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Sub-Plenary Lecture

SP: Reasoning, proof and proving in mathematics education

Regular Lecture based on the work of Survey Team 2

Maria Alessandra Mariotti
University of Pisa, Italy

Team Chair: *Maria Alessandra Mariotti*, University of Pisa, Italy
Team Members: *Celia Hoyles*, University of London, United Kingdom
Hans Niels Jahnke, University of Duisburg-Essen, Essen, Germany
A.S. Mishchenko, Moscow State University, Russia
Ren, Zizhao, National Education Examinations Authority, Beijing, P.R. China

"A Mathematician is a machine for turning coffee into theorems"
(Paul Erdös, 1913-1996)

Introduction

For a long time, mathematical proof has been at the core of an active debate in the community of mathematics educators: often blamed as responsible for pupils' difficulties, but also recognised as a crucial aspect of mathematics activity.

In the recent past the role and the place that proof occupies in the mathematics curriculum have often changed. For instance, in the United States, after a period of 'banishment' proof has got a central position in the Principles and Standards (Knuth, 2000).

> "Reasoning and Proof as fundamental aspects of mathematics.
> Reasoning and Proof are not special activities reserved for special times or special topics in the curriculum but should be a natural, ongoing part of classroom discussions, no matter what topic is being studied."
> (Principles and standards for school mathematics, NCTM, p. 342)

Nevertheless, certainly the idea of "proof for all" is not one that most teachers endorse, and even where there is a longstanding tradition of including proof in the curriculum (for instance in my own country, Italy, but also in France, and in Japan), the considerable difficulties encountered have lead many teachers to abandon this practice.

Thus the debate is certainly still open. In our opinion there are at least three main questions to be addressed:

- Is proof so crucial in the mathematics culture that it is worthwhile to include it in school curricula?
- What are the meanings of proof and proving in school mathematics and how are these meanings introduced into curricula in different countries? Important aspects include students' conceptions on proof, students' achievements, and teachers' conceptions on proof.
- How has research in mathematics education approached the issue of proof. In particular, is it possible to overcome the difficulties in introducing pupils to proof so often described by teachers?

Starting with a brief discussion on the status of proof in the mathematics culture we will attempt to provide a quick overview of proof in the reality of schools. A few snapshots from recent research studies will be presented together with possible directions for the future.

Proof in the mathematics culture

A historical and epistemological analysis serves to highlight the role of proof in the evolution and systematisation of mathematics knowledge throughout the centuries. Mathematics cannot be reduced to theoretical systems, but certainly its theoretical nature constitutes a fundamental component of it, as clearly expressed by Hilbert and Cohn Vosssen in the introduction to their book *Geometry and the imagination*.

> "In mathematics … we find two tendencies present.
> On the one hand, the tendency towards abstraction seek to crystallize the logical relations in the maze of material that is being studied, and to correlate the material in a systematic and orderly manner.
> On the other hand, the tendency towards intuitive understanding foster a more immediate grasp of the objects, a live rapport with them, so to speak, which stress the correct meaning of their relations."
> (Hilbert & Cohn Vossen, 1999)

A dual nature characterises mathematics: on the one hand intuitive understanding and on the other hand systematic order within logical relations.

Actually theoretical perspectives in mathematics have old roots. This led us to the classic book Euclid's Elements and its particular way, the deductive way, of presenting the 'corpus' of knowledge which has characterised mathematics exposition since then. Heath, in his edition of the Euclid's Elements, reports the following passage from Proclus.

> "Now it is difficult, in each science, both to select and arrange in due order the elements from which all the rest proceeds, and into which all the rest is resolved. (…) In all these ways Euclid's system of elements will be found to be superior to the rest."
> (Heath, 1956, vol. I p. 115-116)

The crucial point seems to be the appropriate order in which a set of known properties should be expressed and communicated. The problem of the transmission of knowledge was solved by Euclid in a very peculiar way. The elements were transmitted according to"logical arguments". This method soon became the style of rationality, not only in mathematics, but also more generally for discourses in any 'science' (Vegetti, 1983), This is not the case of other cultures, for instance in China, as we shall see in the following.

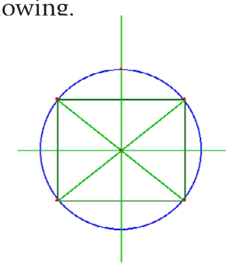


Figure 1.
The figure itself *was* the whole theory.

Let us briefly sketch the story, giving some examples from the history.

1. A historical reconstruction of geometrical argumentations in early Greek mathematics of the pre-Euclidean period (Becker 1975, 24/5) hypothesises that the theorems which Thales (ca. 600 B.C.) knew – such as "A circle is bisected by any of its diameters", "If two straight lines intersect the opposite angles are equal", "The angles at the base of any isosceles triangle are equal", "The diagonals of a rectangle are equal and bisect each other, that is, an angle inscribed in a semicircle is a right angle." – were summarized in a single figure. Apparently, the theorems weren't proved in a Euclidean sense, starting from axioms and definitions. Rather, the figure itself provided the conditions of their validity and, by symmetry, the mode of reasoning. That is, the figure itself *was* the whole theory. One could call this an early and pronounced example of Peirce's idea of 'diagrammatic reasoning' (Dörfler, 2005).

2. With Euclid's *Elements* (ca. 300 B.C.) Greek geometry and number theory was transformed into a deductive system and the very notion of proof in a modern sense came into being. Theorems have to be derived by purely logical conclusions from axioms and other theorems which have been proven before. Nevertheless, Euclid's notion of proof cannot be separated from his practice of working with figures. This is especially true of relations of incidence and position, as pointed out in the 19th century (cfr. the work of Pasch), but it is also true of other aspects of Euclid's theory (Netz, 2000).

3. In Chinese mathematics, as found in the compilation *Nine chapters on mathematical procedures* and its commentary by Liu Hui (ca. 300 A.D.), algorithmic procedures and theoretical arguments are inseparably linked. As Chemla (1996) has shown analysing of the procedures for determining the area of a circle, 'proof' cannot be separated from calculation as in Greek mathematics: in Chinese mathematics, processes of argumentation are dependent on the specific mathematical practice.

