Conspectus
of the minicourse "Topology of manifolds"

for graduated students
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Preface

This lectures was delivered at May,16 - June, 15 2004 for graduated students of Harbin Institute of Technology. Not all topics from the program was delivered and not all of them was included in the following text. So this conspectus should be considered as a very preliminary version of the minicourse.

Programme

1. Initial notions of smooth manifolds.
   (a) Non linear coordinate systems.
   (b) Coordinate homeomorphisms, the Jacobi matrix, jacobian.
   (c) Charts, atlas of charts. Smooth structure of the class C^k. Class C^∞, real and complex analytic manifolds.
   (d) Invariance of the dimension of the smooth manifolds.
   (e) Smooth functions on manifolds.
   (f) Smooth mapping from manifold to a manifold. The Jacobi matrix of the smooth mapping.
   (g) Examples. Sphere, projective space, torus. Cartesian product of manifolds.
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   (i) Description of manifold by a graph.
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2. Submanifolds.
   (a) Examples of different types of submanifolds.
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   (a) Linear approximation of a submanifold. Tangent subspace.
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   (c) Tangent vector as a sheaf of osculating curves.
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4. Some application of the theory of manifolds.
   (a) The mapping degree of orientable manifolds, the main algebra theorem.
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5. Vector fields, the Lie brackets, the Lie algebra structure, integrable distributions, foliations.
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   (a) The structure groups
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10. K-theory.
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1 Initial notions of smooth manifolds.

1.1 Non linear coordinate systems.

Let us consider an n-dimensional Euclidean space which is usually denoted by $\mathbf{R}^n$. We assume that this space is provided with Cartesian coordinates $x^1, \ldots, x^n$ which permit a unique determination of the position of any point in $\mathbf{R}^n$ by associating with it a set of real numbers, the coordinates relative to a fixed orthogonal basis formed by mutually orthogonal unit vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_n$.

The idea of describing a point in an Euclidean space by a set of real numbers (which may also be considered as the coordinates of the radius vector emerging from the origin to the point) underlies analytic geometry which solves various geometric problems by purely algebraic methods. This important approach was first, introduced (explicitly) into mathematics by des Cartes in whose honor we now say "Cartesian coordinates". Algebraization of geometry has played a key role in the development not only of geometry as such but also of mathematics as a whole.

We shall not concentrate on the problems which are easily and elegantly solved by algebraic-analytic methods (viz. classification of second-order surfaces in a three-dimensional space) and refer the reader to numerous courses of algebra and analytic geometry. Let us only recall that Cartesian coordinates in $\mathbf{R}^n$ are closely related to the concept of the Euclidean scalar product, a bilinear form which associates with each pair of vectors $\xi, \eta \in \mathbf{R}^n$ a real number usually denoted by $\langle \xi, \eta \rangle$. This operation is symmetric and linear in each argument, and the form itself is positive definite. In a Cartesian coordinate system we have

$$\langle \xi, \eta \rangle = \xi^1 \eta^1 + \ldots + \xi^n \eta^n,$$

where

$$\xi = \left(\xi^1, \ldots, \xi^n\right),$$
$$\eta = \left(\eta^1, \ldots, \eta^n\right).$$

Simple examples however show that Cartesian coordinates are not always the most convenient ones to solve analytically many particular problems. We shall demonstrate this by writing the equations of curves on a plane in Cartesian coordinates x, y. Of course, for rather simple curves, viz. a circle or ellipse, the analytic expressions in Cartesian coordinates are simple. Indeed, the equation of a circle of radius R with centre at point A is

$$\left(x - A^1\right)^2 + \left(y - A^2\right)^2 = R^2,$$

where $A = \left(A^1, A^2\right)$. The equation of an ellipse is also simple:

$$\frac{\left(x - A^1\right)^2}{a^2} + \frac{\left(y - A^2\right)^2}{b^2} = R^2,$$

where a and b are the principal semi-axes.

However, in various applications and concrete mechanical and physical problems we often deal with smooth curves (say, trajectories of the motion of a particle in a force field) whose equations in Cartesian coordinates are rather cumbersome. For example, the equation

$$\sqrt{x^2+y^2} - e^{\lambda\left(\tan^{-1}\frac{y}{x}\right)} = 0$$

defines a spiral in Cartesian coordinates. Although this equation is rather simple, it could be written in a simpler form in so called polar coordinates (r, φ) related to the Cartesian coordinates (x, y) by

$$\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi. \end{aligned} \tag{1}$$

In polar coordinates the equation of a spiral becomes $r = e^{\lambda\varphi}$, thereby clearly demonstrating the character of the motion along this trajectory.

Below we shall return to a polar coordinate system, and here we just note that the introduction of such coordinates (called curvilinear coordinates) is not a caprice of mathematicians trying to devise new entities but a practical necessity, sometimes quite useful in particular calculations. In this connection, we shall briefly discuss a problem where polar coordinates appear to be quite useful.

1.1.1 A practical example

Let us consider the motion of a particle on a plane in a central force field. Suppose the centre is at the point O and (r, φ) are polar coordinates on the plane. Let $\mathbf{r}$ be the radius vector of the moving particle (this vector originates from O), $r = |\mathbf{r}|$ be its length, and t be the time (the motion parameter); then the coordinates r and φ are functions of time. Consider at a point $\mathbf{r}(t)$ with polar coordinates $r = |\mathbf{r}|$, φ two orthogonal unit vectors: a vector $\mathbf{e}_r$,. along the radius vector of the particle (note that the relation $\mathbf{r} = r \cdot \mathbf{e}_r$ holds in this case) and a vector $\mathbf{e}_\varphi$ orthogonal to $\mathbf{e}_r$ and so directed that the polar angle φ increases. Differentiation of a radius vector $\mathbf{r}(t)$ with respect to time will be denoted by a dot. As is known from mechanics the motion of a particle (its mass is taken equal to 1 for simplicity) in a central force field on a plane is described by the following differential equation:

$$\ddot{\mathbf{r}} = f(r)\mathbf{e}_r,$$

where f is a smooth function of a single argument r. Incidentally, here is a useful exercise for the reader: write this differential equation in Cartesian coordinates on a plane.

The motion of a particle can be described by two functions: $r = r(t)$ and $\varphi = \varphi(t)$, that is, in a polar coordinate system. It is a simple matter to verify that when a material particle moves in a central force field the quantity $r^2\varphi$ is conserved. This is one of Kepler's laws which he discovered while studying the motion of the planets of the Solar system (at that time Kepler already employed

the tables of the coordinates of the planets on the celestial sphere as a function of time). This conserving quantity can be given clear geometrical meaning.

Kepler introduced a convenient notion of areal velocity u as the time rate of change of the area $s(t)$ swept out by the radius vector $r(t)$, i.e. $\upsilon = \frac{ds(t)}{dt}$. In terms of the areal velocity Kepler's law can be formulated as follows: the radius vector sweeps out equal areas in equal times; in other words, the areal velocity is constant, $\frac{ds(t)}{dt} = const$. We can also prove (the proof is omitted here) that this law is one of the formulations of the principle of conservation of angular momentum. The reader can easily see that this law is much simpler to derive in polar coordinates rather than in Cartesian coordinates (though calculations may, of course, be performed in the latter as well).

Solution of particular problems in mechanics and physics has called for the invention of other curvilinear coordinate systems: cylindrical, spherical, and so on. Close examination of all the ways above of associating a point in space with a set of real numbers (the coordinates of this point) shows that this association is based on a general idea admitting a reasonable formalization which comprises all the "curvilinear" coordinates mentioned (inverted commas are used for the word curvilinear, since we have not yet defined the concept strictly, but consider only graphic examples).

1.1.2 Cartesian and curvilinear coordinates

Let us consider an arbitrary domain in a Euclidean space $\mathbf{R}^n$. We recall that, just as in mathematical analysis, by a domain we mean an arbitrary set U in a Euclidean space whose every point P is contained in U together with a ball of sufficiently small radius with centre at P. Consider also a second copy of the Euclidean space, which is denoted by $\mathbf{R}^n_1$. To define the coordinates of the point P in the domain U is to associate with this point a set of numbers, called coordinates. Obviously, an arbitrary association (i.e. the one without additional requirements) will not lead to a good result in that such a correspondence may be devoid of sense (it is desirable that mathematical concepts should be of some use, for instance in computations, just as was the case with Cartesian coordinates). Here is an example of senseless association: the same set of numbers, say $(0, 0, 0, \ldots, 0)$, is associated with each point P in U. Thus, we arrive at the first requirement to the association: it is desirable that distinct sets of numbers (coordinates) should correspond to different points of the domain. The example just mentioned does not satisfy this requirement (all the points of the domain C have the same "coordinates", zeros).

Thus, our aim is to associate with each point P of a domain U a set of n real numbers. Apparently, this operation gives rise to a set of n functions $\big(x^1(P), \ldots, x^n(P)\big)$ defined in the domain U; here $\big(x^1, \ldots, x^n\big)$ are coordinates in the Euclidean space $\mathbf{R}^n_1$.

