On Compact and Fredholm Operators over C^*-algebras and a New Topology in the Space of Compact Operators

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Abstract

It is shown that the class of Fredholm operators over an arbitrary unital C^*–algebra, which may not admit adjoint ones, can be extended in such a way that this class of compact operators, used in the definition of the class of Fredholm operators, contains compact operators both with and without existence of adjoint ones. The main property of this new class is that a Fredholm operator which may not admit an adjoint one has a decomposition into a direct sum of an isomorphism and a finitely generated operator.

In the space of compact operators in the Hilbert space a new IM-topology is defined. In the case when the C^*–algebra is a commutative algebra of continuous functions on a compact space the IM-topology fully describe the set of compact operators over the C^*–algebra without assumption of existence bounded adjoint operators over the algebra.

1 Introduction

In the paper [2] M. Atiyah and G. Segal have considered families of Fredholm operators parametrized by points of a compact space K which are continuous in a topology weaker than the uniform topology, i.e. the norm topology in the space of bounded operators $B(H)$ in a Banach space H.

Therefore, it is interesting to ascertain whether the conditions, characterized families of Fredholm operators, from the paper [2] precisely describe the families of Fredholm operators which forms a Fredholm operator over the C^*–algebra $\mathcal{A} = C(K)$ of all continuous functions on K.

It is not supposed by the authors of the paper [2] that an operator over algebra $\mathcal{A}$ admits the adjoint one or in their terms continuity of the adjoint family.

The aim of this paper is a fully clarification in the question of description of the class of Fredholm operators which in general case do not admit the adjoint operator. For the first time, operators which play the role of Fredholm operators and may not have the adjoint ones were considered in the paper [6]. Since the main class of operators considered in the paper [6] is the class of pseudodifferential operators, for any element of which the adjoint operator automatically is a bounded one, then existence of the adjoint operator was not the actual question for the main goals of this paper. However, in the paper [2] authors have considered operators, which may not have the adjoint one, in the form of families of operators continuous in the compact-open topology the adjoint families of which, in general case, may not be a continuous one. In the present paper we show that the class of Fredholm operators over arbitrary C^*–algebra, which may not admit the adjoint ones, can be extended in a such way that the class of compact operators used in the definition of the class of Fredholm operators contains compact operators both with and without existence the adjoint ones.

In the case when the C^*–algebra is a commutative algebra of continuous functions on a compact space appropriate topologies in the classic spaces of Fredholm and compact operators in the Hilbert space are constructed which fully describe the sets of Fredholm and compact operators over the C^*–algebra without assumption of existence bounded adjoint operators over the algebra. A comparison with the class of operators considered in the paper [2] is given and it is shown that the class of operators from the present paper strictly includes the class of operators from the paper [2].

2 A Notion of Compact Operator over C^*–algebra

Let A be a unital C^*–algebra. We shall consider so called Hilbert C^*–modules over the algebra A. The simplest Hilbert modules are the free finitely generated A–modules

$$A^n = \underbrace{A \oplus A \oplus \cdots \oplus A}_{n \text{ times}}$$

and the A-module $l_2(A) = A^\omega$. All such modules have a convenient description. Any element x of a module A^α, $\alpha \in [1..\omega]$ is a sequence $x = \{x_1, x_2, \ldots, x_k, \ldots\}$, $x_k \in A$, $1 \leq k < 1 + \alpha$, such that the sum

$$\langle x, x\rangle = \sum_{k=1}^{\alpha} x_k x_k^* \in A \tag{1}$$

converges in the algebra A. It is clear that if $\alpha < \omega$ then the sum (1) automatically converges. The elements $e^k \in A^\alpha$, $e^k_j = \delta^k_j$ form a free basis in the module A^α both for finite α and for infinite α, in the sense that any element $x \in A^\alpha$ can be represented as a converged sum

$$x = \sum_{k=1}^{\alpha} x_k e^k. \tag{2}$$

In general, a Hilbert C^*–module M is a Banach space. We say that C^*–module M is a finitely generated C^*–module if M is a finitely generated C^*–module in the algebraic sense. In other words, there exists a free C^*–module

A^n, $n < \omega$, and an algebraic epimorphism

$$f : A^n \longrightarrow M \longrightarrow 0. \tag{3}$$

It is easily verified that the epimorphism f is a bounded map. Indeed, if $x \in A^n$, $x = \{x_k\}$ then

$$\begin{gathered} ||f(x)||^2 = ||\langle f(x), f(x)\rangle|| = \left\| \left\langle \left(\sum_k x_k f(e^k)\right), \left(\sum_j x_j f(e^j)\right) \right\rangle \right\| = \\ = || \sum_{k,j} \left(x_k \langle f(e^k), f(e^j)\rangle x_j^*\right) || \leq || \sum_{k,j} ||\langle f(e^k), f(e^j)\rangle|| \cdot \left(x_k x_j^*\right) || \leq \\ \leq \sum_{k,j} ||\langle f(e^k), f(e^j)\rangle|| \cdot \left\|x_k x_j^*\right\| \leq \sum_{k,j} ||\langle f(e^k), f(e^j)\rangle|| \cdot ||x_k|| \cdot ||x_j^*|| \leq \\ \leq \sum_{k,j} ||\langle f(e^k), f(e^j)\rangle|| \cdot ||x||^2 \leq n^2 C ||x||^2, \end{gathered} \tag{4}$$

where

$$C = \max_{k,j} ||\langle f(e^k), f(e^j)\rangle||. \tag{5}$$

A Hilbert C^*–module is called a projective finitely generated C^*–module if it is isomorphic to a direct summand of a finite free C^*–module $L_n(A) = A^n$.

Theorem 1 [7](Theorem 1.1, p. 69.) *Let M — be a finitely generated Hilbert A–module. Then M is a projective A–module, i.e. M is isomorphic to a direct summand of a finite free A–module $L_n(A)$.*

So, we can give the following definition

Definition 1 *Let* **End** $(l_2(A))$ *be a Banach algebra of all bounded A–operators of a Hilbert A–module $l_2(A)$. An A–operator $K : l_2(A) \longrightarrow l_2(A)$ is called a finitely generated A–operator if it can be represented as a composition of bounded A–operators f_1 and f_2:*

$$K : l_2(A) \xrightarrow{f_1} M \xrightarrow{f_2} l_2(A),$$

where M — is a finitely generated Hilbert C^*–module. The set $\mathcal{FG}(A) \subset$ **End** $(l_2(A))$ *of all finitely generated A–operators forms a two side ideal. By definition, an A–operator K is called a compact if it belongs to the closure $\mathcal{K}(l_2(A)) = \overline{\mathcal{FG}(A)} \subset$* **End** $(l_2(A))$*, which also forms two side ideal.*

In general, the set $\mathcal{FG}(A) \subset$ **End** $(l_2(A))$ is not closed subset. For example, in classical case, when $\mathcal{A} = \mathbf{C}$, the set $\mathcal{FG}(A)$ consists of all finite dimensional operators, while not all compact operators are finite dimensional.