4. With the emergence and development of symbolic algebra in the renaissance and early modern times the established notion of proof was again substantially modified. Isaac Newton, for instance, saw no problem in 'proving' the rule that the integral of x^n is equal to $(n+1)^{-1} x^{n+1}$ simply by working through a numerical example. Nor did he hesitate to state the rule without specifically acknowledging the exception $n = -1$, since it could easily be seen that the formula does not apply in this case (Newton, 1667, 206 ff). Also, he did not prove the right implication. Rather, he showed that the derivative of $(n+1)^{-1} x^{n+1}$ is x^n. It became accepted practice in analysis and algebra in the 17th and 18th centuries that theorems might "suffer exceptions" which, as a rule, one does not need to point out. Mathematical thinking was dominated by manipulations of *indeterminates* and the accepted notion of proof reflected this practice. A valid proof was nothing else than a correct manipulation of algebraic symbols. It was not before the 19th century that – under the influence of Cauchy and Weierstrass – the domain of validity of a theorem was exactly specified, and the modern notion of proof in analysis and algebra emerged step by step. Again, we see that the notion of proof is dependent of the way mathematics is practised. It is plausible to think that Peirce's concept of diagrammatic reasoning can be applied to this case, too.

5. Nowadays, proof plays a different role in different sub-disciplines of mathematics. Of course, there are broad areas of mathematics in which the role and meaning of proof is unquestioned and adequately described by the Euclidean scheme (with Hilbertian refinements). However, with the growing role of computers we have witnessed new developments with new types of proof in certain areas. We mention only the computer proof of the four colour theorem and "zero-knowledge proofs". Above all, there is a growing amount of purely experimental work in mathematics with publications containing results which are suggested by computer experiments, but in many cases not proven (cf. the controversy between Jaffe & Quinn (1993) and Thurston (1994)). Today, numerical analysis contains a large amount of numerical experiments and a lot of algorithms whose optimality and limits of validity are not proven. Surely, numerical analysts try to prove as many results as possible, but the requirements of numerical practice are so extensive that a restriction to proved algorithms is not possible.

6. In applied disciplines such as, say, theoretical physics, the meaning of a proof might be different from its usual meaning in pure mathematics. Consider again the example of Isaac Newton. When he derived Kepler's laws of planetary motion from his supposed law of mass attraction he based empirically well-established laws upon an uncertain hypothesis. At his time, Newton's gravitational law was far from being generally accepted or just plausible. Thus, Newton's proof did not transfer truth from the assumption (the law of gravitation) to the conclusion (Kepler's laws) as is the notion in established fields of mathematics. On the contrary, the gravitational law was justified by Newton's proof because one could deduce Kepler's laws from it. Thus, what is proved may serve to legitimise the assumptions from which it is derived. On the other hand, Newton's proof put Kepler's laws in a broader theoretical context and, by this, made it possible to draw new conclusions, explain additional empirical facts, and formulate new predictions. The proof was a medium of *generalisation*. This encompasses also an effect of a more qualitative nature in that it opens up a *new perspective* on Kepler's laws. Their status was changed from a purely kinematical description to a dynamic view of nature.

Thus, epistemological and historical analyses show a rich variety of meanings and uses of mathematical proof, to which corresponds a complexity in the educational field, where epistemological distinctions, articulating different *functions of proof* (Bell, 1976; Hanna, 2000; de Villiers, 1990), have proved useful and have found a shared consensus across different studies.

There is a dialectical relationship between proof in the scientific practice of mathematics and proof in the educational realm. By division of labour most of the epistemological problems of proof are settled when a mathematician attempts to prove a theorem. Mathematicians seem to be mainly concerned with the *mathematical complexity* of the theorem in question. Of course, they have to evaluate proofs and theorems, but coming to terms with the mathematical complexity is the foremost problem. In the educational realm, however, it is the *epistemological complexity* which matters. For the students proof is above all a problem of meaning, and educators have to devise teaching contexts which make proof meaningful to them. We will come back to this point in the following.

Proof in the curriculum. A comprehensive account of proof at school

It is difficult to have a complete overview of the situation in the different countries; in most cases, no specific studies are available; what we can provide are snapshots coming from the most complete large-scale study carried out in the recent past (Hoyles, 1997; Hoyles & Küchemann 2003).

Proof cannot be considered as any other mathematics topic, such as trigonometry or functions. Thus very rarely one can find "proof" explicitly listed among other topics, in the official programs of national Curricula. The position and the status of proof in education is a complex issue and ideas about it can emerge only from an accurate analysis across different topics and the specific guidelines accompanying the list of the mathematics contents related to them. However, even more can be drawn from direct observation of classroom activities. As discussed above, different aspects (functions) of proof may emerge in different mathematical contexts and in relation to different social interactions among the basic elements of the didactic system. Actually, the presence of an explicit reference to proof is a crucial point. On the other hand the lack of such an explicit reference may not mean that proof is not required, or it is not much valued. Nevertheless, the absence of mentioning proofs might be a hint of the fact that the didactic issue of proof is not in focus. It is impossible here to sketch the story of proof in curricula, although this could be of great interest. Let me share with you some general observations and remarks.

In any country, and in any moment of history, the mathematical curriculum reflects the cultural changes of society, and of the mathematical community in particular (Chevallard, 1995). Of course, different positions can be found in different countries, and changes are sensible only over a long period. In fact, the effects of research studies begin to appear in the attempt at shaping national curricula according to experts' suggestions.

Consider, for instance, the impact of the Bourbaki inspired movement in the sixties: it is possible to notice how the education systems were affected in different countries. For instance, in some quarters in Russia (the Soviet Union at that time) a stormy enthusiasm of over-indulgence into rigorous exposition of material in school mathematics arose at the end of the 1960's, and a radical grandiose reform of the school mathematical education was realized. Old teachers were driven away, new textbooks were written in the manner of Bourbaki and remained in use for more then 20 years. All didactic literature was directed to popularization of the formal logical study of mathematics in school, even with very young pupils. A typical example at the elementary level sounded in the following way: "Let us consider a statement: "the river x flows into the Caspian Sea". To find out if this statement is true if a) x = Volga, b) x = Don".

Actually, some mathematicians found themselves involved in a widespread movement aimed to innovate and improve the teaching of mathematics, although not always so strongly influenced by the Bourbakian perspective.

Hans Freudenthal was one of them and the foundation of the Institute – now the Freudenthal Institute – shows the deep impact that the innovation had in certain countries.

It is important to stress that, at that time, mathematics education research was centred on issues related to curriculum design and innovation: changing the mathematical content was seen as the crucial contribution to solving didactic problems, and following the new trends of the discipline appeared to be the required solution.

