Condition 4.3 holds. If A is elliptic operator than there is an (elliptic) operator $B = b(D)$ such that $AD - \mathbf{id}$ is the operator of order -1.

The final generalization for elliptic operators is the substitution of a sequence of pseudodifferential operators for a single elliptic operator. Let $\xi_1, \xi_2, \ldots, \xi_k$ be a sequence of vector bundles over the manifold M and let

$$0\longrightarrow\pi^*(\xi_1)\xrightarrow{a_1}\pi^*(\xi_2)\xrightarrow{a_2}\ldots\xrightarrow{a_{k-1}}\pi^*(\xi_k)\longrightarrow 0 \tag{38}$$

be a sequence of symbols of order $(m_1, \ldots, m_{k-1})$. Suppose the sequence (38) forms a complex, that is, $a_s a_{s-1} = 0$. Then the sequence of operators

$$0\longrightarrow\Gamma^\infty(\xi_1)\xrightarrow{a_1(D)}\Gamma^\infty(\xi_2)\longrightarrow\ldots\longrightarrow\Gamma^\infty(\xi_k)\longrightarrow 0 \tag{39}$$

in general, does not form a complex because we can only know that the composition $a_k(D)a_{k-1}(D)$ is a pseudodifferential operator of the order less then $m_s + m_{s-1}$.

If the sequence of pseudodifferential operators forms a complex and the sequence of symbols (38) is exact away from a neighborhood of the zero section in T^*M then the sequence (39) is called an *elliptic complex* of pseudodifferential operators.

4.3 Fredholm operators

The bounded operator $K : H \longrightarrow H$ is said to be *compact* if any bounded subset $X \subset H$ is mapped to a precompact set, that is, the set $\overline{F(X)}$ is compact. If $\dim \mathbf{Im}\,(K) < \infty$ then K is called a finite-dimensional operator. Each finite-dimensional operator is compact. If $\lim_{n\to\infty} \|K_n - K\| = 0$ and the K_n are compact operators, then K is again a compact operator. Moreover, each compact operator K can be presented as $K = \lim_{n\to\infty} K_n$, where the K_n are finite-dimensional operators.

The operator F is said to be Fredholm if there is an operator G such that both $K = FG - 1$ and $K' = GF - 1$ are compact.

Theorem 4.4. *Let F be a Fredholm operator. Then*

(1) $\dim \mathbf{Ker}\, F < \infty$, $\dim \mathbf{Coker}\, F < \infty$ *and the image,* $\mathbf{Im}\, F$, *is closed. The number* $\mathbf{index}\, F = \dim \mathbf{Ker}\, F - \dim \mathbf{Coker}\, F$ *is called the index of the Fredholm operator F.*

(2) $\mathbf{index}\, F = \dim \mathbf{Ker}\, F - \dim \mathbf{Ker}\, F^*$, *where F^* is the adjoint operator.*

(3) there exists $\varepsilon > 0$ such that if $\|F - G\| < \varepsilon$ then G is a Fredholm operator and

$$\mathbf{index}\, F = \mathbf{index}\, G,$$

(4) if K is compact then $F + K$ is also Fredholm and

$$\mathbf{index}\,(F + K) = \mathbf{index}\, F. \tag{40}$$

(5) If F and G are Fredholm operators, then the composition FG is Fredholm and

$$\mathbf{index}\,(FG) = \mathbf{index}\, F + \mathbf{index}\, G.$$

The notion of a Fredholm operator has an interpretation in terms of the finite-dimensional homology groups of a complex of Hilbert spaces. In general, consider a sequence of Hilbert spaces and bounded operators

$$0 \longrightarrow C_0 \xrightarrow{d_0} C_1 \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} C_n \longrightarrow 0. \tag{41}$$

We say that the sequence (41) is a *Fredholm complex* if $d_k d_{k-1} 0$, $\mathbf{Im}\, d_k$ is a closed subspace and

$$\dim(\mathbf{Ker}\, d_k / \mathbf{Coker}\, d_{k-1}) = \dim H\,(C_k, d_k) < \infty.$$