These functions are usually required to be continuous and even smooth (at least for almost all points of the domain U), that is, a small change in the position of P should lead to a small change in its coordinates, and a smooth deformation of P should generate a smooth variation of its coordinates.

So, let us consider two copies of Euclidean space: $\mathbf{R}^n$ with Cartesian coordinates $\left(y^1, \ldots, y^n\right)$ and $\mathbf{R}_1^n$ with Cartesian coordinates $\left(x^1, \ldots, x^n\right)$; let U be a domain in $\mathbf{R}^n$.

Remark. The Euclidean space $\mathbf{R}_1^n$ could be considered as an "arithmetic space" by identifying its points with real sequences of length n.

Definition 1 *A continuous coordinate system in a domain U of Euclidean space $\mathbf{R}^n$ is said to be a system of functions*

$$\left\{x^1 = x^1\left(y^1, \ldots, y^n\right), \ldots, x^n = x^n\left(y^1, \ldots, y^n\right)\right\}$$

which maps the domain U continuously and bijectively onto a certain domain V of $\mathbf{R}_1^n$. In other words, the system of functions $\left(x^1(P), \ldots, x^n(P)\right)$ defines a mapping, sometimes called a homeomorphism of U onto V.

Definition 1 is a formal expression of our desire that as a point P moves continuously in U its coordinates should also change continuously. The functions $\left(x^1(P), \ldots, x^n(P)\right)$ are called the coordinates of point P with respect to the coordinate mapping $f : U \to V$.

For instance, the coordinate mapping $f : U \to V$ may be chosen in the form of an identity mapping defined by the linear functions, $\left\{x^1 = y^1, \ldots, x^n = y^n\right\}$.

Sometimes we shall write a point P with coordinates $\left(x^1(P), \ldots, x^n(P)\right)$ in the form $P\left(x^1, \ldots, x^n\right)$ assuming that the coordinate mapping $f : U \to V$ has already been defined and fixed.

1.2 Coordinate homeomorphisms, the Jacobi matrix, jacobian.

Among all continuous coordinate mappings of special interest are those that define a smooth mapping of a domain U onto V, i.e. when all functions $\left\{x^1\left(y^1, \ldots, y^n\right), \ldots, x^n\left(y^1, \ldots, y^n\right)\right\}$ are continuously smooth functions of their arguments $\left(y^1, \ldots, y^n\right)$. But the smoothness of the coordinate mapping f without the assumption of the smoothness of the inverse mapping f^{-1} does not lead to a meaningful coordinate system. Therefore, we now turn to defining coordinate systems in which f and f^{-1} are both smooth. To this end, we shall need a new concept, the Jacobi matrix of a smooth mapping.

Let $f : U \to V$ be a smooth mapping defined by a family of functions

$$\left\{x^1\left(y^1, \ldots, y^n\right), \ldots, x^n\left(y^1, \ldots, y^n\right)\right\}.$$

Definition 2 *The Jacobi matrix of a mapping f is a functional matrix*

$$df = \frac{\partial x}{\partial y} = \begin{pmatrix} \frac{\partial x^1}{\partial y^1} & \cdots & \frac{\partial x^1}{\partial y^n} \\ \cdots & \cdots & \cdots \\ \frac{\partial x^n}{\partial y^1} & \cdots & \frac{\partial x^n}{\partial y^n} \end{pmatrix},$$

composed of partial derivatives of the coordinates $\left(x^1(P), \ldots, x^n(P)\right)$. The determinant of this matrix is denoted by $J(f)$ and called the Jacobian of the mapping f.

Remark. We hope that the notation df for the Jacobi matrix will not be confused with the differential of a smooth function f, since this differential (when interpreted appropriately) coincides with the Jacobi matrix in this particular case. Let us note once more that a Jacobi matrix is a variable matrix, i.e. it depends on a point P in a domain U. Similarly the Jacobian $J(f)$ is a smooth function on U.

Definition 3 *A regular coordinate system in a domain U of Euclidean space $\mathbf{R}^n$ is a system of smooth functions $\left\{x^1\left(y^1,\ldots,y^n\right),\ldots,x^n\left(y^1,\ldots,y^n\right)\right\}$ which map bijectively the domain U onto a domain V in $\mathbf{R}^n$ and are such that the Jacobian $J(f)$ is not zero at all points of U.*

Let us note that the condition that the Jacobian does not vanish at all points of U means that the inverse mapping f^{-1} is not only continuous, but also smooth. This follows from the implicit function theorem. Thus, a regular coordinate system is defined by two smooth mutually inverse mappings establishing a homeomorphism between the domains U and V. Definition 3 makes formal our desire that when a point P changes smoothly in U its coordinates should also change smoothly; moreover, smooth variation of a "coordinate point" Q in V should also result in smooth variation of the point P induced by the mapping . The definitions presented above clearly show that the very concept of a "smooth and regular coordinate system" automatically implies that at least two copies of a standard Euclidean space should be considered. Certain domains of these copies are identified by a continuous and bijective mapping with an additional requirement of smoothness (in both directions).

These definitions can be interpreted from another point of view. We could assume that a Cartesian coordinate system is initially introduced in a domain U of Euclidean space $\mathbf{R}^n$ (via an identity mapping of U onto V under natural identification of both copies, $\mathbf{R}^n$ and $\mathbf{R}_1^n$). Then, the introduction in U of another coordinate system defined by a regular mapping f (i.e. a smooth, one-to-one mapping with a non-zero Jacobian) may be considered as a coordinate transformation: we simply pass from the initial Cartesian coordinate system to a new one in the same domain U.

Definition 4 *A regular coordinate system in a domain U is sometimes called a curvilinear coordinate system in U.*

Consider two arbitrary curvilinear coordinate systems in a domain U:

$$\left(x^1(P),\ldots,x^n(P)\right) \text{ and } \left(z^1(P),\ldots,z^n(P)\right).$$

This means that two regular mappings

$$f: U \to V \subset \mathbf{R}_1^n\left(x^1,\ldots,x^n\right) \text{ and } g: U \to W \subset \mathbf{R}_2^n\left(z^1,\ldots,z^n\right)$$

are defined which map smoothly and bijectively the domains U to V and U to W, respectively. In other words, each point P in U is associated with two

sets of curvilinear coordinates $\{x^i(P)\}$ and $\{z^i(P)\}$, $1 \leq i \leq n$. Since these correspondences are bijective, we may consider a correspondence which relates the coordinates $\{x^i(P)\}$ of point P to the coordinates $\{z^i(P)\}$, this operation defining the mapping $\psi_{x,z} : V \to W$, i.e. $\psi_{x,z} : x^i(p) \to z^i(p)$, $1 \leq i \leq n$. The mapping $\psi_{x,z}$ is called coordinate transformation in the domain U. Under this transformation the initial curvilinear coordinates $\{x^i(P)\}$ of point P change to new curvilinear coordinates $\{z^i(P)\}$.

Lemma 1 *The transformation $\psi_{x,z}$ is a bijective and smooth mapping of the domain V onto W with a non-zero Jacobian.*

Proof. That $\psi_{x,z}$ is one-to-one directly follows from Definition 3. The smoothness of $\psi_{x,z}$ follows from the fact that the composition of two smooth mappings is also a smooth mapping. It remains to verify that the Jacobian $J(\psi_{x,z})$ of $\psi_{x,z}$ is non-zero at each point of the domain W.

Indeed, the mapping $\psi_{x,z}$ splits into the composition of two mappings:

$$\psi_{x,z} = g \circ f^{-1} : V \to W.$$

The Jacobi matrix of $\psi_{x,z}$ splits into the product of Jacobi matrices of the mappings f^{-1} and g. One has $d\psi_{x,z} = \frac{dz}{dx}$. Consider the derivative $\frac{\partial z^i}{\partial x^j}$. Since

$$z^i = z^i\left(y^1, \ldots, y^n\right) = z^i\left(y^1\left(x^1, \ldots, x^n\right), \ldots, y^n\left(x^1, \ldots, x^n\right)\right)$$

where the functions $\{y^\alpha(x^1, \ldots, x^n), \quad 1 \leq i \leq n\}$ define the smooth mapping $f^{-1} : V \to U$, we obtain from the formula for differentiation of a composite function

$$\frac{\partial z^i}{\partial x^j} = \sum_{k=1}^{n} \frac{\partial z^i}{\partial y^k} \frac{\partial y^k}{\partial x^j}$$

which means that the Jacobi matrix $d\psi_{x,z}$ splits into the product of two matrices dg and df^{-1}. We have used here the formula which expresses the elements of the product of two matrices in terms of the elements of each matrix.

It remains to establish a relation between the Jacobi matrices df and $d(f^{-1})$. Since the composition $f^{-1} \circ f$ is an identity mapping of the domain U into itself (see the definition of a regular coordinate system), we find from the formula just proved that $d(f^{-1} \circ f) = d\left(f^{-1}\right) \circ df = E$, where E is an identity matrix of order n, i.e. finally $\left(d(f^{-1})\right) = (df)^{-1}$. Thus, we have proved that the identity $d\psi_{x,z} = (df) \cdot (df)^{-1}$ holds for the matrix $d\psi_{x,z}$, i.e. $J(\psi_{x,z}) = J(g)/J(f)$, and since the Jacobians $J(g)$ and $J(f)$ are both non-zero, $J(\psi_{x,z})$ is also non-zero. The lemma is proved.