The effects of the New Math revolution are well known, and this is not the place to come back to that discussion. As far as proof and theoretical thinking is concerned, the experience of the didactic transposition of Bourbaki inspired approaches, characterized by the aim of presenting mathematics from a structural perspective, is of interest mainly to understand the appearance in the curriculum of some elements of logic, intended to promote pupils' introduction to theoretical thinking, and generally speaking to 'correct' mathematical reasoning. The long term effect of such a cultural 'innovation' can be seen in the persistence of "truth tables" and "Boolean algebra" at the very beginning of some textbooks (Italian textbooks, for example).

International studies on proof

The rapid evolution of research in the field of mathematics education led not only to overcoming the negative influence of the structural approach, but more generally to criticizing didactic practices which were judged to be too formal and therefore to constitute an obstacle to understanding. From this perspective we can interpret the strong criticism of the so-called "two columns proof[1]", and the guidelines contained in the NCTM Standards published in 1989.

While, in the nineteen sixties and early seventies there was a worldwide normative attempt to align mathematics teaching in the school with a formal conception of proof, later on – especially under the influence of I. Lakatos' *Proofs and Refutations* (1976) – there has been a significant reorientation towards communication and understanding in classroom practice (Balacheff, 1987, 1991). The heart of this reorientation which reflected trends in mathematics, in the philosophy of science and in mathematics teaching could be called a *shift towards a pragmatic view of proof* (Hanna & Jahnke 1993, 422). The very term 'pragmatic view' is not meant to indicate a liberal attitude of 'anything goes', but to designate in a rigorous way the insight that the role and norms of mathematical proof are dependent of the specific mathematical practice in which proof is embedded. Proof is subject to historical and cultural change and its role and meaning are different in the different areas of mathematics. Thus we can say that, at least in some countries, the story of the status of proof and proving in the curriculum shows an alternate favour, depending on both cultural factors and on the research studies on the impact of changes of curriculum on students. In the U.K., we find a clear example of the impact that research studies, based on large scale (nation-wide) surveys describing students' view of a subject, may have on the reform of the curriculum. After the massive change following the imposition of the National Curriculum in the 1980's, proving and proof was one area of the mathematics curriculum that was radically altered in the face of persistent student difficulties and lack of motivation, as it emerged from research studies.

The main response to evidence of children's poor grasp of formal proof in the 60's and 70's was the development of a process-oriented approach to proof. Many argued, (Bell, 1976, Cockcroft, 1982), that students should have opportunities to test and refine their own conjectures, thus gaining personal conviction of their truth along with gaining experience of presenting generalisations and evidence of their validity. (Hoyles, 1997)

A survey on students' views on proof

Clearly, there are potentially considerable advantages in a process-oriented approach in terms of motivation and the active involvement of students in problem solving and proving. Indeed many researchers (see, for example, de Villiers, 1990), argue for such a shift in emphasis, suggesting that students develop an inner compulsion to understand *why* a conjecture is true if they have first been engaged in experimental activity where they have '*seen*' it to be true. The implications of these changes are discussed in a paper by Hoyles (1997). The author presents some of the findings of a nation-wide survey into student conceptions which point to a strong curriculum effect. Most notable was the evidence that student responses were strongly connected, in terms of format and language, to the investigations part of the curriculum they were now studying. Students

1 An account and an interesting discussion of the emergence of the two columns proving custom can be found in Herbst, 2002.

appeared to be imposing a 'type' of proof on every question regardless of whether it is appropriate. Moreover, students seemed to have shifted their notion of proving from one ritual to another – from a *formal* ritual to a *social* ritual – something added on at the end of an investigation. As the author says, it was salutary to trace the extent of the intended or unintended curriculum influences which indicates that student responses cannot simply be 'blamed' on the student. Their types of behaviour cannot be ascribed solely to properties of age, ability or even individual interactions with mathematics. The meanings students have appropriated about proof have been shaped and modified by the way the curriculum has been organised. For example, data concerning responses in geometry – much worse, from a mathematical perspective, from those in algebra, and in contrast to results from other countries – can be explained by the almost complete disappearance of geometry in the curriculum. Nonetheless, this finding casts doubt on how far proof can be considered as a unitary mathematical 'object' with its own hierarchy separated from any domain of application.

A recent study

A new survey study has been recently carried out in the UK, focusing on the influence of the curriculum and more generally on students' progress in recognising and constructing mathematical arguments (Hoyles and Küchemann, 2002).

For this project, survey instruments and questionnaires were developed and used to gather large-scale longitudinal data on a nation-wide large sample (n=1512) of high-attaining students in England, from age 13 1/2 years to age 15 1/2 years, as well as information about their teachers and schools. The longitudinal data were analysed descriptively to produce a rich picture of students' explanations and proofs in algebra and geometry and of how these changed over time. The data were also analysed using multilevel modelling to isolate student, teacher and school factors that seemed to promote mathematical reasoning. Using a mixture of quantitative and qualitative methods, the study reported clear progress in reasoning in response to standard items in algebra and geometry. However, on less standard items difficulties persisted, for instance, in basing explanations on perception in geometry or on numerical evidence in algebra. Besides, robust gender differences were identified along with subtle influences of curriculum and school organisation.

Overall, the study reports only limited progress and some regression in students' articulation of explanations based on mathematical structures; appreciation of circularity in arguments; use of analysis and deduction; and in distinguishing logical implication and its converse. Although there is clear progress in calculations involving several deductions and some progress in distinguishing perceptual from logical reasoning, students still have problems with the organisation of written argument and the understanding of 'reasons' in mathematics. The influence of the curriculum on students' responses is sometimes apparent, for example when new geometrical properties are introduced, students often discount their earlier correct heuristics.

Learning to reason is not automatically transferable across items and domains, is not linear (this is probably not surprising), and is not necessarily retained.

The longitudinal analysis allows to describe individual trajectories and show how single snapshots of student understanding can be misleading. The project also illustrated the power and utility of mixed research methods that incorporate longer and more detailed case studies to interpret classroom and teacher influences, alongside statistical analyses of different types.

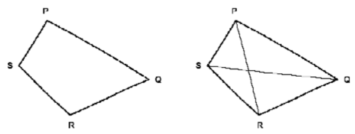
Progress on core items

In the study a key role was played by the so called *core items*, i.e. items that were set each year for 3 years. As a matter of fact, progress was analysed by considering students' scores on the core items. With progress defined as an increase in score (of any amount), 73% of students (N = 1512) progressed on the core items from Yr 8 to Yr 10, 5% stayed the same and 22% regressed, giving a 'net progress' of 51%. Similarly, the net progress from Yr 8 to Yr 9 and Yr 9 to Yr 10 was 23% and 34% respectively. From the case studies, this low net progress was interpreted as partly due to the pressures of the test situation along with influences of the curriculum.