Then the *index of Fredholm complex* (41) is defined by the following formula:

$$\mathbf{index}\,(C, d) = \sum_k (-1)^k \dim H(C_k, d_k).$$

Theorem 4.5. *Let*

$$0 \longrightarrow C_0 \xrightarrow{d_0} C_1 \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} C_n \longrightarrow 0 \tag{42}$$

be a sequence satisfying the condition that each $d_k d_{k-1}$ is compact. Then the following conditions are equivalent:

(1) There exist operators $f_k : C_k \longrightarrow C_{k-1}$ such that $f_{k+1}d_k + d_{k-1}f_k = 1 + r_k$ where each r_k is compact.

(2) There exist compact operators s_k such that the sequence of operators $d'_k = d_k + s_k$ forms a Fredholm complex. The index of this Fredholm complex is independent of the operators s_k.

4.4 Sobolev spaces

Consider an arbitrary compact manifold M and a vector bundle ξ. One can define a Sobolev norm on the space of the sections $\Gamma^\infty(M, \xi)$, using the formula

$$\|u\|_s^2 = \int\limits_{\mathbf{R}^n} \overline{u}(x)(1 + \Delta)^s u(x) dx,$$

where Δ is the Laplace-Beltrami operator on the manifold M with respect to a Riemannian metric. The Sobolev norm depends on the choice of Riemannian metric, inclusion of the bundle ξ in the trivial bundle uniquely equivalent norms. Hence, the completion of the space of sections $\Gamma^\infty(M, \xi)$ is defined correctly. We shall denote this completion by $H_s(M, \xi)$.

Theorem 4.6. *Let M be a compact manifold, ξ be a vector bundle over M and $s_1 < s_2$. Then the natural inclusion*

$$H_{s_2}(M, \xi) \longrightarrow H_{s_1}(M, \xi) \tag{43}$$

is a compact operator.

Theorem 4.7. *Let*

$$a(D) : \Gamma^\infty(M, \xi_1) \longrightarrow \Gamma^\infty(M, \xi_2) \tag{44}$$

be a pseudodifferential operator of order m. Then there is a constant C such that

$$\|a(D)u\|_{s-m} \leq C\|u\|_s, \tag{45}$$

that is, the operator $a(D)$ can be extended to a bounded operator on Sobolev spaces:

$$a(D) : H_s(M, \xi_1) \longrightarrow H_{s-m}(M, \xi_2). \tag{46}$$

Using theorems 4.6 and 4.7 it can be shown that an elliptic operator is Fredholm for appropriate choices of Sobolev spaces.

Theorem 4.8. *Let $a(D)$ be an elliptic pseudodifferential operator of order m as in (44). Then its extension (46) is Fredholm. The index of the operator (46) is independent of the choice of the number s.*

4.5 Index of elliptic operators

An elliptic operator $\sigma(D)$ is defined by a symbol

$$\sigma : \pi^*(\xi_1)\longrightarrow\pi^*(\xi_2) \tag{47}$$

which is an isomorphism away from a neighborhood of the zero section of the cotangent bundle T^*M. Since M is a compact manifold, the symbol (47) defines a triple $(\pi^*(\xi_1), \sigma, \pi^*(\xi_2))$ which in turn defines an element

$$[\sigma] \in K(D(T^*M), S(T^*M)) = K_c(T^*M),$$

where $K_c(T^*M)$ denotes the K–groups with compact supports.

Theorem 4.9. *The index* **index** $\sigma(D)$ *of the Fredholm operator* $\sigma(D)$ *depends only on the element* $[\sigma] \in K_c(T^*M)$. *The mapping*

$$\textbf{index} \ : K_c(T^*M)\longrightarrow\mathbf{Z}$$

is an additive homomorphism.