If the mapping $f : U \to V$ defines curvilinear coordinates in U, the mapping $f : U \to V$ defines curvilinear coordinates in U (through Cartesian coordinates in U). We shall often use this simple remark, when passing from the mapping f to f^{-1}.

Let a set of smooth functions $\{x^i(P)\}, 1 \leq i \leq n$, be given on a domain U. A question arises: does this set define a regular coordinate system in U?

Lemma 2 *Let the family of smooth functions $\{x^i(P)\}, 1 \leq i \leq n$ be such that the Jacobian of this system of functions,*

$$J(f = \{x^i(P), 1 \leq i \leq n\}),$$

is non-zero in the domain U. Then, for each point P in U there exists an open neighbourhood such that system of functions $\{x^i(P)\}$ defines in this neighbourhood a regular coordinate system (such a coordinate system may be called a local coordinate system).

Proof. The lemma does not presuppose that the set of functions $\{x^i(P)\}$ defines (at least locally) a one-to-one mapping of the domain U onto a domain V of Euclidean space $\mathbf{R}^n_1$. Using the implicit function theorem (and the existence theorem for inverse mapping), we see that a non-zero Jacobian implies the existence (at least in an open neighbourhood) of an inverse mapping which is also smooth. Thus, the proof of the lemma follows from the definition of a regular coordinate system.

Let us note that the set of functions satisfying Lemma 2 may not define a regular coordinate system in the whole domain U, i.e. the smooth mapping f^{-1} of the domain V onto U may not exist. Here is a simple example.

1.2.1 Example

Let a two-dimensional Cartesian plane with punctured origin O be chosen as the domain U and let the mapping f (defined by two functions $x^1(P)$ and $x^2(P)$) represent the smooth mapping $f(y^1, y^2) = \big(x^1(y^1, y^2), x^2(y^1, y^2)\big)$, where $x^1(y^1, y^2) = (y^1)^2 - (y^2)^2$ and $x^2(y^1, y^2) = 2y^1y^2$, i.e. if we put $z = y^1 + iy^2$, $w = x^1 + ix^2$ (i is the imaginary unit), then $w = z^2$. This mapping transforms a complex number z into this number squared (for convenience, one may assume that the two copies of the Euclidean plane, $\mathbf{R}^2(y)$ and $\mathbf{R}^2(x)$, are identified with each other).

The same mapping can easily be written in the polar coordinates (r, φ) to give $f(r, \varphi) = (r^2, 2\varphi)$. Calculate the Jacobian $J(f)$ (we shall calculate it, for example, in the initial Cartesian coordinate system y^1, y^2 on $\mathbf{R}^2(y)$). The Jacobi matrix df is of the form

$$df = \begin{pmatrix} 2y^1 & 2y^2 \\ -2y^2 & 2y^1 \end{pmatrix}^T,$$

i.e. $J(f) = 4((y^1)^2 + (y^2)^2) > 0$.

We see that the Jacobian is positive at all points of U (since the origin is punctured). Hence, according to Lemma 2, our mapping establishes a local (regular) coordinate system in an open neighbourhood of each point in U. At the same time, the mapping f does not have the inverse mapping f^{-1} because f is not bijective. Indeed, every point $w = x^1 + ix^2 \in \mathbf{R}^2(x)$, other than the origin, always has exactly two inverse images under the mapping f: namely, the points (r, φ) and $(r, \varphi + \pi)$ which are, of course, distinct points of the domain U.

Thus, if a set of functions is chosen as a regular coordinate system in an Euclidean domain U, we should not only verify that the Jacobian of this system is non-zero (at every point of U), but also that the mapping defined by this set is bijective. Note also that in the above example the Jacobian of the system of functions vanishes as the point P tends to zero.

1.3 Definition of manifold

A metric space M is called an *n-dimensional manifold* (or simply *manifold*) if any point P of the space is contained in a neighbourhood $U \subset M$ homeomorphic to a domain V of an Euclidean space $\mathbf{R}^n$. This condition can be formulated in brief as follows: an n-dimensional manifold M is locally homeomorphic to a domain in an Euclidean space $\mathbf{R}^n$, in which case the dimension of M is said to be equal to n or $\dim M = n$. Thus, if M is an n-dimensional manifold, we can find in M a system of open sets $\{U_\alpha\}$ numbered by finitely (or infinitely) many indices α and a system of homeomorphisms $\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbf{R}^n$ of sets U_α on the domains V_α. The system $\{U_\alpha\}$ must cover the space M, i.e. $M = \bigcup\limits_\alpha U_\alpha$, and the domains V_α may in general, intersect one another.

Fix a Cartesian coordinate system $(x^1, ..., x^n)$ in a Euclidean space $\mathbf{R}^n$. Then for any point $P \in U_\alpha$ the Cartesian coordinates of the point $\varphi_\alpha(P) \in V_\alpha$ can be considered as a numerical $k = 1, ..., n$. The system of functions

$$x_\alpha^k = x_\alpha^k(P) = x_\alpha^k\left(\varphi_\alpha(P)\right)$$

given on an open set U_α is called a *local coordinate system*, and the open set U_α together with a local coordinate system defined on it is called a chart of a manifold M. Thus, a chart is a pair $(U_\alpha, \varphi_\alpha)$, and we shall denote it, for brevity, only by the first symbol, U_α. A set of charts $\{U_\alpha\}$ covering the entire manifold M is called an *atlas*. It is convenient to number local coordinates of a point $P \in M$ by an additional index α characterizing the chart $U_\alpha : x_\alpha^k = x_\alpha^k(P)$. Since the point P can belong simultaneously to several charts, it has several sets of local coordinates.

The same manifold M can admit distinct atlases. Even though the charts, as open sets, remain unchanged, we can alter the local coordinate system in a chart by choosing another coordinate homeomorphism. In particular, the following, lemma holds.

Lemma 3 *Let M be an n-dimensional manifold and let U be its chart with a coordinate homeomorphism φ and a local coordinate system $(x^1, ..., x^n)$. If $U' \subset U$ is an open subset of U, a coordinate homeomorphism φ' and a local coordinate system $(y^1, ..., y^n)$ can also be defined on U'. More over one can put $\varphi'(P) = \varphi(P)$, $y^k(P) = x^k(P)$ for $P \in U'$.*

Proof of the lemma 3 follows from the fact that the homeomorphism $\varphi : U \to V$ maps homeomorphically any open subset $U' \subset U$. It is sufficient therefore to take the restriction of φ to U' as φ' and the restriction of coordinate functions x^k to the same subset U' as y^k.

Lemma 3 shows that using a given atlas $\{U_\alpha\}$ we can construct a new atlas consisting of finer charts. On the other hand, the union of two atlases $\{U_\alpha\}$ and $\{U'_\beta\}$ is again an atlas of the manifold.

Thus, there exists a maximal atlas consisting of all the charts of a given manifold. A maximal atlas may be considered as a union of all atlases on a manifold.

Now we shall prove another useful lemma.

Lemma 4 *Let $\{U_\alpha\}$ and $\{U'_\beta\}$ be two atlases on a manifold M. Then there exists a third atlas which refines these two atlases.*

To prove the lemma, we put $W_{\alpha\beta} = U_\alpha \cap U'_\beta$. According to the lemma 3, a local coordinate system can be introduced on each open set $W_{\alpha\beta}$. On the other hand, $W_{\alpha\beta} \subset U_\alpha$ and $W_{\alpha\beta} \subset U'_\beta$, so the system of sets $\{W_{\alpha\beta}\}$ covers M. Hence, $\{W_{\alpha\beta}\}$ is an atlas refining both $\{U_{\alpha\}}$ and $\{U'_\beta\}$.

1.4 Functions on manifolds, transition functions.

Any continuous function $f : M \to R^1$ defined on an n-dimensional manifold M in the neighbourhood of each point $P \in M$ can be identified with an ordinary continuous real-valued function $h(x^1, ..., x^n)$ of n independent real variables $(x^1, ..., x^n)$, the function h being defined in a domain of a Euclidean space R^n. Indeed, let U be a chart containing a point , let $\varphi : U \to V \in R^n$ be a coordinate homeomorphism of this chart, and let $\left(x^1(P), ..., x^n(P)\right)$ be a local coordinate system in U. If $\mathbf{x} = (x^1, ...x^n)$ is a vector with coordinates $(x^1, ...x^n)$, we put $h(x^1, ...x^n) = f\left(\varphi^{-1}(\mathbf{x})\right)$. Conversely, if h is a continuous function of n real variables defined in a domain $V \subset \mathbf{R}^n$, we can associate with h a continuous function f defined in the domain U of the manifold $M : f(P) = h\left(x^1(P)...., x^n(p)\right)$.