The longitudinal analysis on the core items indicated progress from Yr 8 to Yr 10, along with a sex difference when the baseline mathematics test scores were included in the analysis: girls started from a lower base on the core proof items than boys in Yr 8, caught up in Yr 9 but were overtaken again in Yr 10, with overall progress from Yr 8 to Yr 10 not being significantly different for girls and boys.

Progress on individual items – illustrative results

Overall, the net progress on individual core items was small, of the order of 20%, with the highest at 32% and the lowest at just 2%. Consider the following item.



G1 Tim sketches a quadrilateral. He draws the diagonals of the quadrilateral.
Tim notices that one of the diagonals has cut the area of the quadrilateral in half. He says:
"Whatever quadrilateral I draw, at least one of the diagonals will always cut the area of the quadrilateral in half".
Is Tim right? Explain your answer

G1 is a non-standard geometry item and was designed to test whether or not students would succumb to a perceptual proof (i.e. argue that a false statement was true on the basis of a misleading diagram). In Yr 8, 39% of students claimed the statement was true, reducing to 26% in Yr 10, and with a net progress in score of 20%. Students' counter examples in Yr 10 tended to be more compelling than in Yr 8, but there was little evidence of a shift to a more analytical approach (e.g. starting with and just focussing on the relevant properties), with interviews suggesting the inappropriate use of recently taught geometry facts.

Overall, the study reports an improvement in the use of algebra, in spite of a big gulf between a numerical and the equivalent algebraic task and the persisting strong attraction of "pattern spotting". Data show students' reluctance to abandon an empirical way of thinking, while interviews suggest that students *needed* to calculate and were insecure about using number relationships even when they apparently understood them.

As far as geometry is concerned, students appeared competent at multi-step calculations, at least those based on fairly basic geometric knowledge and they improved markedly over the years. But their explanations tended to be vague and circular; many students were unsure about what is meant by a mathematical reason, and prone to perceptual reasoning, even in Yr 10.

Comparisons with Taiwanese students

Unfortunately, "comparable" data, concerning other countries are not available. Nevertheless some of the parts of the Longitudinal Proof project presented the previous section, were replicated in Taiwan. Though the samples were not exactly similar – the Taiwanese sample was not randomly selected as was the case with the UK sample – some interesting similarities and differences are notable. Similarities and differences can be found, which can be interpreted referring to and comparing the curricula of the two countries. But is interesting to note that differences appear that could not easily be foreseen, which illustrates that curricula cannot capture all the factors in play. Take the following example.

In answer to G1 in Year 8, many more Taiwanese students agreed with the false conjecture (45% as compared to 39%), and fewer produced an explicit counter example (e.g. by making a drawing). This comparison may simply be the result of sample differences, but colleagues in Taiwan suggest the following interpretation: Taiwanese students are not familiar with *refuting* an incorrect property.

From the experience described in this local study crucial variable related to the complex of relationships among the different elements of the didactic system arises. In particular, the impact of the didactic contract (Brousseau, 1997) set up in the classroom in relation to proof and proving is such a variable.

Proof in the classroom: The key role of teachers

Proof is not the same in the classroom as in the mathematical community. As Dreyfus (2000) and Yackel and Cobb (1996) remind us, what is interesting is to shift the focus to a key element of the didactic system: the teacher.

> "Teachers, as representative of the mathematical community in class, have the key role in establishing the various socio-mathematical norms in general and those related to justifications, argumentation and proofs in particular."
>
> (Yackel and Cobb, 1996)

Very few pilot studies have been carried out concerning teachers and proof (Barkai, Tsamir, Tirosh & Dreyfus, 2002; Dreyfus, 2000). Results show a great variability in teachers' evaluation of potential arguments.

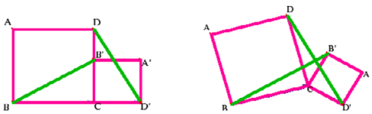
> "Different beliefs are likely to produce widely differing interpretations of guidelines, but also widely differing consequences teachers may draw for their classroom practice."
>
> (Dreyfus, 2000)

Interesting results come from the comparison between two nation-wide surveys. The first one investigates teachers' attitudes towards mathematics education (Nagasaki, 2001), and in particular the relevance attributed to proof, the second one investigates students' achievements on proof. Although not at the highest rate, proof seems to be perceived by the teachers as an important part of mathematics education and there is a shared agreement to introduce proof in compulsory education. In contrast, only about 20-40%

of students in lower secondary school achieves expected objectives on proof. It seems that there is some tacit support to include proof, even if it is difficult to understand. Because of the importance of proof, in spite of its difficulty, and although students cannot perform satisfactorily, the shared opinion is that they should have an idea of proof, even if it remains a vague image.

The report contains examples of the items used in the survey (for 12-13 year olds). It is interesting to compare these examples with those used in the UK, for instance for the 2nd grade of the lower secondary school (13 year olds). Consider the following item.

Square ABCD and square A′B′C′D′ have a common vertex C. Provide an answer to the next questions.



1. Three points B, C, D' are on a straight line like the figure on the left, holding the relation BB' = DD'. In order to prove this, we will show that triangle BCB' is congruent to triangle DCD'. Write the condition for congruence of triangles that will be used in the proof.
2. Even if the three points are not on a straight line like in the figure on the right, they hold the relation BB' = DD'. Write your reason to this.

Results: 1. rate for correct answer 56.8%, 2. rate for reasonable answers 38.1%.

The data available do not allow a reasonable comparison with other countries. However, even this single example shows the intrinsic complexity of a comparison. The difference between this item and item G1, presented above, is evident: it reflects different expectations of the researcher who designed the questionnaire.

Actually, it would be highly valuable to set up a large scale nation-wide survey across different countries and allowing for a comparison. Certainly the design of a common questionnaire, that could be used to test students of different countries, presents great complexity, but I believe that the effort needed to coordinate the work of different researchers, referring to different school systems, but also belonging to different cultures themselves, would be of great interest, and likely to highlight unexpected issues. In fact, culture has a deep impact on how tasks are selected and answers are conceived of by the researchers. Thus new perspectives coming from cross cultural interaction could reveal new interesting questions, but would mainly make some implicit aspects become apparent. It is unbelievable how much one can learn from diversity.