In addition,

$$\textbf{index}\,\sigma(D) = p_*[\sigma], \tag{48}$$

where

$$p_* : K_c(T^*M)\longrightarrow K_c(pt) = \mathbf{Z}$$

is the direct image homomorphism induced by the trivial mapping $p : M\longrightarrow pt$.

4.6 The Atiyah homology theory

The naive idea is that cohomology K–group (with compact supports) of the total space of cotangent bundle of the manifold M, $K_c(T^*M)$, should be isomorphic to a homology K–group due to a Poincare duality, $K_*(T^*M) \approx K_*(M)$. This identification can be arranged to be a natural transformation of functors

$$\begin{aligned} D : K_c(T^*M) \ &\approx \ K_*(M), \\ D(\sigma) : K^*(M \times N) \ &\longrightarrow \ K^*(N) \\ D(\sigma)(\xi) \ &= \ \textbf{index}\,(\sigma \otimes \xi) \in K^*(N). \end{aligned} \tag{49}$$

Therefore, the homology K–groups, $K_*(M)$ can be identified with the collection of triples $\sigma = (H, F, \rho)$, where H is a Hilbert space, F is a Fredholm operator, $\rho : C(M)\longrightarrow B(H)$ is a representation of the algebra $C(M)$ of all continuous functions to the algebra $B(H)$ of bounded operators, such that for any function $f \in C(M)$ the operator $Ff - fF : H\longrightarrow H$ is compact. If ξ is a vector bundle on M then the space $\Gamma(\xi)$ is a finitely generated projective module over $C(M)$. Therefore the operator $F \otimes \mathbf{id}_\xi : H \otimes_{C(M)} \Gamma(\xi) \longrightarrow H \otimes_{C(M)} \Gamma(\xi)$ is Fredholm. Hence, one obtains a natural transformation

$$\begin{aligned} \sigma : K^*(M \times N) &\longrightarrow K^*(N), \\ \sigma(\xi) &= \mathbf{index}\,(F \otimes \mathbf{id}_\xi) \in K^*(N). \end{aligned} \tag{50}$$

This definition was an ancestor of KK–theory.

5 C^*–algebras, Hilbert C^*–modules and C^*–Fredholm operators

5.1 Hilbert C^*–modules

The simplest case of C^*–algebras is the case of commutative C^*–algebras. The Gelfand–Naimark theorem ([1]) says that any commutative C^*–algebra with unit is isomorphic to an algebra $C(X)$ of continuous functions on a compact space X.

This crucial observation leads to a simple but very useful definition of Hilbert modules over the C^*–algebra A. Following Paschke ([21]), the Hilbert A–module M is a Banach A–module with an additional structure of inner product $\langle x, y\rangle \in A,\ x, y \in M$ which possesses the natural properties of inner products.

If ξ is a finite-dimensional vector bundle over a compact space X, then $\Gamma(\xi)$ is a finitely generated projective Hilbert $C(X)$–module. And conversely, each finitely generated projective Hilbert module P over the algebra $C(X)$ is isomorphic to a section module $\Gamma(\xi)$ for some finite-dimensional vector bundle ξ. Therefore

$$K_0(C(X)) \cong K(X).$$

5.2 Fredholm operators, Calkin algebra

A finite-dimensional vector bundle ξ over a compact space X, can be described as a continuous family of projectors, that is a continuous matrix-valued function $P = P(x),\ x \in X$, $P(x) \in \mathbf{Mat}(N, N),\ P(x)P(x) = P(x),\ P(x) : \mathbf{C}^N \longrightarrow \mathbf{C}^N$. This means that $\xi = \mathbf{Im}\,P$. Here $\mathbf{Mat}(N, N)$ denotes the space of $N \times N$ matrices.