More generally, let $f : M_1 \to M_2$ be a continuous mapping of an n-dimensional manifold M_1 into an m-dimensional manifold M_2 . Suppose $Q_0 = f(P_0)$, $P_0 \in M_1$ and $Q_0 \in M_2$. Then in a small neighbourhood $U \ni P_0$ the mapping f can be identified with a continuous vector function h of n independent variables.

Indeed, let $U' \ni Q_0$ be a chart of the manifold M_2 and let $(y^1, ..., y^n)$ be a local coordinate system. Since the mapping f is continuous, there exists, according to Lemma 3, a chart U of point P_0 such that $f(U) \subset U'$.

Suppose $(x^1, ..., x^n)$ is a local coordinate system in U. Since points of the chart U are in a one-to-one correspondence with their coordinates $\left(x^1(P), ..., x^n(P)\right)$, and points Q of the chart U' are also in a one-to-one correspondence with their coordinates $\left(y^1(Q), ..., y^n(Q)\right)$ the equality $Q = f(P)$ means

$$y^k(Q) = y^k\left(f(P)\right) = y^k\left(h\left(x^1(P), ..., x^n(P)\right)\right) = h^k\left(x^1(P), ..., x^n(P)\right).$$

The functions $h^k(x^1, ..., x^n)$ are continuous, and the mapping f is uniquely reconstructed in U by these functions.

In particular any continuous function f on a manifold M in a local coordinate system can be represented by a real-valued function h of n independent variables.

If we alter the local coordinate system, the function h will also be modified. What is the law of modification of h under coordinate transformation? Let $(x^1, ..., x^n)$ and $(y^1, ..., y^n)$ be two local coordinate systems. Without loss of generality, we may assume that these coordinate systems are defined in the same chart U. Suppose h and h' are functions of coordinates $\left(x^1, ..., x^n\right)$ and $\left(y^1, ..., y^n\right)$, respectively, which represent the function f. Then

$$f(P) = h\left(x^1(P), ..., x^n(P)\right) = h'\left(y^1(P), ..., y^n(P)\right). \tag{2}$$

Since the coordinates $\left(y^1, ..., y^n\right)$ are also continuous functions in U, they can in turn be represented as functions of n independent variables $(x^1, ..., x^n)$, i.e.

$$\begin{cases} y^1(P) = y^1\left(x^1(P), \ldots, x^n(P)\right), \\ \vdots \\ y^n(P) = y^n\left(x^1(P), \ldots, x^n(P)\right). \end{cases} \tag{3}$$

In these equations we deliberately use the same symbol y to denote both the coordinate of point P and its representation as a function of $(x^1, ..., x^n) : y^k = y^k(x^1, ..., x^n)$. Then from equation 2 we obtain the identity

$$h(x^1, \ldots, x^n) = h'\left(y^1(x^1, \ldots, x^n), \ldots, y^n(x^1, \ldots, x^n)\right) \tag{4}$$

The functions $y^k = y^k(x^1, ..., x^n)$ on the right-hand side of relations (3) are called the *functions of the coordiaate transformation* or the *transition functions*, provided the set U is a domain in an Euclidean space.

Definition 5 *Let M be an n-dimensional manifold, $\{U_\alpha\}$ be its atlas, φ_α be the coordinate homeomorphisms, and $\{x^k_\alpha\}$ be a set of local coordinate systems. In each intersection of two charts $U_{\alpha\beta} = U_\alpha \cap U_\beta$ two local coordinate systems $\{x^k_\alpha\}$ and $\{x^k_\beta\}$ have the relation $x^k_\alpha(P) = x^k_\alpha\left(x^1_\beta(P), \ldots, x^n_\beta(P)\right)$, $P \in U_{\alpha\beta}$. The functions $x^k_\alpha = x^k_\alpha(x^1_\beta, \ldots, x^n_\beta)$ are called functions of coordinate transformation or functions of transition from the coordinates $\{x^k_\alpha\}$ to the coordinates $\{x^k_\beta\}$.*

Transition functions are not defined in the entire domain V_β, but only in its part $V_{\beta\alpha} = \varphi_\beta(U_{\alpha\beta})$ where it is meaningful to speak about two coordinate systems.

The transition functions $x^k_\alpha = x^k_\alpha\left(x^1_\beta, \ldots, x^n_\beta\right)$ map the domain $V_{\beta\alpha}$ onto $V_{\alpha\beta}$ in $\mathbf{R}^n$:

$$\begin{aligned} x_\alpha &= \{x^k_\alpha(x^1_\beta, \ldots, x^n_\beta)\} = \\ &= \{x^k_\alpha(x_\beta)\} = x_\alpha(x_\beta) = \\ &= \varphi_\alpha \varphi_\beta^{-1}(x_\beta) = \varphi_{\alpha\beta}(x_\beta). \end{aligned} \tag{5}$$

The mappings $\varphi_{\alpha\beta} : V_{\beta\alpha} \to V_{\alpha\beta}$ given by equation (5) are, in fact, another writing of transition functions, and represent a homeomorphism of the domain $V_{\beta\alpha}$ onto $V_{\alpha\beta}$. Note that if $\alpha = \beta$, then $U_{\alpha\beta} = U_\alpha = U_\beta$, $V_{\alpha\beta} = V_{\beta\alpha} = V_\alpha = V_\beta$ and

$$x^k_\alpha = x^k_\alpha\left(x^1_\beta, \ldots, x^n_\beta\right) = x^k_\beta.$$

1.5 Examples

Here are simple examples of manifolds.

Example 1. Consider a circle $\mathbf{S}^1 \in \mathbf{R}^2$ defined by the equation $x^2+y^2=1$. Let us cover $\mathbf{S}^1$ with an atlas consisting of four charts

$$\begin{array}{rcl} U_1 & = & \{(x,y) \in \mathbf{S}^1 : y > 0\}, \\ U_2 & = & \{(x,y) \in \mathbf{S}^1 : y < 0\}, \\ U_3 & = & \{(x,y) \in \mathbf{S}^1 : x > 0\}, \\ U_4 & = & \{(x,y) \in \mathbf{S}^1 : x < 0\}. \end{array} \tag{6}$$

The corresponding domains V_1, V_2, V_3, and V_4 on the real axis $\mathbf{R}^1$ coincide and are equal to the open interval (-1, 1).

Homeomorphisms φ_1 and φ_2 are constructed as projections of the circle onto the x-axis:

$$\varphi_1(x,y) = \varphi_2(x,y) = x,$$

and homeomorphisms φ_3 and φ_4 as projections onto the y-axis:

$$\varphi_3(x,y) = \varphi_4(x,y) = y.$$

In order to prove that the mappings φ_k, $k = 1, \ldots, 4$, are homeomorphisms, it is sufficient to write explicitly the inverse mappings

$$\begin{array}{rcl} \varphi_1^{-1}(x) & = & (x, \sqrt{1-x^2}) \in \mathbf{S}^1, \\ \varphi_2^{-1}(x) & = & (x, -\sqrt{1-x^2}) \in \mathbf{S}^1, \\ \varphi_3^{-1}(x) & = & (\sqrt{1-y^2}, y) \in \mathbf{S}^1, \\ \varphi_3^{-1}(x) & = & (-\sqrt{1-y^2}, y) \in \mathbf{S}^1. \end{array} \tag{7}$$

and demonstrate that they are continuous. Then we obtain on the circle four local coordinate systems, each consisting only of one coordinate:

$$\begin{array}{rcl} x_1 & = & \varphi_1(x,y) = x, \\ x_2 & = & \varphi_2(x,y) = x, \\ x_3 & = & \varphi_3(x,y) = y, \\ x_4 & = & \varphi_4(x,y) = y. \end{array} \tag{8}$$

Certain points are supplied with two local coordinate systems. For instance, for points P of the intersection $U_1 \cap U_2$ the coordinates $x_1(P)$ and $x_3(P)$ are defined.

There are other ways of introducing an atlas on a circle. For example we can consider polar coordinates (r, φ) on a plane. The equation of a circle in these coordinates is very simple: $r = 1$. Strictly speaking, polar coordinates on a plane are not a coordinate system. We introduce therefore two charts on a circle S^1, namely

$$\begin{array}{rcl} U_1 & = & \{(x,y) \in \mathbf{S}^1 : x \neq -1\}, \\ U_2 & = & \{(x,y) \in \mathbf{S}^1 : x \neq 1\}. \end{array} \tag{9}$$

Let $\varphi_1(P) = \varphi_1(x, y)$ be the value of φ in the interval $(-\pi, \pi)$ and $\varphi_2(P) = \varphi_2(x, y)$ be the value of φ in the interval $(0, 2\pi)$, i.e. $V_1 = (-\pi, \pi)$, $V_2 = (0, 2\pi)$.

Obviously, the local coordinates $\varphi_1 = \varphi_1(P)$ and $\varphi_2 = \varphi_2(P)$ coincide for points of the upper semicircle and do not coincide for points of the lower semicircle, that is, $\varphi_1(x, y) = \varphi_2(x, y)$ for $y > 0$ and $\varphi_1(x, y) = \varphi_2(x, y) - 2\pi$ for $y < 0$.