> "Cross cultural comparison also leads researchers and educators to a more explicit understanding of their own implicit theories about how children learn mathematics. Without comparison, we tend not to question our own traditional teaching practices and we may not even be aware of the choices we have made in constructing the educational process."
> (Stigler and Perry 1988, p. 199).

A cultural perspective to explain differences

Different perspectives related to different cultures can be a powerful tool of interpretation. A comparative study (Knipping, 2002), analysing French and German contexts in relation to arguments and proof, will be used to illustrate how cultural difference may affect school practice and curricula. Consider the following quotation taken from an interview with a student of a German-French school (Knipping, 2003). Pupils had experienced both French and German mathematics teachers (in two different academic years).

> **Sophie**, classe de seconde "Oui, pour ma part je trouve que les mathématiques en allemand sont beaucoup plus concrètes que les mathématiques enseignées par les professeurs français, euh les professeurs français s'attachent plus à démontrer des théorèmes, ou à faire beaucoup de démonstrations alors que les professeurs allemands vont plus... directement au principal, et ils ne s'attachent pas à donner des choses, qui en fait sont superflues …"[2]

Knipping analyses the didactic context into which proofs are inserted. Data and results concern the introduction, the development and the justification of Pythagoras' theorem, together with its application in the solution of exercises. Differences among a French and a German didactic style are highlighted and described both in terms of proof processes and in terms of the functions of proof, as they are lived by the participants in the classroom activity.

Comparing the role of proof in German and French teaching contexts has uncovered different teaching patterns and different functions of proofs. As Knipping clearly discusses, in the observed German teaching the function of proof is to "understand why", and generally speaking to get an insight that makes students grasp, at the same time, a property and its reasons. In contrast, in French teaching it is important to "defend why" a statement is true. There is a general habit to divide the arguments into "sound bites" in a chain of reasoning, which reaches a public status in the class, reinforced by writing them down on the blackboard. Proving in French teaching is seen as an activity which characterises the whole teaching: even when students are not explicitly asked to prove something, they are implicitly asked to state the conditions of validity of a statement or a solution.

We may presume that this characterises distinct relations to knowledge and rationality as ingrained in culture. According to Knipping (2001), the German attitude towards proof that the teacher shows can be interpreted as an outcome of a hermeneutic tradition that has influenced education, and in particular mathematics education since Humboldt's reforms in 1810 in Prussia, which turned the ideas of enlightenment, defending Cartesian principles in reasoning, into another approach (Jahnke 1990).

2 "In my opinion, I find that mathematics in German is much more concrete than mathematics taught by the French teachers. Yeah, French teachers care more about proving theorems, to do a lot of proofs. In contrast, German teachers go more ...directly to the core, and they do not care to give things that in fact are useless ..." (translated by the author)

Research studies on students conceptions of proof

Two main streams can be considered, one related to the analysis of problems related to proof as they emerge from school practice, and another one related to proposals for introducing pupils to proof, generally speaking projects for innovation, within which the role of proof is recognized and addressed. There are at least two possible, opposite, perspectives, which often seem unable to communicate to each other, and which might be related to cultural differences (Balacheff, 1999). They differ with respect to the aspects they have in focus.

On the one hand, starting form the analysis of students productions in solving problems, different ways of thinking, related to the observable behaviours, are described and classified. On the other hand, starting from an epistemological perspective, difficulties and problems encountered by students are related to the specific nature of proof.

Analysis and classification of students' behaviour: Proof schemes

An example of the first type analysis is provided by the research study carried out by Harel and Sowder. The authors describe the solutions given to problems (mainly in linear algebra), classifying the different arguments provided by students. The large scale investigation came out with a taxonomy of what the authors call *proof schemes*, obtained by a highly refined classification, fully described in Harel and Sowder (1998). Any kind of argument is considered a proof, and convincing and the key elements in play are persuading. A teaching project (PUPA) elaborating this model according to specific pedagogical assumptions was set up, a description can be found in (Harel, 2001).

A historio-epistemological perspective: Argumentation versus proof

"Argumenter, démontrer, expliquer: continuité ou rupture?" is the title of a seminal work by R. Duval (1992). Starting from an epistemological perspective, Duval analyses the nature and the role of argumentation as well as those of proof in mathematics. Duval holds a very radical position; he focuses on one crucial point: the difference between the semantic level, where the epistemic value of a statement is fundamental, and the theoretical level, where only the validity of a statement is concerned, i.e. only the logical dependence of a statement on the axioms and the theorems of the theory, independent from the epistemic value that may attribute to the propositions in play.

As a consequence of this analysis, Duval stresses the *cognitive distance* between argumentation and proof and, consequently, the relevance of this issue from an educational point of view. A similar analysis, although not so radical, can be found in Balacheff (1987).

In spite of the strength of Duval's arguments, they are still debated. For instance, the contrast with a position like that of Harel and Sowder, just presented, is evident. Crucial questions arise:

Is it possible to overcome the rupture between argumentation and proof? Or, is there any real rupture between the two?

Some results, coming from a research project aimed at introducing pupils to proof, open a new perspective. Data were collected from a long term teaching experiment, centred on open-ended problems: pupils were asked to produce a conjecture and then to prove it. The phase of producing a conjecture showed the appearance of a number of arguments aimed to support or reject a statement. The analysis of the subsequent proof showed an essential continuity with these arguments. This is what the authors called *Cognitive Unity*.

> "During the production of the conjecture, the student progressively works out his/her statement through an intensive argumentative activity functionally intermingled with the justification of the plausibility of his/her choices. During the subsequent statement-proving stage, the student links up with this process in a coherent way, organizing some of previously produced arguments according to a logical chain."
> (Boero et al., 1996)

Although the notion of Cognitive Unity within avery peculiar teaching experiment, and as part of a very peculiar teaching project, it showed its potential, both in explaining traditional difficulties and in suggesting new directions of investigation. In particular, this seminal work has provided a new tool of analysis, which in my opinion seems to be very promising.

Overcoming the dichotomy: The notion of Cognitive Unity

The notion of Cognitive Unity, developed with the aim of describing and interpreting students' approaches to theorems, can be used as an analytical tool in investigating the relationship between argumentation and mathematical proof, taking as an underlying assumption the parallel proposed by Balacheff:

> "Je résumerai en une formule la place que je crois possible pour l'argumentation en mathématique, allant dans le sens du concept d'unité cognitive des théorème, forgé par nos collègue italiens:
> L'argumentation est à la conjecture ce que la démonstration et au théorème[3]"
> (Balacheff, 1999)

In this way, the analysis proposed by Duval is not refused but further articulated with the aim of identifying the key elements of a comparison between argumentation and proof.