Hence if $\eta = \ker P$ then

$$\xi \oplus \eta = \overline{\mathbf{N}}. \tag{51}$$

Then $\xi \oplus \sum_{k=1}^{\infty} \overline{\mathbf{N}}_k \approx \sum_{k=1}^{\infty} \overline{\mathbf{N}}_k \approx H \times X$ where H is a Hilbert space, or $\xi \oplus H \otimes X = H \times X$. Hence, there is a continuous Fredholm family

$$\begin{aligned} &F(x) : H \longrightarrow H, \\ &\mathbf{Ker}\,(F) = \xi, \\ &\mathbf{Coker}\,(F) = 0. \end{aligned} \tag{52}$$

And conversely, if we have a continuous family of Fredholm operators $F(x) : H \longrightarrow H$ such that $\dim \mathbf{Ker}\, F(x) = \mathbf{const}$, $\dim \mathbf{Coker}\, F(x) = \mathbf{const}$ then both $\xi = \mathbf{Ker}\, F$ and $\eta = \mathbf{Coker}\, F$ are locally trivial vector bundles. More generally, for an arbitrary continuous family of Fredholm operators $F(x) : H \longrightarrow H$ there is a continuous compact family $K(x) : H \longrightarrow H$ such that the new family $\widetilde{F}(x) = F(x) + K(x)$ satisfies the conditions $\dim \mathbf{Ker}\, \widetilde{F}(x) = \mathbf{const}$, $\dim \mathbf{Coker}\, \widetilde{F}(x) = \mathbf{const}$ consequently defining two vector bundles ξ and η which generate an element $[\xi] - [\eta] \in K(X)$ not depending on the choice of compact family. This correspondence is actually one-to-one. In fact, if two vector bundles ξ and η are isomorphic then there is a compact deformation of $F(x)$ such that $\mathbf{Ker}\, F(x) = 0$, $\mathbf{Coker}\, F(x) = 0$, that is $F(x)$ is an isomorphism, $F(x) \in G(H)$.

The remarkable fact discovered by Kuiper ([22]) is that the group $G(H)$ as a topological space is contractible, i.e. the space $\mathcal{F}(H)$ of Fredholm operators is a representative of the classifying space $\mathbf{BU}$ for vector bundles. In other words, one can consider the Hilbert space H and the group of invertible operators $GL(H) \subset B(H)$. The Kuiper theorem says that

$$\pi_i(GL(H)) = 0, \ 0 \leq i < \infty. \tag{53}$$

5.3 K–theory for C^*–algebras, Chern character

Generalization of K–theory for C^*–algebra A. $K_A(X)$ is the Grothendieck group generated by vector bundles whose fibers M are finitely generated projective A–modules, and the structure groups $\mathbf{Aut}_{\,A}(M)$. $K^*_A(X)$ are the corresponding periodic cohomology theory.

For example, let us consider the quotient algebra $\mathcal{Q}(H) = B(H)/K(H)$, the so-called Calkin algebra, where $B(H)$ is the algebra of bounded operators of the Hilbert space H, $K(H)$ is the algebra of compact operators. Let $p : B(H) \longrightarrow \mathcal{Q}(H)$ be the natural projector. Then the Fredholm family $F(x) : H \longrightarrow H$ generates the family $\overline{F} : X \longrightarrow \mathcal{Q}(H)$, $\overline{F}(x) = p(F(x))$, $\overline{F}(x)$ is invertible that is

$$\overline{F} : X \longrightarrow G(\mathcal{Q}(H)).$$

So, one can prove that the space $G(\mathcal{Q}(H))$ represents the classifying space $\mathbf{BU}$ for vector bundles. In other words,

$$K^0(X) \cong K^1_{\mathcal{Q}(H)}(X).$$

A generalization of the Kuiper theorem for the group $GL^*_A(l_2(A))$ of all invertible operators which admit adjoint operators. Let $\mathcal{F}$ be the space of all Fredholm operators. Then

$$K^*(X) \approx [X, \mathcal{F}]. \tag{54}$$

Let $\mathcal{Q} = B(H)/\mathcal{K}$ be the Calkin algebra, where $\mathcal{K}$ is the subalgebra of all compact operators. Let $G(\mathcal{Q})$ be the group of invertible elements in the algebra $\mathcal{Q}$. Then one has a homomorphism

$$[X, \mathcal{F}] \longrightarrow [X, \mathcal{Q}],$$

hence a homomorphism

$$K^0(X)\longrightarrow K^1_{\mathcal{Q}}(X)$$

which is an isomorphism.