Lemma 1 *The ideal $\mathcal{K}(l_2(A))$ is a proper ideal.*

Proof. It is sufficient to prove that the identity operator $\mathbf{id} \in \mathbf{End}\,(l_2(A))$ does not belong to $\mathcal{K}(l_2(A))$. Or, to prove that the distance (in the sense of the operator norm) between this operator and the set $\mathcal{FG}(A)$ is a positive number. In other words, it is sufficient to prove that any finitely generated A–operator is not invertible. Indeed, if a finitely generated A–operator $K : l_2(A)\xrightarrow{f_1}M\xrightarrow{f_2}l_2(A)$ is an invertible A–operator then that means that the A–operator f_2 is an epimorphism. Since C^*–module M is a finitely generated C^*–module then there exists an epimorphism $p : L_n(A)\longrightarrow M$. Then the A–operator $f_2 \circ p : L_n(A)\longrightarrow l_2(A)$ is an epimorphism. But this is impossible. ∎

Let $l_2(A) = (L_n(A))^\perp \oplus L_n(A)$ be an orthogonal decomposition which is given by a pair of projectors

$$p_n, q_n : l_2(A)\longrightarrow l_2(A), p_n + q_n = \mathbf{id}, \mathbf{Im}\, p_n = L_n(A). \tag{6}$$

Any A–operator $f : l_2(A)\longrightarrow l_2(A)$ forms a matrix composed from the bounded operators

$$f = \begin{pmatrix} q_n f q_n & q_n f p_n \\ p_n f q_n & p_n f p_n \end{pmatrix} : (L_n(A))^\perp \oplus L_n(A)\longrightarrow (L_n(A))^\perp \oplus L_n(A). \tag{7}$$

Theorem 2 *A bounded A–operator $K : l_2(A)\longrightarrow l_2(A)$ is a compact A-operator iff for any $\varepsilon > 0$ there exists a number N such that for any $m > N$ we have*

$$||q_m K|| \le \varepsilon. \tag{8}$$

Proof. Let us assume that the property (8) holds. Let $K_m = p_m K$. Since

$$K_m : l_2(A)\xrightarrow{f_1 = p_m K} L_m(A)\xrightarrow{f_2 = i} l_2(A) \tag{9}$$

then the operator K_m is a finitely generated A–operator, i.e. $K_m \in \mathcal{FG}(A)$. Since for any $\varepsilon > 0$ there exists a natural number N such that for any $m > N$

$$||K - K_m|| = ||K - p_m K|| = ||q_m K|| \le \varepsilon,$$

then $K \in \overline{\mathcal{FG}(A)}$, i.e. the operator K is a compact A-operator.

Inverse, Let K be a compact A–operator. It follows from the definition 1 that there exists a finitely generated A–operator $K' \in \mathcal{FG}(A)$ such that

$$||K - K'|| \le \frac{\varepsilon}{2}. \tag{10}$$

The finitely generated A–operator K' can be represented as a composition

$$K' : l_2(A)\xrightarrow{f_1}M\xrightarrow{f_2}l_2(A), \tag{11}$$

in which, without loss of generality, we can assume that $M = L_n(A)$ with the basis $e_1, e_2, \ldots, e_n$. In the other words, the operator f_1 can be described as linear combination of bounded functionals

$$f_1(x) = \sum_{j=1}^{n} e_j \varphi^j(x), \quad ||\varphi_j|| \le C. \tag{12}$$

Correspondingly, the operator f_2 is given by a set of vectors $y_j = f_2(e_j) \in l_2(A)$. Thus, the operator K' can be represented by the formula

$$K'(x) = \sum_{j=1}^{n} y_j \varphi^j(x). \tag{13}$$

Under the formula (10) the operator K' has the following matrix form:

$$K' = \begin{pmatrix} q_m K' q_m & q_m K' p_m \\ p_m K' q_m & p_m K' p_m \end{pmatrix} : (L_m(A))^{\perp} \oplus L_m(A) \longrightarrow (L_m(A))^{\perp} \oplus L_m(A). \tag{14}$$

We have:

$$q_m K'(x) = \sum_{j=1}^{n} q_m(y_j) \varphi^j(x). \tag{15}$$

Then

$$\begin{aligned} \|q_m K'(x)\| &\leq \| \sum_{j=1}^{n} q_m(y_j) \varphi^j(x) \| \leq \\ &\leq \sum_{j=1}^{n} \|q_m(y_j)\| \cdot \|\varphi^j\| \cdot \|x\|. \end{aligned} \tag{16}$$

Since the number of vectors y_j is finite then there exists a number N such that for any $m > N$ $\|q_m(y_j)\| \leq \frac{\varepsilon}{2nC}$. Then for any $m > N$ we have

$$\|q_m K'(x)\| \leq \frac{\varepsilon}{2} \|x\|, \tag{17}$$

i.e. $\|q_m K'\| \leq \frac{\varepsilon}{2}$. Taking in account the inequality (10) we obtain the desired inequality

$$\|q_m K\| \leq \varepsilon. \tag{18}$$

∎

Corollary 1 *Let $K : l_2(A) \longrightarrow l_2(A)$ be a compact A-operator. Then for any $\varepsilon > 0$ there exists a number N such that for any $m > N$ we have*

$$\|q_m K q_m\| \leq \varepsilon. \tag{19}$$

Proof. We are interested in the operator $q_m K' q_m$ from the formula (14).

We have:

$$q_m K' q_m(x) = \sum_{j=1}^{n} q_m(y_j) \varphi^j(q_m(x)). \tag{20}$$

Then

$$\begin{aligned} \|q_m K' q_m(x)\| &\leq \| \sum_{j=1}^{n} q_m(y_j) \varphi^j(q_m(x)) \| \leq \\ &\leq \sum_{j=1}^{n} \|q_m(y_j)\| \cdot \|\varphi^j\| \cdot \|q_m\| \cdot \|x\|. \end{aligned} \tag{21}$$

Since the number of vectors y_j is finite then there exists a number N such that for any $m > N$ $||q_m(y_j)|| \leq \frac{\varepsilon}{2nC}$. Then for any $m > N$ we have

$$||q_m K' q_m(x)|| \leq \frac{\varepsilon}{2}||x||, \tag{22}$$

i.e. $||q_m K' q_m|| \leq \frac{\varepsilon}{2}$. Taking in account the inequality (10) we obtain the desired inequality

$$||q_m K q_m|| \leq \varepsilon. \tag{23}$$

∎

3 Fredholm Operators over C*-algebra

Definition 2 *A bounded A–operator $F : l_2(A) \longrightarrow l_2(A)$ is called a Fredholm A–operator if there exists a bounded A–operator $G : l_2(A) \longrightarrow l_2(A)$ such that*

$$\mathbf{id} - FG \in \mathcal{K}(l_2(A)), \quad \mathbf{id} - GF \in \mathcal{K}(l_2(A)). \tag{24}$$

Definition 3 *We say that a bounded A–operator $F : l_2'(A) \longrightarrow l_2''(A)$ admits an inner (Noether) decomposition if there is a decomposition of the preimage and the image*

$$l_2'(A) = M_1 \oplus N_1, \quad l_2''(A) = M_2 \oplus N_2, \tag{25}$$

where C^*–modules N_1 and N_2 are finitely generated Hilbert C^*–modules, and if F has the following matrix form

$$F = \begin{pmatrix} F_1 & F_2 \\ 0 & F_4 \end{pmatrix} : M_1 \oplus N_1 \longrightarrow M_2 \oplus N_2, \tag{26}$$

where $F_1 : M_1 \longrightarrow M_2$ is an isomorphism.

Definition 4 *We put by definition index $F = [N_2] - [N_1] \in K(A)$.*

Definition 5 *We say that a bounded A–operator $F : l_2'(A) \longrightarrow l_2''(A)$ admits an external (Noether) decomposition if there exist finitely generated C^*–modules X_1 and X_2 and bounded A–operators E_2, E_3 such that the matrix operator*

$$F_0 = \begin{pmatrix} F & E_2 \\ E_3 & 0 \end{pmatrix} : l_2'(A) \oplus X_1 \longrightarrow l_2''(A) \oplus X_2 \tag{27}$$

is an invertible operator.