The very first analysis was limited to what may be called the referential field. A more refined analysis has been carried out by Pedemonte (2002), showing the complex relationship between the structure of an argumentation and the structure of the related proof: Toulmin's model (Toulmin, 1958) provides a powerful framework for this analysis.

An example of the structural distance between argumentation and proof

I would like to discuss the case of induction because it provides good opportunity to come back to the notion of proof scheme, and thus to relate to different research studies. Inductive types of argumentation are quite common, but the development of recursion has been difficult, raising a strong debate, so that only recently mathematicians have obtained an agreement about its acceptability.

3 I would summarize in a formula the place that I find possible for argumentation in mathematics, according to the notion of Cognitive Unity as it was introduced by our Italian colleagues: argumenttation relates to conjecture, like proof does to a theorem" (translated by the author)

According to Harel's analysis (Harel, 2001), two different types of arguments can be recognized both leading to a general statement. The first one is a case of an *empirical* proof scheme, whilst the second one is a *transformational* proof scheme.

Generalization may be achieved by recognizing a general pattern in the result itself (*result pattern generalization*), for instance, observing the regularity of the result of a calculation.

Generalization may be derived from the process that leads to the results *(process pattern generalization)*, for instance, observing a chain of steps relating the results to each other.

Although in both cases the arguments supporting the general statement are obtained from an inductive process, i.e. from the verification of a limited number of particular cases, in the former case the examples function as generic elements on which the arguments can be applied. In the latter case, examples are provided, but the passage from one step to the following is in focus. A more refined analysis reveals a further distinction between what Harel calls Quasi Induction and Mathematical Induction (MI): While both are instantiations of the transformational proof scheme, the latter is an abstraction of the former. In MI, on the other hand, the student views $P(n) \rightarrow P(n+1)$ as a variable inference form, a placeholder for the entire sequence of inferences. In spite of its brevity, this analysis clearly shows the variety of argumentations that may be produced and which have different relationships with the mathematical proof by induction. In particular, both the continuity and the cognitive gap between quasi induction and MI should be carefully investigated.

The distance is related to the fact that in the case of proof, the standards of acceptability for an argument are pretty strict and mainly related to a well defined and clearly stated set of paradigms of arguments. Among these paradigms the deductive reasoning is perhaps the most accredited one. From this perspective, a main characteristic of proof is its social dimension, i.e. proof makes sense with respect to a community which shares (more or less implicitly) the criteria of acceptability.

At school, this social dimension related to sharing the standards of the community of mathematicians must be articulated within the social dimension of the classroom community: the crucial role of the teacher comes to the fore, at the same time representing the mathematics community and the classroom community.

From a different perspective, the possible discrepancy between argumentation and proof, was recently analysed by Raman (2002), in terms of private versus social perspectives. Here the author expresses the two poles of the dialectics as *private* and *public* aspects, and identifies, in what she calls the *key idea*, the possible link between the two poles. The results, reported in Raman's work, open a new direction of investigation, i.e. the comparison between experts and novices. The analysis is consistent with the perspectives of cognitive unity research; a fine grained analysis of data, if carried out using Toulmin's model, in analogy with what was done by Pedemonte, could provide further insight into the potential continuity between the private and the public aspect of proof, as well the potential gap between them, both in the case of experts and that of novices.

Proposals for introducing pupils to proof

Results coming from survey studies as well as from research work focused on students' conceptions of proof provide a motivation and a base for projects whose aims are more or less directly related to the aim of introducing pupils to proof.

Different research studies clearly suggest an early start of a practice of proving. Accordingly, a number of research projects, at the primary and the lower secondary school levels, are widely based on thoughtfully selected open-ended problems investigated by children and collectively discussed by the whole class. The aim is very often that of establishing a "mathematical community in the classroom" (Arsac, 1992; Davis & Maher, 1993; Yackel & Cobb, 1996, Bartolini Bussi, 1991). Different approaches share as the common assumption that reasoning and arguments contribute to knowledge construction (Boero et al, 1995; Dueck, 1999). This widely accepted perspective expresses the need of coordinating psychological and sociological perspectives, i.e. developing a model where education is interpreted as entering and participating in a culture rather than as being subject to transmission of knowledge. New knowledge emerges from pupils' activities. However in the collective activity of the classroom it is systematized into a mathematical framework, and social norms determine what is considered acceptable and in particular mathematically acceptable.

At the primary school level the nature of "mathematical" is hardly questionable: "What makes the "objects" of Trevonda less mathematical than those of Jameel?" do the authors (Yackel & Cobb, 1996), analysing the transcripts of a classroom discussion, ask themselves. As a consequence, beyond the social norms, controlling what students are expected to do, socio-mathematical norms are established in the classroom, and as part of them the criteria for acceptability are negotiated.

> "The understanding that students are expected to *explain* their solutions is a *social norm*, whereas the understanding of what counts as an acceptable mathematical explanation is a *socio-mathematical norm.*"
> (Yackel, 2001)

Similarly, the development of young children's understanding of mathematical argumenttation constituted the key objective of another teaching experiment, designed to create classroom environments within which the sense making is a cultural norm, and a particular outcome of this culture is expected to be the emergence of argumentation, justification and proving in children's discourse.

The teacher's actions accomplish several goals: among others to calling attention to the argumentative support for conclusions thus contributing to the class' understanding of what is taken to be argumentative support. (Maher, 1996, 1998)

Consistent with this perspective, but more explicitly oriented towards framing mathematical arguments within a theoretical system, are the experiments carried out in Italy by the research group directed by Bartolini Bussi. The detachment from the conception of empirical verification as the only tool suitable to resolve conflict situations is carefully managed by the teacher by means of the collective construction of *germ-theories* (a germ-theory is an embryo of theory that has an expansive power and the potential for developing into a fully-fledged one). Within a selected field of experience (Boero et al., 1995), the solution of a rich collection of problems provides a basis on which a *germ theory* is established.

An example of a germ theory

Consider the field of experience of gears (Bartolini Bussi et al., 1999). From a variety of experiences a basic principle arises; this principle is not explicitly named a postulate, but its status is stated as a principle.

In the protocols, various argumentations are produced to justify individual statements, based on the stated principle. The following example aims to show the kind of argument that can be produced and how it can be related to proof in a germ theory. Consider the following problem, proposed to 5th grade class.