The Chern character

$$\mathbf{ch}_A : K^*_A(X)\longrightarrow H^*(X; K^*_A(\mathbf{pt})\otimes \mathbf{Q})$$

is defined in a tautological way: let us consider the natural pairing

$$K^*(X)\otimes K^*_A(\mathbf{pt})\longrightarrow K^*_A(X) \tag{55}$$

which generates the isomorphism

$$K^*(X)\otimes K^*_A(\mathbf{pt})\otimes \mathbf{Q}\xrightarrow{\theta} K^*_A(X)\otimes \mathbf{Q} \tag{56}$$

due to the classical uniqueness theorem in axiomatic homology theory. Then, the Chern character is defined as the composition

$$\begin{aligned}\mathbf{ch}_A : K^*_A(X)\subset K^*_A(X)\otimes \mathbf{Q}\xrightarrow{\theta^{-1}} K^*(X)\otimes (K^*_A(\mathbf{pt})\otimes \mathbf{Q}) \xrightarrow{\mathbf{ch}} \\ \xrightarrow{\mathbf{ch}} H^*(X; K^*_A(\mathbf{pt})\otimes \mathbf{Q}).\end{aligned} \tag{57}$$

Therefore, the next theorem is also tautological:

Theorem 5.1. *If X is a finite CW–space, the Chern character induces the isomorphism*

$$\mathbf{ch}_A : K^*_A(X)\otimes Q\longrightarrow H^*(X; K^*_A(\mathbf{pt})\otimes Q).$$

5.4 Non-simply connected manifolds and canonical line vector bundle

Let π be a finitely presented group which can serve as a fundamental group of a compact connected manifold M, $\pi = \pi_1(M, x_0)$. Let $B\pi$ be the classifying space for the group π. Then there is a continuous mapping

$$f_M : M\longrightarrow B\pi \tag{58}$$

such that the induced homomorphism

$$(f_M)_* : \pi_1(M, x_0)\longrightarrow \pi_1(B\pi, b_0) = \pi \tag{59}$$

is an isomorphism. One can then construct the line vector bundle ξ_A over M with fiber A, a one-dimensional free module over the group C^*–algebra $A = C^*[\pi]$ using the representation $\pi\subset \mathbf{C}^*[\pi]$. This canonical line vector bundle can be used to construct the so-called assembly map

$$\mu : K_*(B\pi)\longrightarrow K_*(\mathbf{C}^*[\pi]) \tag{60}$$

5.5 Symmetric equivariant C^*–signature

Let M be a closed oriented non-simply connected manifold with fundamental group π. Let $B\pi$ be the classifying space for the group π and let

$$f_M : M \longrightarrow B\pi,$$

be a map inducing the isomorphism of fundamental groups.

Let $\Omega_*(B\pi)$ denote the bordism group of pairs (M, f_M). Recall that $\Omega_*(B\pi)$ is a module over the ring $\Omega_* = \Omega_*(\text{ pt })$.

One can construct a homomorphism

$$\sigma : \Omega_*(B\pi) \longrightarrow L_*(\mathbf{C}\pi) \tag{61}$$

which for every manifold (M, f_M) assigns the element $\sigma(M) \in L_*(\mathbf{C}\pi)$, the so-called symmetric $\mathbf{C}\pi$–signature, where $L_*(\mathbf{C}\pi)$ is the Wall group for the group ring $\mathbf{C}\pi$.