Example 2. The circle $\mathbf{S}^1$ considered in Example 1 is a rather complicated manifold. The simplest example is represented by Euclidean space $\mathbf{R}^n$. We may take an atlas consisting of only one chart $U = \mathbf{R}^n$, the coordinate homeomorphism φ is the identity mapping $\varphi : U \to V = \mathbf{R}^n$, the local coordinate system is Cartesian coordinates of points in $\mathbf{R}^n$. Similarly, any domain $U \subset \mathbf{R}^n$ is an n-dimensional manifold whose atlas also consists of one chart with a Cartesian coordinate system.

Example 3. Let $f : \mathbf{R}^n \to \mathbf{R}^1$ be a continuous function and $\Gamma_f \subset \mathbf{R}^{n+1}$ be its *graph*, i.e. the set of points

$$(x_1, ..., x^n, x^{n+1}) : x^{n+1} = f(x^1, ..., x^n).$$

The space Γ_f is an n-dimensional manifold with an atlas consisting of one chart $U = \Gamma_f$. The coordinate homeomorphism $\varphi : U \to V = \mathbf{R}^n$ is defined as a projection along the last coordinate:

$$\varphi(x^1, ..., x^n, x^{n+1}) = (x^1, ..., x^n) \in \mathbf{R}^n.$$

Then the inverse mapping φ^{-1} is given by

$$\varphi^{-1}(x^1, ..., x^n) = (x^1, ..., x^n, f(x^1, ..., x^n))$$

and is, apparently, a continuous mapping.

Example 4. Let us consider an *n-dimensional* sphere $\mathbf{S}^n$ of unit radius defined as a set of points in $\mathbf{R}^{n+1}$ satisfying the equation

$$(x^1)^2 + ... + (x^n)^2 = 1.$$

We shall demonstrate that an n-dimensional sphere is an n-dimensional manifold. The open sets

$$\begin{array}{rcl} U_i^+ & = & \{(x^1, ..., x^{n+1}) \in S^n : x^i > 0\}, \\ U_i^- & = & \{(x^1, ..., x^{n+1}) \in S^n : x^i < 0\} \end{array} \tag{10}$$

can be taken as an atlas. We obtain $2n + 2$ open sets covering the entire sphere $\mathbf{S}^n$. Indeed, if the point $P = (x^1, ..., x^{n+1})$ belongs neither to charts U_i^+ nor to charts U_i^- for all i, then the inequalities $x^i \leq 0$ and $x^i \geq 0$ are satisfied, i.e. $x^i = 0$, $i = 1, 2, \ldots, n + 1$. Then $(x^1)^2 + \cdots + (x^n)^2 = 0$, that is the point P does not lie on the sphere $\mathbf{S}^n$. The coordinate homeomorphisms φ_i^+ and φ_i^- are defined as projections of the Euclidean space $\mathbf{R}^{n+1}$ onto $\mathbf{R}^n$ along the coordinate x^i. In this case the domains V_i^+ and V_i^- coincide and are equal to a unit ball, and the coordinate homeomorphisms are given by

$$\varphi_i^+(x^1, ..., x^{n+1}) = \varphi_i^- \left(x^1, ..., x^{n+1}\right) = (x^1, ..., x^{i-1}, x^{i+1}, ..., x^{n+1}) \in \mathbf{R}^n.$$

The inverse homeomorphisms are defined by the formulas

$$(\varphi_i^+)^{-1}(y^1,...,y^n) = (y^1,...,y^{i-1},\sqrt{1-(y^1)^2-...-(y^n)^2},y^i,...,y^n) \in \mathbf{R}^{n+1},$$

$$(\varphi_i^-)^{-1}(y^1,...,y^n) = (y^1,...,y^{i-1},-\sqrt{1-(y^1)^2-...-(y^n)^2},y^i,...,y^n) \in \mathbf{R}^{n+1},$$

and are apparently continuous.

Example 5. Let us consider a projective plane $\mathbf{RP}^2$. By this we mean a space the points of which are straight lines through the origin in $\mathbf{R}^3$. Define the distance between two straight lines as the least angle between them. Then $\mathbf{RP}^2$ becomes a metric space. We now prove that $\mathbf{RP}^2$ is a two-dimensional manifold. It is convenient to describe any straight line $P \in \mathbf{RP}^2$ by three homogeneous coordinates $[x:y:z]$ which admit multiplication by a number $\lambda \neq 0$. Homogeneous coordinates do not vanish simultaneously, i.e. $x^2+y^2+z^2>0$. Cover $\mathbf{RP}^2$ with three charts:

$$\begin{array}{rcl} U_1 & = & \{[x:y:z]:x\neq 0\}, \\ U_2 & = & \{[x:y:z]:y\neq 0\}, \\ U_3 & = & \{[x:y:z]:z\neq 0\}. \end{array} \tag{11}$$

Let $V_1 = V_2 = V_3 = R^2$. The mappings $\varphi_k : U_k \to V_k = \mathbf{R}^2$ are taken as coordinate homeomorphisms, namely

$$\begin{array}{rcl} \varphi_1(x,y,z) & = & (y/x, z/x), \\ \varphi_2(x,y,z) & = & (x/y, z/y), \\ \varphi_3(x,y,z) & = & (x/z, y/z). \end{array} \tag{12}$$

Thus, we have constructed three local coordinate systems

$$x_1^1 = y/x, \quad x_1^2 = z/x;$$

$$x_2^1 = x/y, \quad x_2^2 = z/y;$$

$$x_3^1 = x/z, \quad x_3^2 = y/z.$$

To verify that the mappings φ_k are homeomorphisms, it is sufficient to construct the inverse mappings

$$\varphi_1^{-1}(x_1^1,x_1^2) = [1:x_1^1:x_1^2],$$

$$\varphi_2^{-1}(x_2^1,x_2^2) = [x_2^1:1:x_2^2],$$

$$\varphi_3^{-1}(x_3^1,x_3^2) = [x_3^1:x_3^2:1]$$

and prove that they are continuous.

1.6 Smooth structure of the class C^k. Class C^∞, real and complex analytic manifolds.

Let us now return to the representation of a continuous function f defined on an n-dimensional manifold M as a function h of n independent variables, local coordinates of a point of the manifold. It is known that a narrower class of function - differentiable functions - is of great significance in mathematical analysis. We now transfer this important concept to functions defined on a manifold. If a function $h\left(x^1,\ldots,x^n\right)$ is continuously differentiable, we cannot say the same about the function $h'\left(y^1,\ldots,y^n\right)$ representing f in another local coordinate system $\left(y^1,\ldots,y^n\right)$. Indeed, the functions h and h' are related by equation (4). Thus, the condition that h' is also continuously differentiable is that the functions of coordinate transformation $x^k = x^k(y^1,...,y^n)$ should be continuously differentiable. If these functions are not continuously differentiable, there exists a function f such that its representation h in the coordinates $\left(x^1,\ldots,x^n\right)$ is a continuously differentiable function, while the representation h' in the coordinates $\left(y^1,\ldots,y^n\right)$ is not. As an example, we consider the function $f(P) = x^k(P)$, $P \in U \subset M$. Then $h\left(x^1,\ldots,x^n\right) \equiv x^k$, apparently, a continuously differentiable function, while $h'\left(y^1,\ldots,y^n\right) = x^k\left(y^1,\ldots,y^n\right)$ does not possess this property.

We thus arrive at the following definition.

Definition 6 *A smooth n-dimensional manifold is an n-dimensional manifold M with an atlas $\{U_\alpha\}$ having local coordinate systems $\{x^k_\alpha\}$ satisfying the condition: the transition functions $x^k_\alpha = x^k_\alpha\left(x^1_\beta,\ldots,x^n_\beta\right)$ are continuously differentiable for any pair of charts U_α and U_β in the entire domain of definition. One says that the atlas $\{U_\alpha\}$ with fixed local coordinate systems $\{x^k_\alpha\}$ represents a smooth structure on the manifold M.*

This definition enables a class of continuously differentiable functions to be distinguished among all functions valid on a manifold M.

Definition 7 *A function $f : M \to \mathbf{R}^1$ defined on a smooth manifold M is called continuously differentiable at a point $P_0 \in M$ (with respect to a smooth structure on the manifold M) if in any local coordinate system $(x^1_\alpha,\ldots,x^n_\beta)$ (the charts $U_\alpha \ni P_0$ belong to a fixed atlas which represents the smooth structure) the function f can be represented as a continuously differentiable function $h(x^1_\alpha,\ldots,x^n_\alpha)$ of n independent variables in a neighbourhood of the point $\left(x^1_\alpha(P_0),...,x^n_\alpha(P_0)\right)$.*

Note that the condition of continuously differentiability of the transition function in Definition 6 is essential for Definition 7. As we have already pointed out, if the transition functions were not (continuously) differentiable, the condition of continuously differentiability of f at the point $P_0 \in M$ would depend on the choice of the chart U_i containing P_0.