> "We have often met planar wheels in pairs. What if there were three wheels? How could they be positioned? You can build possible situations by drawing or by cutting. Remember you must always give the necessary explanations and write down your observations."
>
> Elisabetta's protocol

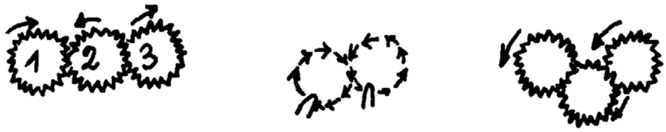


Figure 1: A sequence of three drawings produced by Elisabetta

1) *Wheel n. 1 turns, but we do not know in which direction; let us say that it turns clockwise, then Wheel n. 2 turns anticlockwise, this is sure, and n. 3, how do you think this one turns?*
   I know how: it turns like n.1. Do you know why? Because they have to be in gear in the opposite direction. We could do this with fingers too, remember. I've drawn two wheels with arrows in opposite directions.
   Yet, if we think hard, n. 1 could turn clockwise and n. 2 clockwise too, couldn't they? They could not, they would not be in gear.
2) *Let us try to draw the wheels in another way.*
   Yes, they are on the same plane; but if I try to turn one we would see that two turn in the same direction and the other turns in the opposite direction. They cannot be in gear, because the first turns clockwise, the second anticlockwise and the third clockwise too, but the first and the third touch each other and so they are not in gear.

> „The first two wheels turn in opposite directions; and this is OK, but there is a third wheel that is in gear with both; it is a kind of block as the teeth should break. Actually the two wheels go in opposite directions and a tooth would push a tooth of the wheel one way but there is the tooth of the other wheel that pushes this tooth the other way. Conclusion: if the wheels are put in this way they can't turn."
>
> Davide's protocol

In both the graphical and the mental experiments a property drawn from physical experience is used: *'two wheels in gear turn opposite ways'.* For the pupils, this principle does have the status of 'postulate' of a germ theory. Within this framework, only a 'small' step is needed, to shift the arguments to the status of mathematical proofs, thus explicitly building the reference theory. Within this theory, pupils' argumentations about the impossible motion assume the status of proofs by *reductio ad absurdum*. The mental experiments show all their power, as they allow the dynamic exploration of gears that do not actually work and permit production of statements and argumentation for any number of wheels. Analogous examples can be found in (Boero et al., 1996), where the field of experience of shadows functions to foster a rich context where the need of explanation leads to modelling and conceptualising, according to the main assumption on the teaching-learning process, which is modelled by a systemic interaction between the production of conjectures and mathematical systematisation.

The contribution of technologies

In recent years a number of different contributions on the theme of proof shared the choice of a Dynamic Geometry Environment (DGE) as their context. Some years ago a Special Issue of *Educational Studies in Mathematics Education* (vol. 44, 2000) was devoted to this theme. Certainly, the availability of graphing capabilities "has given a new impetus to mathematical exploration, and has brought a welcome new interest in the teaching of geometry" In fact, "Dynamic software has the potential to encourage both exploration and proof, because it makes so easy to pose and test conjectures" (Hanna, 2000, pag. 13).

But if it seems clear that dynamic figures may contribute to setting up conjectures, providing the students with a strong evidence that a property is true, their contribution to finding a proof, i.e. validating that conjecture, seems less clear. The possible contribution to introducing students to a theoretical perspective, i.e. to construct a meaning of proof, appears to be even more critical. It could be natural and reasonable if the student jump to the conclusion that exploration via dragging is sufficient to guarantee the truth of what can be observed (Mason, 1991; Healy & Hoyles (2001). Thus the critical point concerning the relationship between empirical evidence and theoretical reasons arises in this new context. As, using the notion of *milieu* (Brousseau, 1997), Laborde points out:

> "[…] a DGE itself without an adequately organized milieu would not prompt the need of proof. And it becomes evident the need of establishing a rich milieu with which the student is interacting during the solving process and the elaboration of a proof."
> (Laborde, 2000, p. 154)

The papers included in the ESM special issue were intended to discuss this question and to contribute to clarifying potentials and limits of DGE.

Although it is impossible here to give a full account of the discussion, I would like to focus on a specific point, which in my view is a crucial one: the relationship between the *dragging tool* and theoretical control within geometry.

Dragging tool and logical control

DGEs, as opposed to paper and pencil environments, contain within them the seeds for a geometry of relations: entering a DGE offers the opportunity of experiencing the break between these two worlds and to experience this break at the level of actions (Laborde, 2000). But I would like to go further by stressing the fact that actions are mediated by tools which, according to a Vygotskian perspective, can become "semiotic tools", exploited by the teacher according to her didactic objective related to making students develop mathematical meanings.

Consider the dragging tool, as it is used in a DGE, like for instance *Cabri*.

The dragging tool can be activated by the user through the mouse and can determine the motion of different objects on the screen. Two main kinds of motions are possible, as a consequence of the dragging mode: direct and indirect motion.

The "indirect motion" of an element occurs when a construction has been accomplished; in this case, dragging the basic points from which the construction originates will determine the motion of the new elements obtained through the construction. This motion will be consistent with the properties stated by the tools used in the construction. In other words, the use of dragging allows one to directly experience *motion dependency* which can be interpreted in terms of *logical dependency* within the geometrical theory.

Such a semiotic analysis highlights the link between the dragging tool and the meaning of theoretical control that is the complex of meanings related to the notion of theorem. Thus a *statement* that can be *proved* within a specific *theory* (Mariotti et al., 1997). In spite of the centrality of this interpretation for an effective use of dragging in exploration, both for posing and proving conjectures, its difficulty is well documented (Hoelz, 1996; Hazzan & Goldenberg, 1998; Chazan & Yerushalmy, 1998). As a consequence, it becomes crucial to face the didactic problem consisting in relating phenomena, visually and kinetically perceived on the screen, and logical dependency between geometrical properties.

According to a semiotic process, triggered by the teacher's actions, meanings should evolve from personal meanings, concerning the idea of dependent movement as it emerges from pupils' own experience in a DGE, to mathematical meanings, concerning the mathematical idea of logical dependence between hypothesis and thesis, as expressed in a theorem.

Evidence from different studies (see for instance Jones, 2000) indicates that using dynamic geometry software does provide students with access to the world of geometry, including definitions and explanations based on the logical relationships between properties.

Similarly, taking geometrical constructions in a DGE as the field of experience, a long term teaching experiment has been carried out, aiming at introducing students to theoretical thinking. Results from different classes provide evidence of both the complexity and the feasibility of this project. The validation test based on the dragging mode has been used as an instrument of semiotic mediation to introduce the meaning of theoretical control. More generally, different tools offered in the Cabri environment were used as instruments of semiotic mediation to make the meaning of "theorem" evolve (Mariotti, 2000, 2001, 2002).