The homomorphism σ satisfies the following conditions:

(a) σ is homotopy invariant,

(b) if N is a simply connected manifold and $\tau(N)$ is its signature then

$$\sigma(M \times N) = \sigma(M)\tau(N) \in L_*(\mathbf{C}\pi).$$

We shall be interested only in the groups after tensor multiplication with the field $\mathbf{Q}$, in other words, in the homomorphism

$$\sigma : \Omega_*(B\pi) \otimes \mathbf{Q} \longrightarrow L_*(\mathbf{C}\pi) \otimes \mathbf{Q}.$$

However,

$$\Omega_*(B\pi) \otimes \mathbf{Q} \approx H_*(B\pi; \mathbf{Q}) \otimes \Omega_*.$$

Hence one has

$$\sigma : H_*(B\pi; \mathbf{Q}) \longrightarrow L_*(\mathbf{C}\pi) \otimes \mathbf{Q}.$$

Thus, the homomorphism σ represents the cohomology class

$$\overline{\sigma} \in H^*(B\pi; L_*(\mathbf{C}\pi) \otimes \mathbf{Q}).$$

Then, for any manifold (M, f_M) one has

$$\sigma(M, f_M) = \langle L(M) f_M^*(\overline{\sigma}), [M] \rangle \in L_*(\mathbf{C}\pi) \otimes \mathbf{Q}. \tag{62}$$

Hence, if $\alpha : L_*(\mathbf{C}\pi) \otimes \mathbf{Q} \longrightarrow \mathbf{Q}$ is an additive functional and $\alpha(\overline{\sigma}) = x \in H^*(B\pi; \mathbf{Q})$ then

$$\text{sign}_x(M, f_M) = \langle L(M) f_M^*(x), [M] \rangle \in \mathbf{Q}$$

should be the homotopy-invariant higher signature. This gives a description of the family of all homotopy-invariant higher signatures. Hence, one should study the cohomology class

$$\overline{\sigma} \in H^*(B\pi; L_*(\mathbf{C}\pi) \otimes \mathbf{Q}) = H^*(B\pi; \mathbf{Q}) \otimes L_*(\mathbf{C}\pi) \otimes \mathbf{Q}$$

and look for all elements of the form $\alpha(\overline{\sigma}) = x \in H^*(B\pi; \mathbf{Q})$.

5.5.1 Combinatorial description of symmetric $\mathbf{C}\pi$–signature

Here we give an economical description of algebraic Poincare complexes as a graded free $\mathbf{C}\pi$–module with the boundary operator and the Poincare duality operator. Consider a chain complex of $\mathbf{C}\pi$–modules C, d:

$$C = \bigoplus_{k=0}^{n} C_k,$$
$$d = \bigoplus_{k=1}^{n} d_k,$$
$$d_k : C_k \longrightarrow C_{k-1}$$

and a Poincare duality homomorphism

$$D : C^* \longrightarrow C, \quad \deg D = n.$$

They form the diagram

$$\begin{array}{ccccccc}
C_0 & \xleftarrow{d_1} & C_1 & \xleftarrow{d_2} & \cdots & \xleftarrow{d_n} & C_n \\
\uparrow D_0 & & \uparrow D_1 & & & & \uparrow D_n \\
C_n^* & \xleftarrow{d_n^*} & C_{n-1}^* & \xleftarrow{d_{n-1}^*} & \cdots & \xleftarrow{d_1^*} & C_0^*
\end{array}$$

with the following properties:

$$\begin{aligned}
d_{k-1}d_k &= 0, \\
d_k D_k + (-1)^{k+1} D_{k-1} d_{n-k+1}^* &= 0, \\
D_k &= (-1)^{k(n-k)} D_{n-k}^*.
\end{aligned} \tag{63}$$

Assume that the Poincare duality homomorphism induces an isomorphism of homology groups. Then the triple (C, d, D) is called a *algebraic Poincare complex.* This definition permits the construction of the algebraic Poincare complex $\sigma(X)$ for each triangulation of the combinatorial manifold X: $\sigma(X) = (C, d, D)$, where

$$C = C_*(X; \mathbf{C}\pi)$$

is the graded chain complex of the manifold X with local system of coefficients induced by the natural inclusion of the fundamental group $\pi = \pi_1(X)$ in the group ring $\mathbf{C}\pi$, d is the boundary homomorphism,

$$D = \oplus D_k, \quad D_k = \frac{1}{2}\left(\cap[X] + (-1)^{k(n-k)}(\cap[X])^*\right),$$

where $\cap[X]$ is the intersection with the open fundamental cycle of the manifold X.