Example 6. We now consider the following atlas on a manifold M. Let $M = \mathbf{R}^1$ be a real axis and let the atlas consist of two identical charts $U_\alpha =$

$U_\beta = M = \mathbf{R}^1$, but with distinct coordinate systems. On U_α we define the coordinate $x_\alpha = x$, $x \in \mathbf{R}^1$, and on U_β the coordinate $x_\beta = x^3$. Then the transition functions are

$$x_\beta = x_\beta(x_\alpha) = (x_\alpha)^3, \tag{13}$$

$$x_\alpha = x_\alpha(x_\beta) = \sqrt[3]{x_\beta} \tag{14}$$

While the transitions function (13) is continuously differentiable (a polynomial), the function (14) has a discontinuous derivative. Thus, according to the definition 3, the manifold M with the atlas $\{U_\alpha, U_\beta\}$ is not a smooth manifold.

Remark. If an atlas on a manifold M consists of only one chart (i.e. M is homeomorphic to a domain in Euclidean space), M is a smooth manifold (with respect to this atlas).

Definition 8 *Let on a manifold M there be given two atlases $\{U_\alpha\}$ and $\{U'_\beta\}$ such that M is smooth with respect to each atlas. Two atlases $\{U_\alpha\}$ and $\{U'_\beta\}$ are called equivalent (or represent the same smooth structure on the manifold M) if every function of coordinate transformation from any local coordinate system in $\{U_\alpha\}$ to any local coordinate system in $\{U'_\beta\}$ is continuously differentiable.*

A substantiation of this definition lies in the fact that any function f on the manifold M is continuously differentiable in the atlas $\{U_\alpha\}$ if and only if it is continuously differentiable in $\{U'_\beta\}$. Thus, from the point of view of continuously differentiable functions on a manifold M equivalent atlases "have equal rights", so that any of the equivalent atlases can he used to represent a function as a continuously differentiable real-valued function of independent variables (coordinates of a point). Definition 8 admits another formulation: two atlases $\{U_\alpha\}$ and $\{U'_\beta\}$ are equivalent if M is a smooth manifold with respect to a new atlas equal to the union of two initial atlases, $\{U_\alpha\} \cup \{U'_\beta\}$.

We often have to deal with narrower classes of functions. Recall that a real-valued function $h(x^1, ...x^n)$ is smooth of classes $C^r (r = 1, 2, ..., \infty)$ in a neighbourhood of a point $(x^1_0, ..., x^n_0)$ if in this neighbourhood all partial derivatives of h up to order r exist and are continuous. For $r = \infty$ this means that the function h has continuous partial derivatives of any order. Consequently, we shall impose on atlases of a manifold M the conditions formulated in the following definition.

Definition 9 (6a) *A manifold M with a fixed atlas $\{U_\alpha\}$ is called a smooth manifold of the class $C^r (r = 1, 2, ..., \infty)$ (or C^r-manifold) if all the transition functions are smooth of class C^r at all points of the domain of their definition.*

Definition 10 (7a) *Let M be a C^r-manifold and let f be a continuous function on this manifold. The function f is called smooth of class C^s, $s \leq r$ (or C^s-function) in a neighbourhood of a point $P_0 \in M$ if any representation of f as a function h of local coordinates $(x^1, \ldots, x^n)$ (from a fixed atlas) is a C^s-function in neighbourhood of the point $(x^1(P_0), ..., x^n(P_0))$. The function f is smooth of class C^s if it is of class C^s in a neighbourhood of each point P_0 in the domain of definition.*

Example 7. Let us modify Example 6 by choosing the coordinate $x_\beta = x\,|x|$ in the second chart U_β. Then M is a C^1-manifold, but it is not C^2-manifold.

Remark. Below, if not stated otherwise, we shall consider only C^∞-manifolds and C^∞-functions on these manifolds.

In examples 1-4 we have considered manifolds with such atlases that these manifolds are always of class C^∞.

Geometry may also deal with more strict conditions on atlases and their transition functions. For example, if all transition functions are real-analytic, i.e. in a neighbourhood of each point in their domain these functions can be expanded into convergent Taylor series, the manifold is called *a real-analytic manifold*. A real-analytic manifold is C^∞-manifold.

A more important class of manifolds is represented by complex-analytic manifolds. Let M be a $2n$-dimensional manifold, $\{U_\alpha\}$ its atlas, and $\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbf{R}^{2n}$ its coordinate homeomorphisms. Identify a 2n-dimensional Euclidean space $\mathbf{R}^{2n}$ with an n-dimensional complex linear space $\mathbf{C}^n$, assuming that complex coordinates of a point $(z^1, \ldots, z^n)$ give rise to $2n$ real coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$, $z^k = x^k + iy^k$. Then, $2n$ coordinate functions $\big(x^1_\alpha(P), \ldots, x^n_\alpha(P), \quad y^1_\alpha(P), ..., y^n_\alpha(P)\big)$ in the chart U_α are transformed into n complex-valued functions $\big(z^k_\alpha(P) = x^k_\alpha(P) + iy^k_\alpha(P)\big)$. The functions $z^k_\alpha(P)$ are called complex coordinates of a point in the chart U_α. In the intersection of two charts $U_\alpha \cap U_\beta$ the transition functions are given by

$$\begin{cases} x^1_\alpha = x^1_\alpha(x^1_\beta, \ldots, x^n_\beta, y^1_\beta, \ldots, y^n_\beta), \\ \vdots \\ x^n_\alpha = x^n_\alpha(x^1_\beta, \ldots, x^n_\beta, y^1_\beta, \ldots, y^n_\beta), \\ y^1_\alpha = y^1_\alpha(x^1_\beta, \ldots, x^n_\beta, y^1_\beta, \ldots, y^n_\beta), \\ \vdots \\ y^n_\alpha = y^n_\alpha(x^1_\beta, \ldots, x^n_\beta, y^1_\beta, \ldots, y^n_\beta). \end{cases} \tag{15}$$

These functions can be represented as complex-valued functions of n independent variables

$$\begin{cases} z^1_\alpha = z^1_\alpha(z^1_\beta, \ldots, z^n_\beta), \\ \vdots \\ z^1_\alpha = z^1_\alpha(z^1_\beta, \ldots, z^n_\beta). \end{cases} \tag{16}$$

Functions (16) are called transition functions or functions of transformation of complex coordinates.

A manifold M with a fixed atlas $\{U_\alpha\}$ and local complex coordinate systems $(z^1_\alpha, \ldots, z^n_\alpha)$ is called *a complex-analytic manifold*, provided all transition functions (16) are complex-analytic, i.e. they can be expanded in convergent Taylor series of complex variables in a neighbourhood of each point in the corresponding domain of definition.

As an example of a manifold admitting a complex-analytic structure, we shall consider a two-dimensional sphere $\mathbf{S}^2$ with a specially defined atlas. We can construct the stereographic projection of the sphere $\mathbf{S}^2 = x^2 + y^2 + z^2 = 1$ from

the north pole $P_0 = (0,0,1)$ onto the coordinate plane (x,y). This projection, denoted by φ_α, maps all the points of $\mathbf{S}^2$ except for the pole P_α (i.e. the open set $U_\alpha = \mathbf{S}^2 \backslash (P_\alpha))$ homeomorphically onto the entire plane $V_\alpha = \mathbf{R}^2$. In Cartesian coordinates the homeomorphism φ_α is of the form $\varphi_\alpha(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$. We introduce therefore in the chart U_α one complex coordinate $w_\alpha = \frac{x+iy}{1-z}$ expressed in terms of the Cartesian coordinates on a sphere. Furthermore, we shall consider the south pole $P_\beta = (0,0,-1)$ and the stereographic projection φ_β from the south pole onto the same coordinate plane (x,y). The projection φ_β maps homeomorphically the set $U_\beta = S^2 \backslash (P_\beta)$ onto the entire plane $V_\beta = \mathbf{R}^2$. In Cartesian coordinates φ_1 takes the form $\varphi_\beta(x,y,z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$. Introduce in the chart U_1 a complex coordinate $w_\beta = \frac{x+iy}{1+z}$.

Then, in the intersection $U_\alpha \cap U_\beta$ we obtain $w_\alpha w_\beta = \frac{x^2+y^2}{1-z^2} = 1$. Hence,

$$\begin{array}{rcl} w_\alpha & = & w_\alpha(w_\beta) = \frac{1}{w_\beta}, \\ w_\beta & = & w_\beta(w_\alpha) = \frac{1}{w_\alpha}. \end{array} \tag{17}$$

Functions (17) are complex-analytic, so the sphere $\mathbf{S}^2$ is a complex-analytic manifold. Each chart U_α and U_β covers the entire sphere $\mathbf{S}^2$, except for one point, and is identified by virtue of the coordinate homeomorphisms φ_α and φ_β with the complex plane $\mathbf{C}^1 = \mathbf{R}^2$. Thus, the sphere $\mathbf{S}^2$ is usually identified with the so-called completed complex plane obtained from $\mathbf{C}^1$ by addition an extra "infinitely distant" point.

An arbitrary smooth manifold need not necessarily be a complex-analytic one. For example, if its dimension is odd, the manifold is not complex-analytic by trivial arguments. Yet, there exist manifolds of even dimension which do not admit a complex-analytic structure either.