Definition 6 *We put by definition index $F = [X_1] - [X_2] \in K(A)$.*

Theorem 3 *A bounded A–operator $F : l_2'(A) \longrightarrow l_2''(A)$ admits an external (Noether) decomposition iff it admits an inner (Noether) decomposition.*

Proof. If we have an inner (Noether) decomposition (26) then we can construct an external decomposition by an A–operator F_0 which has the following matrix form

$$F_0 = \begin{pmatrix} F_1 & F_2 & 0 \\ 0 & F_4 & \mathbf{id} \\ 0 & \mathbf{id} & 0 \end{pmatrix} : M_1 \oplus N_1 \oplus N_2 \longrightarrow M_2 \oplus N_2 \oplus N_1. \tag{28}$$

It is obvious that the operator F_0 is an invertible A–operator.

Now, let an external decomposition (27) is given. Then the operator $E_3 : l_2'(A) \longrightarrow X_2$ is an epimorphism. Since the module X_2 is a projective C^*–module then there exists a decomposition

$$l_2'(A) = M_1 \oplus N_1, \quad M_1 = \mathbf{Ker}\, E_3, \quad E_3' = (E_3)_{|N_1} : N_1 \approx X_2. \tag{29}$$

Analogously, let the inverted operator $G_0 = F_0^{-1}$ has the following matrix form

$$G_0 = \begin{pmatrix} G & G_2 \\ G_3 & G_4 \end{pmatrix} : l_2''(A) \oplus X_2 \longrightarrow l_2'(A) \oplus X_1. \tag{30}$$

The condition $G_0 = F_0^{-1}$ can be rewrited as $F_0 G_0 = \mathbf{id}_{(l_2''(A) \oplus X_2)}$, $G_0 F_0 = \mathbf{id}_{(l_2'(A) \oplus X_1)}$, which have the following matrix forms

$$F_0 G_0 = \begin{pmatrix} F & E_2 \\ E_3 & 0 \end{pmatrix} \begin{pmatrix} G & G_2 \\ G_3 & G_4 \end{pmatrix} = \begin{pmatrix} \mathbf{id} & 0 \\ 0 & \mathbf{id} \end{pmatrix}, \tag{31}$$

$$G_0 F_0 = \begin{pmatrix} G & G_2 \\ G_3 & G_4 \end{pmatrix} \begin{pmatrix} F & E_2 \\ E_3 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{id} & 0 \\ 0 & \mathbf{id} \end{pmatrix}. \tag{32}$$

The conditions (31), (32) can be rewrited as

$$\begin{array}{rcl} \mathbf{id} = FG + E_2 G_3 & : & l_2''(A) \longrightarrow l_2''(A); \\ 0 = FG_2 + E_2 G_4 & : & X_2 \longrightarrow l_2''(A); \\ 0 = E_3 G & : & l_2''(A) \longrightarrow X_2; \\ \mathbf{id} = E_3 G_2 & : & X_2 \longrightarrow X_2; \\ \mathbf{id} = GF + G_2 E_3 & : & l_2'(A) \longrightarrow l_2'(A); \\ 0 = GE_2 & : & X_1 \longrightarrow l_2'(A); \\ 0 = G_3 F + G_4 E_3 & : & l_2'(A) \longrightarrow X_1; \\ \mathbf{id} = G_3 E_2 & : & X_1 \longrightarrow X_1. \end{array} \tag{33}$$

In particular, the operator $G_3 : l_2''(A) \longrightarrow X_1$ is also an epimorphism. Hence, there exists a decomposition

$$l_2''(A) = M_2 \oplus N_2, \quad M_2 = \mathbf{Ker}\, G_3, \quad G_3' = (G_3)_{|N_2} : N_2 \approx X_1. \tag{34}$$

Then the operator F_0 has the following matrix form

$$F_0 = \begin{pmatrix} F_1 & F_2 & * \\ 0 & F_4 & * \\ 0 & E_3' & 0 \end{pmatrix} : M_1 \oplus N_1 \oplus X_1 \longrightarrow M_2 \oplus N_2 \oplus X_2. \tag{35}$$

Indeed, if $x \in M_1$ then $E_3(x) = 0$. Hence, $G_3F(x) = 0$, i.e. $F(x) \in \mathbf{Ker}\, G_3 = M_2$, and $F_0(x) \in M_2$. If $y \in M_2$ then $G_3(y) = 0$, and $E_3G(y) = 0$, i.e. $G(y) \in M_1$. Moreover, if $x \in M_1$ then $x = GF(x)$, and for $y \in M_2$ we have $y = FG(y)$. Hence, the operator F_1 is an invertible A–operator. Since the operators E_3' and G_3' are invertible A–operators then the modules N_1 and N_2 are finitely generated Hilbert C^*–modules. ∎

Corollary 2 *The index constructed by inner or external decomposition does not depend on the method of decomposition.*

Theorem 4 *Let $K : l_2(A) \longrightarrow l_2(A)$ — be a compact operator in the sense of definition 1. Then the operator* $\mathbf{id}+K$ *admits an inner (Noether) decomposition.*

Proof. Under the formula (7) any operator $f : l_2(A) \longrightarrow l_2(A)$ has the following matrix form:

$$f = \begin{pmatrix} q_n f q_n & q_n f p_n \\ p_n f q_n & p_n f p_n \end{pmatrix} : (L_n(A))^{\perp} \oplus L_n(A) \longrightarrow (L_n(A))^{\perp} \oplus L_n(A). \quad (36)$$

Due to the corollary 1 we can find a natural number N such that for any $m > N$

$$\|q_m K q_m\| < 1. \quad (37)$$

The operator $F = \mathbf{id} + K$ can be represented in the following matrix form

$$F = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix} : (L_m(A))^{\perp} \oplus L_m(A) \longrightarrow (L_m(A))^{\perp} \oplus L_m(A), \quad (38)$$

where the operator F_1 has the form $F_1 = \mathbf{id} + q_m K q_m$, and hence, is an invertible A-operator. The invertibility of the operator F_1 allows to represent the matrix (38) in the following form

$$\begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix} = \\ = \begin{pmatrix} \mathbf{id} & 0 \\ F_3F_1^{-1}0 & \mathbf{id} \end{pmatrix} \cdot \begin{pmatrix} F_1 & 0 \\ 0 & F_4 - F_3F_1^{-1}F_2 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{id} & F_1^{-1}F_2 \\ 0 & \mathbf{id} \end{pmatrix}, \quad (39)$$

This proves the theorem. ∎

Theorem 5 *Any Fredholm operator in the sense of definition 2 admits both the inner and external (Noether) decomposition.*

Proof. Let operators $F : l_2'(A) \longrightarrow l_2''(A)$, $G : l_2''(A) \longrightarrow l_2'(A)$ are chosen such that

$$K' = \mathbf{id} - FG \in \mathcal{K}(l_2(A)), \quad K'' = \mathbf{id} - GF \in \mathcal{K}(l_2(A)). \quad (40)$$

In accordance with the theorem 4 there exist decompositions

$$\begin{array}{lll} l_2'(A) = M_1 \oplus N_1, & M_1 = \mathbf{Im}\, p_1, & N_1 = \mathbf{Im}\,(\mathbf{id} - p_1), \\ l_2'(A) = M_2 \oplus N_2, & M_2 = \mathbf{Im}\, p_2, & N_2 = \mathbf{Im}\,(\mathbf{id} - p_2), \end{array} \quad (41)$$

such that the modules N_1 and N_2 are finitely generated C^*–modules, and the matrix of the operator $\mathbf{id} - K''$ has a diagonal form

$$\mathbf{id} - K'' = GF = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} : M_1 \oplus N_1 \xrightarrow{F} l_2''(A) \xrightarrow{G} M_2 \oplus N_2, \qquad (42)$$

where the operator K_1 is invertible. Let us consider the operator

$$P : l_2''(A) \longrightarrow l_2''(A), \quad P(x) = FK_1^{-1}p_2G(x). \qquad (43)$$

We have

$$PP(x) = FK_1^{-1}p_2G \cdot FK_1^{-1}p_2G(x) = FK_1^{-1}p_2K_1K_1^{-1}p_2G(x) = \\ = FK_1^{-1}p_2p_2G(x) = FK_1^{-1}p_2G(x) = P(x), \qquad (44)$$

i.e. the operator P is a projector. This means that the module $l_2''(A)$ can be decomposed in direct sum

$$l_2''(A) = \mathbf{Im}\ P \oplus \mathbf{Ker}\ P = M_3 \oplus N_3, \qquad (45)$$

and in the decomposition (45) the operator F has the following matrix form

$$F = \begin{pmatrix} F_1 & * \\ 0 & F_4 \end{pmatrix} : M_1 \oplus N_2 \longrightarrow M_3 \oplus N_3, \qquad (46)$$

where the operator F_1 is an isomorphism.