Put

$$F_k = i^{k(k-1)} D_k. \tag{64}$$

Then the diagram

$$\begin{array}{ccccccc} C_0 & \xleftarrow{d_1} & C_1 & \xleftarrow{d_2} & \cdots & \xleftarrow{d_n} & C_n \\ \uparrow F_0 & & \uparrow F_1 & & & & \uparrow F_n \\ C_n^* & \xleftarrow{d_n^*} & C_{n-1}^* & \xleftarrow{d_{n-1}^*} & \cdots & \xleftarrow{d_1^*} & C_0^* \end{array} \tag{65}$$

possesses more natural conditions of commutativity and conjugacy

$$\begin{aligned} d_k F_k + F_{k-1} d_{n-k+1}^* &= 0, \\ F_k &= (-1)^{\frac{n(n-1)}{2}} F_{n-k}^*. \end{aligned} \tag{66}$$

Let

$$F = \oplus_{k=0}^{n} F_k, \ \ F, \quad \deg F = n.$$

Over completion to a regular C^*–algebra $C^*[\pi]$, one can define an element of hermitian K–theory using the non-degenerate self-adjoint operator

$$G = d + d^* + F : C \longrightarrow C.$$

Then

$$\text{sign } [C, G] = \text{sign } (C, d, D) \in K_0^h(C^*[\pi]). \tag{67}$$

6 Additional historical remarks

The only candidates which are homotopy invariant characteristic numbers are the higher signatures. Moreover, any homotopy invariant higher signature can be expressed from a universal symmetric equivariant signature of the non-simply connected manifold. Therefore, to look for homotopy invariant higher signatures, one can search through different geometric homomorphisms $\alpha : L_*(\mathbf{C}\pi) \otimes \mathbf{Q} \longrightarrow \mathbf{Q}$. On of them is the so-called Fredholm representation of the fundamental group.

Application of the representation theory in the finite-dimensional case leads to Hirzebruch-type formulas for signatures with local system of coefficients. But the collection of characteristic numbers which can be represented by means of finite-dimensional representations is not very large and in many cases reduces to the classical signature. The most significant here is the contribution by Lusztig ([28]) where the class of representations with indefinite metric is considered.

The crucial step was to find a class of infinite-dimensional representations which preserve natural properties of the finite-dimensional representations. This infinite-dimensional analogue consists of a new functional-analytic construction as a pair of unitary infinite dimensional representations (T_1, T_2) of the fundamental group π in the Hilbert space H and a Fredholm operator F which braids the representations T_1 and T_2 upto compact operators. The triple $\rho = (T_1, F, T_2)$ is called the Fredholm representation of the group π.

From the categorical point of view, the Fredholm representation is a relative representation of the group C^*–algebra $C^*[\pi]$ in the pair of Banach algebras $(B(H), \mathcal{Q}(H)$ where $B(H)$ is the algebra of bounded operators on the Hilbert space H and $\mathcal{Q}(H)$ is the Calkin algebra $\mathcal{Q}(H) = B(H)/\mathcal{K}(H)$. Then, for different classes of manifolds one can construct a sufficiently rich resource of Fredholm representations. For one class of examples from amongst many others, there are the manifolds with Riemannian metric of nonpositive sectional curvature, the so-called hyperbolic fundamental groups. For the most complete description of the state of these problems, one may consult ([29]) and the book ([2]).

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