1.7 Invariance of the dimension of the smooth manifolds.

Let M_1 and M_2 be smooth manifolds and let $f : M_1 \to M_2$ be a continuous mapping. As was already noted, in a neighbourhood of any point $P_0 \in M_1$ the mapping f can be represented as a vector function , $y^k = h^k(x^1, ..., x^n)$, where $(x^1, ..., x^n)$ is a local coordinate system in the neighbourhood of $P_0 \in M_1$ and $(y^1, ..., y^n)$ is a local coordinate system in the neighbourhood of point $Q_0 = f(P_0) \in M_2$.

Definition 11 *The mapping $f : M_1 \to M_2$ smooth manifolds is called a smooth mapping of class $C^r (r = 1, 2, ..., \infty)$ (or C^r-$(x^1, ..., x^n)$ in the neighbourhood of any point $P_0 \in M_1$ and $(y^1, ..., y^n)$ in the neighbourhood of point $Q_0 = f(P_0) \in M_2$ the representation of f as a vector function $y = (y^k) = (h^k(x^1, ..., x^n)) = h(x)$ is a vector function of class C^r.*

Note that the definition of a C^r-manifold has a meaning if only the manifolds M_1 and M_2 are smooth of class not less than C^r.

Let $f : M_1 \to M_2$ be a homeomorphism of manifolds. If f is a C^r- mapping, the inverse mapping f^{-1} need not be a smooth mapping. Therefore, if the inverse mapping $f^{-1} : M_2 \to M_1$ is also a C^r -mapping the homemorphism f is called a *smooth homeomorphism of class* C^r or C^r*-diffeomorphism.* Diffeomorphisms of smooth manifolds play the same role as homeomorphisms of topological spaces.

If $f : M_1 \to M_2$ is a diffeomorphism, the manifolds M_1 and M_2 are called diffeomorphic. The set of all manifolds is subdivided into non-intersecting classes of pairwise diffeomorphic manifolds. Any general property of smooth manifolds, smooth functions or mappings on a manifold can be transferred to any other diffeomorphic manifold. We shall not therefore distinguish between diffeomorphic manifolds.

There are however such properties of manifolds that their "identity" for a pair of diffeomorphisms is not quite obvious. In particular, we have assigned to each manifold a numerical characteristic, the dimension. Do diffeomorphic manifolds have the same dimension?

Theorem 1 *Let $f : M_1 \to M_2$ be a C^r–homeomorphism $(r \geq 1)$ of smooth manifolds. Then* $\dim M_1 = \dim M_2$.

Proof.
Suppose $P_0 \in M_1$ is an arbitrary point, $Q_0 = f(P_0)$, and $g = f^{-1}$ is the inverse mapping. Choose local coordinates $(x^1, ..., x^n)$ in the neighbourhood U_0 of point P_0 and local coordinates $(y^1, ..., y^m)$ in the neighbourhood V_0 of point Q_0. Then the mappings f and g can be represented as vector functions $x = h^{-1}(y)$ and $y = h(x)$, with $h(h^{-1}(x)) \equiv x$ and $h^{-1}(h(y)) \equiv y$. The mapping h consists of m functions $y^l = h^l\left(x^1, \ldots, x^n\right)$ of n independent variables $\left(x^1, \ldots, x^n\right)$. Consider the Jacobi matrix of all partial derivatives of h:

$$dh = \begin{pmatrix} \frac{\partial h^1}{\partial x^1} & \cdots & \frac{\partial h^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial h^m}{\partial x^1} & \cdots & \frac{\partial h^m}{\partial x^n} \end{pmatrix}$$

The matrix dh is a rectangular one of order $(m \times n)$, i.e. it has m rows and n columns.

Lemma 5 *Let $U_0 \subset \mathbf{R}^n$, $V_0 \subset \mathbf{R}^m$, and $W_0 \subset \mathbf{R}^k$ be open Euclidean domains, let $f : U_0 \to V_0$ and $g : V_0 \to W_0$ be continuously differentiable mappings, and let $h : U_0 \to W_0$ be the composition of f and g, i.e. $h(P) = g(f(P))$. Then*

$$dh(P) = dg(f(P)) \cdot df(P), \quad P \in U_0.$$

In other words, the Jacobi matrix of the composition $h = g \circ f$ is the product of the Jacobi matrices of the mappings g and f.

We now apply Lemma 5 to a pair of mappings h and h^{-1} . Let e_0 be the composition of the mappings h and h^{-1} and let e_1 be the composition of h^{-1} and h, i.e. $e_0(Q) = h(h^{-1}(Q)) \quad Q \in V_0$, and $e_1(P) = h^{-1}(h(P) \quad P \in U_0$.

The mappings $e_0 : V_0 \to V_0$ and $e_1 : U_0 \to U_0$ are both identity mappings, so that their Jacobi matrices are identity matrices of order m and n, respectively. In particular, rank$de_0 = m$ and rank$de_1 = n$. On the other hand, by Lemma 5 we have

$$de_0(Q) = dh(h^{-1}(Q)) \cdot dh^{-1}(Q), \qquad Q \in V_0.$$

$$de_1(P) = dh^{-1}(h(P)) \cdot dh(P), \qquad P \in U_0.$$

It is known from linear algebra that the rank of the product of two matrices does not exceed the rank of each matrix. Since rank $dh \leq \min(m, n)$ and rank $dh^{-1} \leq \min(m, n)$ (the matrices dh and dh^{-1} are rectangular!), rank $de_0 \leq \min(m, n)$ and rank $de_1 \leq \min(m, n)$. Hence, $m \leq \min(m, n)$ and $n \leq \min(m, n)$ or $\max(m, n) \leq \min(m, n)$ that is, $m = n$. The theorem 1 is proved.

The concept of dimension has a sense not only for smooth manifolds, but for arbitrary manifolds as well. A question naturally arises: do the dimensions of homeomorphic manifolds coincide? The answer is yes, i.e. *if M_1 and M_2 are two homeomorphic manifolds,then* $\dim M_1 = \dim M_2$.

1.8 Implicit function theorem. Regular points, regular values. Inverse image of regular value.

There is a conventional way, used most frequently in practice, to describe and construct manifolds.

Many examples of manifolds appear as a set of solutions of a non-linear equation given in a Euclidean space. For instance, an n-dimensional sphere $\mathbf{S}^n$ in a Euclidean space $\mathbf{R}^{n+1}$ is defined by the equation $(x^1)^2 + ... + (x^{n+1})^2 = 1$; a pseudosphere $\mathbf{S}_1^2$ is given by $x^2 + y^2 - z^2 = -1$. In general, if $f(x^1, ..., x^n)$ is a continuously differentiable function, the set of solutions of the equation $f(x^1, ..., x^n) - c = 0$ is called a *manifold of level c of the function f*. Thus, the Euclidean space $\mathbf{R}^n$ is decomposed into a union of level manifolds for the function f. In the case of a function of two variables, the solutions of the equation are called level lines for f and in the case of a function of three variables level surface.

To justify the term the level manifold for a function f, one should prove that the level manifold for f is really a manifold. However, this is not always the case.

Nevertheless, a level manifold for a continuously differentiable function f is almost always a manifold.

Theorem 2 *Let $f = f(x^1, ..., x^n)$ be a function of class C^∞ defined in the entire Euclidean space $\mathbf{R}^n$ and let*

$$M_c = \{(x^1, ..., x^n) : f(x^1, ..., x^n) = c\}.$$

If the gradient of f is non-zero at each point of the set M_c, this set is a smooth $(n-1)$-dimensional manifold of class C^∞, and we can choose $(n-1)$ Cartesian coordinates of the ambient Euclidean space $\mathbf{R}^n$ as local coordinates in a neighbourhood of a point $P_0 \in M_c$.

Proof.
The theorem is, in fact, the implicit function theorem formulated in convenient terms. Fix a point $P_0 \in M_c$, $P_0 = (x_o^1, ..., x_0^n)$. Since

$$grad_{P_0} f \neq 0, \quad grad\ f = \left(\frac{\partial f}{\partial x^1}, ..., \frac{\partial f}{\partial x^n} \right),$$

there exists a non-zero partial derivative at point P_0. Without loss of generality, we may assume

$$-\frac{\partial f}{\partial x^n}(x_o^1, ..., x_0^n) \neq 0.$$

Let $Q_0 = (x_o^1, ..., x_0^{n-1})$ be a point in R^{n-1} which is the image of P_0 under projection along the coordinate axis x. According to the implicit function theorem, there exists a neighbourhood $V_0 \ni P_0$ of point Q_0 an interval $(x-\delta, x+\delta)$, and a continuous function $y = y(x^1, ..., x^{n-1})$ of class C^∞ defined in V_0 such that:

1. $f\left(x^1, \ldots, x^{n-1}, y\left(x^1, \ldots, x^{n-1}\right)\right) \equiv c$ in V_0,
2. $x_0^n = y\left(x_0^1, ..., x_0^n\right)$,
3. $\left|x_0^n - y(x^1, ..., x^{n-1})\right| < \delta$ in the domain V_0,
4. any solution $(x^1, ..., x^n) \in V_0 \times (x_0^n - \delta,\ x_0^n + \delta)$ of the equation $f\ (x^1, ..., x^n) = c$ is of the form $x^n = y(x^1, ..., x^{n-1})$.