Now it is necessary to prove that the module N_3 is a finitely generated C^*–module. For, by the theorem 4 the operator $K' = \mathbf{id} - FG$ in the decompositions

$$\begin{array}{l} l_2''(A) = M_4 \oplus N_4, \\ l_2''(A) = M_5 \oplus N_5 \end{array} \qquad (47)$$

has the following matrix form

$$\mathbf{id} - K' = FG : \begin{pmatrix} K_3 & 0 \\ 0 & K_4 \end{pmatrix} : M_4 \oplus N_4 \longrightarrow M_5 \oplus N_5, \qquad (48)$$

where the operator K_3 is an isomorphism. In particular, the operator

$$A = \begin{pmatrix} K_3 & 0 & 0 \\ 0 & K_4 & \mathbf{id} \end{pmatrix} : M_4 \oplus N_4 \oplus N_5 \longrightarrow M_5 \oplus N_5 \qquad (49)$$

is an epimorphism. It is convenient to represent this operator in the following matrix form

$$A = \begin{pmatrix} FG & , & a \end{pmatrix} : l_2''(A) \oplus N_5 \longrightarrow l_2''(A). \qquad (50)$$

We can represent the operator A as composition

$$A = \begin{pmatrix} F & , & a \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & \mathbf{id} \end{pmatrix} : l_2''(A) \oplus N_5 \longrightarrow l_2'(A) \oplus N_5 \longrightarrow l_2''(A). \qquad (51)$$

Hence, the operator

$$B = \begin{pmatrix} F & , & a \end{pmatrix} : l_2'(A) \oplus N_5 \longrightarrow l_2''(A) \tag{52}$$

also is an epimorphism.

By (46) the operator B has the following matrix form

$$B = \begin{pmatrix} F_1 & * & a_1 \\ 0 & F_4 & a_2 \end{pmatrix} : M_1 \oplus N_2 \oplus N_5 \longrightarrow M_3 \oplus N_3. \tag{53}$$

Hence the operator

$$D = \begin{pmatrix} F_4 & , & a_2 \end{pmatrix} : N_2 \oplus N_5 \longrightarrow N_3. \tag{54}$$

is an epimorphism. This means that the module N_3 is a finitely generated C^*–module. ∎

Corollary 3 *Let $K : l_2(A) \longrightarrow l_2(A)$ — be a compact operator in the sense of definition 1 and $F : l_2(A) \longrightarrow l_2(A)$ be a Fredholm A–operator in the sense of definition 2. Then the operator $F + K$ is a Fredholm A–operator in the sense of definition 2 and admits an inner (Noether) decomposition.*

4 Fredholm and Compact Operators over Commutative C^*–algebras

Let $\mathcal{A} = C(X)$ be an algebra of continuous functions on a compact Hausdorff space X. We can identify the Hilbert $\mathcal{A}$–module $l_2(\mathcal{A})$ with the set $[X, H]$ of all continuous maps from the space X into the Hilbert space H. Denote the set of all maps from the space X into the space of bounded linear operators $\mathcal{B}(H)$, continuous in the strong topology, by $[X, \mathcal{B}(H)^s]$. We denote by $\mathcal{E}$ the map

$$\mathcal{E} : [X, \mathcal{B}(H)^s] \to End_{\mathcal{A}}(l_2(\mathcal{A})) \tag{55}$$

defined by the formula

$$(\mathcal{E}(T)(\varphi))(x) = T(x)\varphi(x), \tag{56}$$

where $T \in [X, \mathcal{B}(H)^s]$ is fixed, $x \in X$, and $\varphi \in l_2(\mathcal{A}) = [X, H]$ are arbitrary, and denote by $\mathcal{D}$ the map

$$\mathcal{D} : End_{\mathcal{A}}(l_2(\mathcal{A})) \to [X, \mathcal{B}(H)^s] \tag{57}$$

defined be the formula

$$(\mathcal{D}(B)(x))(a) = (B\varphi)(x), \tag{58}$$

where $B \in End_{\mathcal{A}}(l_2(\mathcal{A}))$ and the map $\varphi \in [X, H]$ is chosen so that $\varphi(x) \equiv a$, $a \in H$.

It was shown in the paper [3] that the definitions (55)–(58) are correct, the maps $\mathcal{E}$ and $\mathcal{D}$ are $\mathcal{A}$-isomorphisms and

$$\mathcal{E}\mathcal{D} = \mathbf{id}_{End_{\mathcal{A}}(l_2(\mathcal{A}))} \quad \mathcal{D}\mathcal{E} = \mathbf{id}_{[X,\mathcal{B}(H)^s]}. \tag{59}$$

Further, it was shown ibidem that each invertible $\mathcal{A}$–operator $S \in GL(l_2(\mathcal{A}))$ under the map $\mathcal{D}$ can be represented as a family of invertible operators $S_x \in GL(H)^s$ continuous in the strong topology such that $\sup_{x\in X} \|S_x^{-1}\| < \infty$, and, conversely, each family of invertible operators $S_x \in GL(H)^s$ continuous in the strong topology such that $\sup_{x\in X} \|S_x^{-1}\| < \infty$ is mapped by $\mathcal{E}$ into an invertible $\mathcal{A}$–operator.

In the paper [4] the F-topology was introduced in the space of Fredholm operators $\mathcal{F}(H)$.

Definition 7 [4] *The following sets form a subbase of the F-topology*

$$U_{\varepsilon,a_1,\dots,a_n,A} = \{B \in \mathcal{F}(H) \mid \|(B-A)a_i\| < \varepsilon \ \ \forall\ i = 1,\dots,n\},$$

$$U_{\varepsilon,V,A} = \{B \in \mathcal{F}(H) \mid \exists R \in GL(H),\ R(V) \subset V,\ \textit{such that}\ \|RB - A\| < \varepsilon\}.$$

Here V denotes a finite dimensional subspace of the Hilbert space H and $a_1,\dots,a_n \in H$.

Let $f : [0,1] \to \mathcal{F}(H)^F$ be any continuous map in the F–topology. Then, *index* $f(x) = const$. On the other hand, there exists a map $f : [0,1] \to \mathcal{F}(H)^s$ continuous in the strong topology such that *index* $f(0) \neq$ *index* $f(1)$ (see [4]), so the F–topology is strictly stronger than the strong topology in the space of Fredholm operators.