It is interesting to remark that in a recent research project the general theoretical framework based on the notion of semiotic mediation has been used to promote a theoretical perspective in a completely different mathematical field, namely that of symbolic manipulation of algebraic expressions. Similarly to the case of geometry, a computational environment was designed to offers specific tools that may function as instruments of semiotic mediation to foster students' evolution of the theoretical meaning of symbolic manipulation (Mariotti & Cerulli, 2002).

Conclusions

I want to start these concluding remarks by going back to the epistemological distinction among different *functions of proof*, on which I think there is a large consensus (see Bell (1976) and de Villiers (1990)):

- *verification* (concerned with the truth of a statement)
- *explanation* (providing insight why it is true)
- *systematisation* (the organisation of various results into a deductive system of axioms, major concepts and theorems)
- *discovery* (the discovery or invention of new results)
- *communication* (the transmission of mathematical knowledge)
- *construction of an empirical theory*
- *exploration* of the meaning of a definition or the consequences of an assumption
- *incorporation* of a well-known fact into a new framework and thus viewing it from a fresh perspective.

This long list clearly exhibits the complexity of the educational task concerning the introduction of pupils to proof, a complexity that cannot be transposed into educational practice without difficulties.

Nevertheless, nowadays (and maybe differently to ten years ago) there is a general consensus about the fact that the development of a sense of proof concerns an important objective of mathematical formation: this objective is strictly intertwined with other objectives (for instance the development of linguistic abilities and competence within different mathematics fields), which require long term strategies of intervention within an encompassing curricular perspective.

The design of curricula, at least in some countries, has been determined by pressures from the world of educational research, and there seems to be a general trend to include proof in the curriculum as highlighted by the change in the NCTM-2000 standards with respect to the 1989 standards, but also, to some extent, by the reform in the UK.

But, in spite of this consensus on the importance and value of proof, the complexity of the idea of proof and the difficulties that must be faced ask for a great caution.

Including proof in the curriculum is only the first step. It is also important to ensure that the goals for doing so and how these goals are operationalised, are clarified and taken into account.

Clearly proof has the purpose of verification – confirming the truth of an assertion by checking the correctness of the logic behind a mathematical argument. But at the same time, if proof simply follows after the conviction of truth rather than contributing to its construction, and is only experienced as a demonstration of something already known to be true, it is likely to remain meaningless and purposeless in the eyes of students (de Villiers, 1990; Hanna & Jahnke, 1993). For a long time an alternative approach has been claimed, characterized by proofs that are acceptable from a mathematical point of view but whose focus is on understanding (what Hanna calls explanatory proofs – Hanna, 1990, p12), rather than on meaningless formal deductive methods. One crucial point in operationalising this approach is that of encouraging student engagement and ownership of the proving activity; that means to add a social dimension to explanatory proving. A culture of validation has to be established in the class, leading students to explain their arguments to peers and to the teacher, as well as to convince themselves of the truth of their arguments (Hoyles, 1997).

In this same vein, interesting suggestions come form the recent research projects, mainly from what is called research for innovation. But the difference between experimenttal classes and reality must not be underestimated, and the problem of disseminating the results, mainly in teachers' training courses, must be taken seriously.

The teacher must be adequately prepared. In particular, I would like to stress the delicate role that the teacher has to play at the primary school level, where students' first beliefs are settled, and most of the basic meanings sprout but remain implicit.

Possible research directions for the future:
In face of the richness and the variety of issues concerning reasoning, proof, and proving, a number of different research directions are possible.

As far as studies on students' conceptions are concerned, it seems useful to enlarge the number of large scale surveys on students' conceptions, but instead of multiplying unrelated studies, we must profit from comparisons between different cultural backgrounds, which can provide deeper insight, highlighting unexpected points of view, e. g. a comparative analysis of different "cultures of proof", as proposed in the schools of different countries, with relation to the specific cultural features of curricula and, more generally, to the cultural values characteristic of each country. Cross-cultural studies may be of great value, and the seminal work of Knipping is a good example highlighting the potentials of such a perspective. In particular, students' conception on proof is tightly related to their beliefs about mathematics.

Moving into the field of beliefs, makes the role of the teacher come to the fore and reminds us not to forget that specific investigations should be devoted to describing teachers' views on proof. Both epistemological and pedagogical perspectives have to be taken into account: what is proof to a teacher? And also, what is a student's proof to a teacher?

Research studies concerning the analysis of argumentation processes and their comparison with the production of mathematical proof appear to be very promising. The construct of Cognitive Unity can be fruitfully applied to describe and to compare cognitive processes related to proof. We need to enlarge the number of case studies; in particular comparisons between experts and novices deserve great attention. This area of research has a natural dimension of investigation concerning the relationships between proof and knowledge construction, in relation with the study of the discursive constitution of both mathematical concepts and procedures constituting an important trend in the current educational research.

Several studies show that competencies of students in devising a proof as well as their understanding of proofs vary across the mathematical subject areas. A common message behind such studies is that not only the competence of interpreting a given and devising a new proof is bound to special areas and situations, but the same is true of the general understanding of what a proof is. Therefore, successful teaching of proof requires the construction of specific contexts. Different fields of experience have been used in specific long term teaching experiments, as discussed above, but also, interdisciplinary working contexts, for instance between mathematics and physics (Hanna & Jahnke, 2002). And a different organization of classroom activities, from dialogue between students to collective discussions orchestrated by the teacher. Certainly, further investigation is required, assuming that such investigation cannot be entirely detached from the classroom, in which the whole didactic system operates, and where beliefs are settled. Classroom investigations are greatly valuable, although methodological difficulties must be taken into account in a serious manner.

Since ancient Greece, Western thought has considered proof to be an essential characteristic of mathematics, and as such proof should be a key component in mathematics education. However, translating this statement into classroom practice is not a simple matter. There has been and there remains differing and constantly developing views on the nature and role of proof and on the norms to which it should adhere. Besides, mathematics education has to take general goals into account, for instance the promotion of mathematical understanding and the furthering of insights into the contribution of mathematics to human understanding of the world around us. All these goals must be tuned with the introduction of theoretical perspectives and in particular with practices of proof.

How to overcome these difficulties is the challenging issue that the future presents to us. The richness of the recent contributions and vivacity of the debate are very promising.

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