Let $F \in End_{\mathcal{A}}(l_2(\mathcal{A}))$ be any Fredholm $\mathcal{A}$–operator. Then for any $x \in X$ $(\mathcal{D}(F))(x) \in \mathcal{F}(H)$ and the map $\mathcal{D}(F) : X \to \mathcal{F}(H)^s$ is continuous in the strong topology. It was shown in [4] that the map

$$\mathcal{D}(F) : X \to \mathcal{F}(H)^F \xrightarrow{\mathbf{id}} \mathcal{F}(H)^s \subset \mathcal{B}(H)^s$$

is continuous in the F–topology and vice versa if a map $f : X \to \mathcal{F}(H)^F$ is continuous in the F–topology then the $\mathcal{A}$–operator $\mathcal{E}(f)$ is a Fredholm $\mathcal{A}$–operator. Thus, the map

$$\mathcal{D}|_{\mathcal{F}(l_2(\mathcal{A}))} : \mathcal{F}(l_2(\mathcal{A})) \to [X, \mathcal{F}(H)^F], \tag{60}$$

where $\mathcal{F}(l_2(\mathcal{A}))$ is the space of Fredholm operators over the algebra $\mathcal{A}$ and $[X, \mathcal{F}(H)^F]$ is the set of continuous maps from the space X into the space of Fredholm operators $\mathcal{F}(H)^F$, with the F-topology, is an isomorphism.

Denote by $\mathcal{U}(H)^s$ the space of unitary operators in H with the strong topology. Due to the formula

$$U(x)^{-1} - U(x_0)^{-1} = U(x)^{-1}\left(U(x_0) - U(x)\right)U(x_0)^{-1}, \tag{61}$$

we can assert that if a map $U : X \to \mathcal{U}(H)^s$ is continuous in the strong topology then the map $U^{-1} : X \to \mathcal{U}(H)^s$ is also continuous in the strong topology.

Theorem 6 *Let X be a compact Hausdorff space and maps $U : X \to \mathcal{U}(H)^s$, $F : X \to \mathcal{F}(H)^F$ are continuous in the strong topology and F-topology, respectively. Then the map $UFU^{-1} : X \to \mathcal{F}(H)^s \subset \mathcal{B}(H)^s$, given by the formula $UFU^{-1}(x) = U(x)F(x)U^{-1}(x)$, is continuous in the F-topology:*

$$UFU^{-1} : X \to \mathcal{F}(H)^F \xrightarrow{\mathbf{id}} \mathcal{F}(H)^s \subset \mathcal{B}(H)^s.$$

Proof. Since the $\mathcal{A}$–operators $\mathcal{E}(U)$, $\mathcal{E}\left(U^{-1}\right)$ are unitary $\mathcal{A}$–operators, and $\mathcal{E}(F)$ is Fredholm $\mathcal{A}$–operator then the operator $\mathcal{E}\left(UFU^{-1}\right) = \mathcal{E}(U)\mathcal{E}(F)\mathcal{E}\left(U^{-1}\right)$ is Fredholm $\mathcal{A}$–operator. Hence, by the isomorphism (60) the map $UFU^{-1} = \mathcal{D}\mathcal{E}\left(UFU^{-1}\right)$ is continuous in the F-topology. ∎

Let us consider the set of compact $\mathcal{A}$–operators $\mathcal{K}(l_2(\mathcal{A}))$. In the paper [6] has been considered the following class of compact operators $\mathcal{K}^*(l_2(\mathcal{A}))$. By definition (see [6]) an $\mathcal{A}$–operator $K : l_2(A) \to l_2(A)$ belongs to the set $\mathcal{K}^*(l_2(\mathcal{A}))$ iff

$$\lim_{n\to\infty} ||Kq_n|| = 0, \tag{62}$$

where the operator q_n is defined by the formula (6). It was shown in ([8], Prop. 2.2.1.) that the set $\mathcal{K}^*(l_2(\mathcal{A}))$ coincides with the closure of the set of linear combinations of elementary operators $\theta_{x,y}(z) := x < y, z >$, where $x, y, z \in l_2(\mathcal{A})$. Hence, any $K \in \mathcal{K}^*(l_2(\mathcal{A}))$ automatically admits the adjoint operator. On the other hand, our notion of compact $\mathcal{A}$–operator does not demand existing of adjoint operator unlike that was assumed in many papers on KK–theory and so we shall distinguish the set $\mathcal{K}(l_2(\mathcal{A}))$ of all compact $\mathcal{A}$–operators and the subset $\mathcal{K}^*(l_2(\mathcal{A})) \subset \mathcal{K}(l_2(\mathcal{A}))$ of compact $\mathcal{A}$–operators which admit adjoint operator.

Theorem 7 [4] *A compact $\mathcal{A}$–operator K admits adjoint operator, i.e. $K \in \mathcal{K}^*(l_2(\mathcal{A}))$, iff the map*

$$\mathcal{D}(K) : X \to \mathcal{K}(H)^u \xrightarrow{\mathbf{id}} \mathcal{K}(H)^s \subset \mathcal{B}(H)^s$$

is continuous in the uniform topology.

The following example shows that there exists a self-adjoint family of compact operators continuous in the strong topology such that the corresponding $\mathcal{A}$–operator does not belong to the set $\mathcal{K}(l_2(\mathcal{A}))$.

Example. Let $X = \{0\} \cup \bigcup_{i=1}^{\infty}\{\frac{1}{i}\} \subset \mathbf{R}$. We define the map $K : X \to \mathcal{K}(H)$ by the following formula

$$K\left(\frac{1}{i}\right)(\xi) = -\xi_i, \qquad K(0) = 0, \tag{63}$$

where $\xi = (\xi_1, \xi_2, \ldots)$ is an element of the standard Hilbert space. Then $K\left(\frac{1}{i}\right)(\xi) \to 0$ as $i \to \infty$ for every $\xi \in H$. But $\mathcal{E}(K) \notin \mathcal{K}(l_2(\mathcal{A}))$. Indeed, if we suppose the contrary then by the theorem 4 the operator $\mathbf{id} + \mathcal{E}(K)$ is a

Fredholm $\mathcal{A}$–operator. Due to the isomorphism (60) the map $\mathcal{D}(\mathbf{id}+\mathcal{E}(K)) = \mathbf{id}+K : X \to \mathcal{F}(H)^F$ is continuous in the F-topology. But for any invertible operator S we have

$$\left\| S\left(\mathbf{id}+K\left(\frac{1}{i}\right)\right) - (\mathbf{id}+K(0))\right\| \geq \left\| \left(S\left(\mathbf{id}+K\left(\frac{1}{i}\right)\right) - \mathbf{id}\right)(e_i)\right\| = 1.$$

That means that the map $\mathbf{id}+K : X \to \mathcal{F}(H)^F$ is not continuous in the F-topology. $\blacksquare$

Thus, the example poses the problem of finding a topology in the space of compact operators $\mathcal{K}(H)$ such that any family continuous in this topology forms a compact $\mathcal{A}$–operator, and vice versa, any compact $\mathcal{A}$–operator maps by the map $\mathcal{D}$ to a family of compact operators continuous in the sought topology.

We define a new IM-topology in the space of compact operators $\mathcal{K}(H)$ in the following way. Let

$$U_{\varepsilon,a_1,\dots,a_n,K} = \{B \in \mathcal{K}(H) \mid \|(B-K)a_i\| < \varepsilon \ \ \forall\, i = 1,\dots,n\},$$

$$U_{\varepsilon,n,S,K} = \{B \in \mathcal{K}(H) \mid \exists R \in GL(H),\ \textit{such that}$$
$$\|R(S+Q_nB) - (S+Q_nK)\| < \varepsilon\},$$

where $\varepsilon > 0$, $S \in GL(H)$, and $Q_n : H = (L_n)^\perp \oplus L_n \longrightarrow (L_n)^\perp \subset H$ is the orthogonal projection along the subspace L_n spanned by the first n orthonormal basis vectors $e_1,\dots,e_n$.

Definition 8 *As a subbase of the IM-topology we take the following sets*

$$U_{\varepsilon,a_1,\dots,a_n,K} \quad \textit{and} \quad U_{\varepsilon,S,K} := \bigcap_{n=0}^{\infty} U_{\varepsilon,n,S,K}.$$

Remark 1 *It follows from the definition of IM–topology that the identity map*

$$\mathcal{K}(H)^{IM} \xrightarrow{\mathbf{id}} \mathcal{K}(H)^s \subset \mathcal{B}(H)^s$$

from the space of compact operators with the IM–topology to the same space with the strong topology is continuous. Since any sets $U_{\varepsilon,a_1,\dots,a_n,K}$ and $U_{\varepsilon,S,K}$ contain the ball $B(K,\varepsilon) = \{Z \in \mathcal{K}(H) \mid \|Z-K\| < \varepsilon\}$, then the map

$$\mathcal{K}(H)^u \xrightarrow{\mathbf{id}} \mathcal{K}(H)^{IM}$$

from the space of compact operators with the norm topology to the same space with the IM–topology is continuous.

Theorem 8 *An $\mathcal{A}$–operator K is compact operator, i.e. $K \in \mathcal{K}(l_2(\mathcal{A}))$, iff the map*

$$\mathcal{D}(K) : X \to \mathcal{K}(H)^{IM} \xrightarrow{\mathbf{id}} \mathcal{K}(H)^s \subset \mathcal{B}(H)^s$$

is continuous in the IM-topology.

Proof. Let $K \in \mathcal{K}(l_2(\mathcal{A}))$. We have to prove that the map $\mathcal{D}(K)$ is continuous map from X to $\mathcal{K}(H)$ with the IM–topology. For, it is sufficient to show that for any $\varepsilon > 0$, for any $S \in GL(H)$, and for any $x_0 \in X$ there exists a neighbourhood $U_{x_0} \subset X$, $x_0 \in U_{x_0}$ such that for any $n \geq 0$

$$\mathcal{D}(K)\,(U_{x_0}) \subset U_{\varepsilon,n,S,\mathcal{D}(K)(x_0)}. \tag{64}$$

Let $l_2(\mathcal{A}) = L_{n,\mathcal{A}}^{\perp} \oplus L_{n,\mathcal{A}}$ be an orthogonal decomposition which is given by a pair of projectors p_n, q_n, $p_n + q_n = \mathbf{id}$, $\mathbf{Im}\, p_n = L_{n,\mathcal{A}}$, $\mathbf{Im}\, q_n = L_{n,\mathcal{A}}^{\perp}$, $\mathcal{D}(q_n)(x) \equiv Q_n$.

Let $s : X \to GL(H)$ be a constant map, $s(x) \equiv S \in GL(H)$, and $\hat{S} = \mathcal{E}(s) \in GL(\mathcal{A})$, i.e. $\mathcal{D}(\hat{S})(x) \equiv S \in GL(H)$.

Let us choose $n \in \mathbf{N}$ such that for all $m > n$ the $\mathcal{A}$–operator $G_m = \hat{S} + q_m K$ is invertible, i.e.

$$G_m = \hat{S} + q_m K \in GL(l_2(\mathcal{A})). \tag{65}$$

Then for any $x \in X$ we have

$$\mathbf{id} = \mathcal{D}(G_m^{-1} G_m)(x) = \mathcal{D}(G_m^{-1})(x)(S + Q_m \mathcal{D}(K)(x)).$$

If we put $R := \mathcal{D}(G_m)(x_0)\mathcal{D}(G_m^{-1})(x) \in GL(H)$ then for any $x \in X$ and $m > n$ we have

$$||\mathcal{D}(G_m)(x_0)\mathcal{D}(G_m^{-1})(x)\,(S + Q_m\mathcal{D}(K)(x)) - (S + Q_m\mathcal{D}(K)(x_0))\,|| = 0.$$

The last equality means that $\forall x \in X$, $\forall m > n$, and $\forall \varepsilon > 0$

$$\mathcal{D}(K)(x) \in U_{\varepsilon,m,S,\mathcal{D}(K)(x_0)}.$$

Since K is a compact $\mathcal{A}$–operator then by the corollary 3 for any Fredholm $\mathcal{A}$–operator F the operator $F + q_l K$ is a Fredholm $\mathcal{A}$–operator. In particular, the operator $\hat{S} + q_l K$ is a Fredholm $\mathcal{A}$–operator for all $l \geq 0$. By the isomorphism (60), the maps $\mathcal{D}(\hat{S} + q_l K) : X \to \mathcal{F}(H)^F$, $\mathcal{D}(\hat{S} + q_l K)(x) = S + Q_l\mathcal{D}(K)(x)$, $l = 1, \ldots, n$, are continuous in the F-topology. Hence, for any fixed finite dimensional subspace $V_l \subset H$, $l = 1, \ldots, n$, there exists a neighbourhood $U_{x_0}^l \subset X$ such that for any $x \in U_{x_0}^l$

$$\mathcal{D}(\hat{S} + q_l K)(x) \in U_{\varepsilon,V_l,S+Q_l\mathcal{D}(K)(x_0)},$$

where $U_{\varepsilon,V_l,S+Q_l\mathcal{D}(K)(x_0)}$ is an open set in F-topology. This means that $\exists R \in GL(H)$, $R(V_l) \subset V_l$, such that

$$||R\,(S + Q_l\mathcal{D}(K)(x)) - (S + Q_l\mathcal{D}(K)(x_0))\,|| < \varepsilon,$$

i.e.

$$\mathcal{D}(K)(x) \in U_{\varepsilon,l,S,\mathcal{D}(K)(x_0)}.$$

Then, if we put $U_{x_0} := \bigcap_{l=1}^{n} U_{x_0}^l$ we obtain the necessary condition (64). This proves that the map $\mathcal{D}(K) : X \to \mathcal{K}(H)^{IM} \xrightarrow{\mathbf{id}} \mathcal{K}(H)^s \subset \mathcal{B}(H)^s$ is continuous in the IM-topology.

To prove the inverse assertion of the theorem let us suppose the contrary. This means that there exists a continuous map $C : X \to \mathcal{K}(H)^{IM}$ such that the operator $\mathcal{E}(C)$ is not a compact $\mathcal{A}$-operator, i.e. $\mathcal{E}(C) \notin \mathcal{K}(l_2(\mathcal{A}))$. Due to the theorem 2 there exist a number $c_1 > 0$ and an increasing sequence of natural numbers $\{n_i\}_{i\in\mathbf{N}}$ such that

$$||q_{n_i}\mathcal{E}(C)|| > c_1. \tag{66}$$

Since

$$||q_{n_i}\mathcal{E}(C)|| = \sup_{x\in X} ||Q_{n_i}C(x)||, \tag{67}$$

then there exist an element $x_i \in X$ and a vector $v^i \in H$, $||v^i|| = 1$, such that

$$\left\|Q_{n_i}C(x_i)(v^i)\right\| > c_1. \tag{68}$$

Let

$$c_0 := \frac{c_1}{2} \quad and \quad \varepsilon_0 := \min\left(c_0, \frac{c_0^2}{||\mathcal{E}(C)||}\right). \tag{69}$$

Since the map C is continuous in the IM–topology which is stronger than the strong topology then we can conclude that $||\mathcal{E}(C)|| < \infty$ and $\varepsilon_0 > 0$.

We assert that we can choose from the sequence $\{x_i\}_{i\in\mathbf{N}}$ a subsequence $\{y_i\}_{i\in\mathbf{N}}$, an increasing subsequence of natural numbers $\{r_i\}_{i\in\mathbf{N}}$, and an orthonormal sequence of vectors $w_i \in H$, $i \in \mathbf{N}$, such that

$$||Q_{r_i}C(y_i)(w_i)|| > c_0 \tag{70}$$

and

$$||Q_{r_i}C(y_j)(w_j)|| < \frac{\varepsilon_0}{2^{i+2}} \quad for\ all \quad j < i. \tag{71}$$

We shall prove our assertion by mathematical induction. Let us put $y_1 := x_1$ and $w_1 := v^1$. Let us suppose that k points $y_1, \ldots, y_k \in X$ and k orthonormal vectors $w_1, \ldots, w_k \in H$ have already been chosen such that the conditions (70) and (71) hold. We consider $2k$ functions $\varphi_i : X \to H$, $\varphi_i(x) \equiv w_i$, and $\mathcal{E}(C)\varphi_i$, $(\mathcal{E}(C)\varphi_i)(x) = C(x)w_i$, $i = 1, \ldots, k$, as elements of $\mathcal{A}$–module $l_2(\mathcal{A}) = [X, H]$. For any $\varepsilon > 0$ there exists a natural number $N(\varepsilon)$ such that for any natural number $l > N(\varepsilon)$ the inequality

$$||q_l\mathcal{E}(C)\varphi_j|| < \varepsilon \tag{72}$$

holds for $j = 1, \ldots, k$. Let $n_m > N\left(\frac{\varepsilon_0}{\sqrt{k}2^{k+3}}\right)$ and $H_1 := span < w_1, \ldots, w_k >$. Let $v^m = v_1^m + v_2^m$, $v_1^m \in H_1$, $v_2^m \in (H_1)^\perp$, be a representation of the vector v^m in accordance with the decomposition $H = H_1 \oplus (H_1)^\perp$. We have

$$c_1 < ||(Q_{n_m}C(x_m))\,(v^m)|| = ||(Q_{n_m}C(x_m))\,(v_1^m + v_2^m)|| \leq \frac{\sqrt{k}\varepsilon_0}{\sqrt{k}2^{k+3}}\,||v_1^m|| + ||(Q_{n_m}C(x_m))\,(v_2^m)||. \tag{73}$$

If we put $y_{k+1} := x_m$, $w_{k+1} := \frac{v_2^m}{\|v_2^m\|}$, and $r_{k+1} := n_m$, then by (73) we obtain the inequality (70) for $i = k+1$:

$$\left\| \left(Q_{r_{k+1}} C(y_{k+1})\right)(w_{k+1}) \right\| \geq \|(Q_{n_m} C(x_m))(v_2^m)\| > c_1 - \frac{\varepsilon_0}{2^{k+3}} > c_0. \quad (74)$$

The inequality (71) for $i = k+1$ follows from the inequality (72):

$$\left\| \left(Q_{r_{k+1}} C(y_j)\right)(w_j) \right\| \leq \|q_{n_m} \mathcal{E}(C)\varphi_j\| \leq \frac{\varepsilon_0}{\sqrt{k} 2^{k+3}}, \quad j \leq k.$$

Let $a = (a_1, a_2, \ldots) \in H$, $\sum_{i=1}^{\infty} a_i \bar{a}_i = 1$. Let us estimate the norm of the element $\sum_{i=s}^{t} a_i \cdot (Q_{r_i} C(y_i))(w_i) \in H$. We have

$$\left\| \sum_{i=s}^{t} a_i \cdot (Q_{r_i} C(y_i))(w_i) \right\|^2 = \left(\sum_{i=s}^{t} a_i \cdot (Q_{r_i} C(y_i))(w_i), \sum_{i=s}^{t} a_i \cdot (Q_{r_i} C(y_i))(w_i) \right) =$$

$$\sum_{i=s}^{t} |a_i|^2 \left(Q_{r_i} C(y_i)(w_i), Q_{r_i} C(y_i)(w_i)\right)$$

$$+2 \sum_{s \leq i < j \leq t} \mathbf{Re} \left(a_i \cdot (Q_{r_i} C(y_i))(w_i), a_j \cdot (Q_{r_j} C(y_j))(w_j)\right) \leq$$

$$\|\mathcal{E}(C)\|^2 \sum_{i=s}^{t} |a_i|^2 + 2 \sum_{s \leq i < j \leq t} \mathbf{Re} \left(a_i \cdot (Q_{r_j} C(y_i))(w_i), a_j \cdot (Q_{r_j} C(y_j))(w_j)\right) \leq$$

$$\|\mathcal{E}(C)\|^2 \sum_{i=s}^{t} |a_i|^2 + 2\|\mathcal{E}(C)\| \sum_{i=s}^{t-1} \sum_{j=i+1}^{t} \frac{\varepsilon_0}{2^{j+2}} =$$

$$\|\mathcal{E}(C)\|^2 \sum_{i=s}^{t} |a_i|^2 + 2\|\mathcal{E}(C)\|\varepsilon_0 \left(\frac{1}{2^{s+1}} - \frac{t-s+2}{2^{t+2}} \right), (75)$$

and

$$\left\| \sum_{i=1}^{t} a_i \cdot (Q_{r_i} C(y_i))(w_i) \right\|^2 \geq$$

$$\left| c_0^2 \sum_{i=1}^{t} |a_i|^2 - 2 \sum_{1 \leq i < j \leq t} \mathbf{Re} \left(a_i \cdot (Q_{r_j} C(y_i))(w_i), a_j \cdot (Q_{r_j} C(y_j))(w_j)\right) \right| \geq$$

$$\left| c_0^2 \sum_{i=1}^{t} |a_i|^2 - 2\|\mathcal{E}(C)\| \frac{\varepsilon_0}{4} \right| \geq \frac{c_0^2}{2}. \quad (76)$$

Let us choose from the sequence $\{y_i\}_{i \in \mathbf{N}}$ a subsequence $\{y_{l(i)}\}$ such that both closures of the subspaces W and $C(W)$ spanned by the vectors $\{w_{l(i)}\}_{i \in \mathbf{N}}$ and $\{Q_{r_{l(i)}} C(y_{l(i)})(w_{l(i)})\}_{i \in \mathbf{N}}$ respectively, have infinite codimension. Due to the inequalities (75) and (76) the map

$$S|_W : W \to C(W), \qquad S(w_{l(i)}) = Q_{r_{l(i)}} C(y_{l(i)})(w_{l(i)}),$$

is bounded and bijective. Hence, by theorem III.11 of [9] the map $S|_W$ is an isomorphism, and we can extend it to an invertible operator $S : H \to H$ by choosing any isomorphism between orthogonal complements of the spaces W and $C(W)$.

Since X is a compact space then there exists a point $x_0 \in X$ such that for any open neighbourhood U_{x_0} of the point x_0 there are infinitely many members of the subsequence $\{y_{l(i)}\}_{i\in\mathbf{N}}$ lying in U_{x_0}. For any open in IM–topology neighbourhood $U_{\frac{1}{d},-S,C(x_0)}$ of the operator $C(x_0)$, $d \in \mathbf{N}$, we denote by $U^d_{x_0} \subset X$ the open neighbourhood

$$U^d_{x_0} := C^{-1}\left(U_{\frac{1}{d},-S,C(x_0)}\right)$$

of the point $x_0 \in X$.

Let $z_d := y_{l(i_d)} \in U^d_{x_0} \cap \{y_{l(i)}\}_{i\in\mathbf{N}}$ which exists by the choice of the point $x_0 \in X$. Then there exists a sequence of invertible operators $G_d \in GL(H)$ such that

$$\begin{aligned}\frac{1}{d} > \left\| G_d\left(-S+Q_{r_{l(i_d)}}C(z_d)\right) - \left(-S+Q_{r_{l(i_d)}}C(x_0)\right)\right\| \geq \\ \left\|\left(G_d\left(-S+Q_{r_{l(i_d)}}C(z_d)\right) - \left(-S+Q_{r_{l(i_d)}}C(x_0)\right)\right)(w_{l(i_d)})\right\| = \\ \left\|\left(-S+Q_{r_{l(i_d)}}C(x_0)\right)(w_{l(i_d)})\right\|. \end{aligned} \tag{77}$$

Hence, for sufficiently large d

$$||C(x_0)(w_{l(i_d)})|| \geq \frac{1}{2}||S(w_{l(i_d)})|| \geq \frac{c_0}{2} \tag{78}$$

But the inequality (78) contradicts the condition that $C(x_0)$ is a compact operator. ∎

Corollary 4 *The uniform topology is stronger than the IM–topology.*

Proof. Let $X = \{0\} \cup \bigcup_{i=1}^{\infty}\{\frac{1}{i}\} \subset \mathbf{R}$ and $\mathcal{A} = C(X)$. We shall construct an $\mathcal{A}$–operator $K \in \mathcal{K}(l_2(\mathcal{A}))$ such that the map $\mathcal{D}(K) : X \to \mathcal{K}(H)^{IM} \xrightarrow{\mathbf{id}} \mathcal{K}(H)^s \subset \mathcal{B}(H)^s$ is not continuous in the uniform topology but by the theorem 8 is continuous in the IM–topology.

Let us define continuous functions $\varphi_i : X \to \mathbf{C}$, $i \in \mathbf{N}$, by the following rule

$$\varphi_i\left(\frac{1}{i}\right) = 1, \quad \varphi_i\left(\frac{1}{j}\right) = 0, \quad for \quad j \neq i, \quad \varphi_i(0) = 0.$$

We define the $\mathcal{A}$–operator $K : l_2(\mathcal{A}) \to l_2(\mathcal{A})$ by the following formula

$$K(\xi) = \left(\sum_{i=1}^{\infty}\varphi_i\xi_i, 0, 0, \ldots\right), \tag{79}$$

where $\xi = (\xi_1, \xi_2, \ldots)$ is an element of the $l_2(\mathcal{A})$, $\xi_i \in \mathcal{A}$, $i = 1\ldots,\infty$. Then $K \in \mathcal{K}(l_2(\mathcal{A}))$ and hence the map $\mathcal{D}(K) : X \to \mathcal{K}(H)^{IM} \xrightarrow{\mathbf{id}} \mathcal{K}(H)^s \subset \mathcal{B}(H)^s$ is continuous in the IM–topology.

By definition (79) of the $\mathcal{A}$–operator K we have

$$\mathcal{D}(K)(0) = 0 \in \mathcal{B}(H).$$

But

$$\left\| \mathcal{D}(K)\left(\frac{1}{i}\right) \right\| \geq \left\| \mathcal{D}(K)\left(\frac{1}{i}\right)(e_i) \right\| = 1,$$

where $e_i \in H, = 1, \ldots, \infty$ is the standard basis of H. That means that the map $\mathcal{D}(K) : X \to \mathcal{K}(H)^{IM} \xrightarrow{\mathbf{id}} \mathcal{K}(H)^s \subset \mathcal{B}(H)^s$ is not continuous in the uniform topology. ∎

Now, let us discuss the representing space for **K**-theory introduced in the paper [2]. In this paper M. Atiyah and G. Segal have considered locally trivial bundles $P \to X$ whose fibers $P_x = \mathbf{P}(H)$ are the projective space of a separable infinite dimensional complex Hilbert space H and structural group is the projective unitary group $\mathcal{PU}(H)^{c.o}$, with the compact-open topology. With the aim to define a twisted **K**-theory they need to replace fiber P_x by a representing space for **K**-theory such that the structural group $\mathcal{PU}(H)^{c.o}$ acts continuously on it by conjugation (ibid. sect.3, p.12). It is well known (see [1] and [5]) that the space $\mathcal{F}(H)^u$ of Fredholm operators in H with the uniform topology is a representing space for **K**-theory. Unfortunately, the unitary group $\mathcal{U}(H)^{c.o}$ (and $\mathcal{PU}(H)^{c.o}$), with the compact-open topology, does not act continuously on $\mathcal{F}(H)^u$ by conjugation. To surmount this obstacle M. Atiyah and G. Segal have suggested (ibid.) to use as a representing space for **K**-theory the following set

$$Fred'(H) = \{(A,B) \in \mathcal{F}(H) \times \mathcal{F}(H) \mid AB - I \in \mathcal{K}(H) \quad and \quad BA - I \in \mathcal{K}(H)\}$$

with the topology induced by the embedding

$$Fred'(H) \hookrightarrow \mathcal{B}(H)^{c.o} \times \mathcal{B}(H)^{c.o} \times \mathcal{K}(H)^u \times \mathcal{K}(H)^u$$

$$(A,B) \to (A, B, AB - I, BA - I),$$

where $\mathcal{B}(H)^{c.o}$ is the space of bounded operators in H, with the compact-open topology, and $\mathcal{K}(H)^u$ is the space of compact operators in H, with the uniform topology. Let X be a compact space. In this case, by Banach-Steinhaus theorem (see [9] Theorem III.9) the continuous maps $X \to \mathcal{B}(H)$ are the same for the compact-open and for the strong operator topologies. Then any continuous map $f : X \to Fred'(H)$ of compact space X into $Fred'(H)$ can be considered as a pair of continuous maps in the strong operator topology

$$A^f : X \to \mathcal{F}(H)^s, \tag{80}$$

$$B^f : X \to \mathcal{F}(H)^s \tag{81}$$

such that for any $x \in X$ the operators $A^f(x)B^f(x) - I$ and $B^f(x)A^f(x) - I$ are compact and the maps

$$A^f B^f - I : X \to \mathcal{K}(H)^u, \tag{82}$$

$$B^f A^f - I : X \to \mathcal{K}(H)^u \tag{83}$$

are continuous in the uniform topology.

Now, we can relax the conditions (82) and (83) to strictly extend the set of admitted maps A^f, in the following way. The maps

$$A^f B^f - I : X \to \mathcal{K}(H)^{IM}, \tag{84}$$

$$B^f A^f - I : X \to \mathcal{K}(H)^{IM} \tag{85}$$

are continuous in the IM-topology. Indeed, by the theorem 8 we have

$$\mathcal{E}(A^f B^f - I), \quad \mathcal{E}(B^f A^f - I) \in \mathcal{K}(l_2(\mathcal{A})), \tag{86}$$

By the theorems 4 and 5 the $\mathcal{A}$–operators $\mathcal{E}(A^f)$ and $\mathcal{E}(B^f)$ are Fredholm $\mathcal{A}$–operators. Due to the isomorphism (60) the maps

$$A^f : X \to \mathcal{F}(H)^F, \tag{87}$$

$$B^f : X \to \mathcal{F}(H)^F \tag{88}$$

are continuous in the F-topology. Hence, the class of continuous maps $X \to \mathcal{F}(H)^F$ is strictly wider than the class of continuous maps $A^f : X \to \mathcal{F}(H)^s$ for which the conditions (80), (81), (82), (83) hold. Moreover, it was shown in the paper [4] that the space $\mathcal{F}(H)^F$ of Fredholm operators, with the F-topology, is a representing space for **K**-theory. Taking in account the theorem 6, we conclude, that we can take the space $\mathcal{F}(H)^F$ as a representing space for **K**-theory in the construction of the twisted **K**-theory.

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