Let

$$U_0 = M_c \cap (V_0 \times (x_0^n - \delta,\ x_0^n + \delta))$$

is a neighbourhood of the point $P_0 \in M_c$. This neighbourhood is the chart in question which contains P_0. Take the restriction of the projection $R^n \to R^{n-1}$ to U_0 as a coordinate homeomorphism

$$\varphi_0(x^1, ..., x^n) = (x^1, ..., x^{n-1}) \in V_0$$

and define the inverse mapping φ_0^{-1} by the relation

$$\varphi_0^{-1}(x^1, ..., x^{n-1}) = (x^1, ..., x^{n-1}, y(x^1, ..., x^{n-1})).$$

It follows from condition (3) that

$$\varphi_0^{-1}(x^1, ..., x^{n-1}) \in V_0 \times (x_0^n - \delta,\ x_0^n + \delta)$$

and from condition (1) that

$$\varphi_0^{-1}(x^1, ..., x^{n-1}) \in M_c.$$

Thus,

$$\varphi_0^{-1}(x^1, ..., x^{n-1}) \in U_0,$$

i.e. the mappings φ_0 and φ_0^{-1}, are continuous and mutually inverse.

We have proved that M_c is an $(n-1)$-dimensional manifold and found in the neighbourhood of each point $P_0 \in M_c$ a local coordinate system formed by some Cartesian coordinates in the Euclidean space $\mathbf{R}^n$. We now prove that the transition functions are smooth. Let P_0 be also contained in another chart U_1 and let Cartesian coordinates $(x^1, ..., x^{i-1}, x^{i+1}, ..., x^n)$ be chosen as local coordinates in U_1. Then in the intersection $U_0 \cap U_1$ the coordinates $(x^1, ..., x^{i-1}, x^{i+1}, ..., x^n)$ are expressed in terms of $(x^1, ..., x^{n-1})$ as

$$\begin{cases} x^1 = x^1, \\ \vdots \\ x^{i-1} = x^{i-1}, \\ x^{i+1} = x^{i+1}, \\ \vdots \\ x^{n-1} = x^{n-1}, \\ x^n = y(x^1, ..., x^{n-1}). \end{cases} \tag{18}$$

Since the function $y = y(x^1, ..., x^{n-1})$ is smooth of class C^∞, all the functions (18) are also of class C^∞. This completes the proof of Theorem 2.

1.8.1 Examples

1. Consider an n-dimensional sphere $\mathbf{S}^n$ given by the equation

$$f\ (x^1, ..., x^n) = \sum_{k=1}^{n+1} (x^k)^2 = 1.$$

The gradient of f is

$$\mathbf{grad} f = \left(2x^1, 2x^2, ..., 2x^{n+1}\right).$$

If the point $P = (x^1, ..., x^{n+1})$ lies on $\mathbf{S}^n$ then not all of its coordinates vanish, that is, one of the gradient coordinates is non-zero. Conditions of Theorem 2 are satisfied, hence $\mathbf{S}^n$ is a C^∞-manifold.

2. Consider an Euclidean space $\mathbf{R}^{n^2}$ of dimension n^2 and represent the points of $\mathbf{R}^{n^2}$ as square matrices of order n with the coordinates $A = (a_{ij})$. Consider also the set $\mathbf{SL}(n, \mathbf{R})$ of all matrices $A \in \mathbf{R}^{n^2}$ with the determinant equal to unity, $\det A = 1$.The set $\mathbf{SL}(n, \mathbf{R})$ is a group with respect to multiplication of matrices and is called a special linear group. We shall demonstrate that $\mathbf{SL}(n, \mathbf{R})$ is a C^∞-manifold of dimension $n^2 - 1$. Consider a function f of n^2 variables, $f(a_{ij}) = \det(a_{ij})$. This function is a polynomial and is therefore of class C^∞. In order to apply Theorem 2, we have to calculate $\mathbf{grad} f$ at all

points of the group $\mathbf{SL}(n, \mathbf{R})$. Let E be an identity matrix. Since $\det E = 1$, $E \in \mathbf{SL}(n, \mathbf{R})$. Calculate $\mathbf{grad} f$ at the point E. One has

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \ldots + (-1)^{n+1} \det A_{1n} \cdot a_{1n}. \tag{19}$$

This expression contains the determinants of the matrices A_{1k} which are polynomials of all variables (a_{ij}) except those in the first row. Then the partial derivative of f with respect to a_{11} is of the form $\frac{\partial f}{\partial a_{11}} = \frac{\partial}{\partial a_{11}}(a_{11} \det A_{11}) = \det A_{11}$. At point E we obtain

$$\frac{\partial f}{\partial a_{11}}(E) = 1. \tag{20}$$

Thus, the gradient of f at the point E is non-zero. We now demonstrate that at arbitrary point $A_0 \in SL(n, R)$ the gradient $\mathbf{grad} f$ is also non-zero. Introduce new variables b_{ij} defined by $(b_{ij}) = B = A_0^{-1} A = A_0^{-1} \cdot (a_{ij})$. If $A = A_0$, then $B = E$ and

$$f(A) = f(A_0 B) = \det A_0 \cdot \det B = \det A_0 \cdot f(B).$$

Differentiating the superposition of functions, we obtain

$$\frac{\partial f}{\partial b_{11}}(E) = \sum_{i,j} \frac{\partial f}{\partial a_{ij}}(A_0) \cdot \frac{\partial a_{ij}}{\partial b_{11}} \tag{21}$$

The left-hand side of Eq. (21) is equal to unity according to formula (20); hence, at least one of the terms on the right-hand side of Eq. (21) is non-zero and therefore one of the partial derivatives $\frac{\partial f}{\partial a_{ij}}(A_0)$, as well as $\mathbf{grad} f$, is also non-zero. Thus, the conditions of Theorem 2 are satisfied, so that the group $\mathbf{SL}(n, \mathbf{R})$ is a smooth manifold of dimension $(n^2 - 1)$.

Theorem 2 can easily be extended to a system of non-linear equations. Note that Theorem 2 admits another formulation. The gradient of a function f can be represented as a column of partial derivatives of f and is therefore the Jacobi matrix of f. Then the condition of nontriviality of $\mathbf{grad} f$ at a point $P_0 \in \mathbf{R}^n$ is equivalent to the condition that the rank of the Jacobi matrix df of the function f is equal to unity, i.e. is maximal.

Consider a system of equations

$$\begin{cases} f^{\,1}(x^1, ..., x^n) = c^1, \\ f^{\,2}(x^1, ..., x^n) = c^2, \\ \vdots \\ f^{\,k}(x^1, ..., x^n) = c^k. \end{cases} \tag{22}$$

which can be written in compact form as $f(x) = c$, where

$$x = (x^1, ..., x^n) \in R^n,$$

$$c = (c^1, ..., c^k) \in R^k,$$

and f is a mapping defined by functions $(f^1, \ldots, f^k)$. The set M_c of solutions of system (22) is called a level manifold for the system of functions $(f^1, \ldots, f^k)$.

Theorem 3 *Let $f : \mathbf{R}^n \to \mathbf{R}^k$ be a C^∞-mapping and let M_c be a set of solutions of the system of equations $f(x) = c$. If the rank of the Jacobi matrix of f is maximal at every point $P_0 \in M_c$ (i.e. rank$df(P_0) = k$), M_c is an $(n-k)$-dimensional smooth manifold of class C^∞, and $(n-k)$ Cartesian coordinates of the surrounding Euclidean space $\mathbf{R}^n$ can be taken as local coordinates in the neighbourhood of each point $P_0 \in M_c$.*

.

The proof of Theorem 23is exactly analogous to that of Theorem 2 with the only difference that instead of one variable x^n we choose k variables $(x^{i_1}, ..., x^{i_k})$. Denoting this group of variables by one symbol, say $y = (x^{i_1}, ..., x^{i_k})$, we obtain the same formulas as in Theorem 2.

4. Let us consider in Euclidean space $\mathbf{R}^4$ with the coordinates (x^1, x^2, x^3, x^4) the system of equations

$$\begin{array}{rcl} (x^1)^2 + (x^2)^2 & = & 1, \\ (x^3)^2 + (x^4)^2 & = & 1. \end{array} \tag{23}$$

The corresponding functions f^1 and f^2 are of the form

$$\begin{array}{rcl} f^1(x^1, x^2, x^3, x^4) & = & (x^1)^2 + (x^2)^2, \\ f^2(x^1, x^2, x^3, x^4) & = & (x^3)^2 + (x^4)^2. \end{array} \tag{24}$$

In order to apply Theorem 3, we shall calculate the Jacobi matrix of the mapping $f = (f^{\,1}, f^{\,2})$

$$df = \begin{pmatrix} \frac{\partial f^{\,1}}{\partial x^1} & \frac{\partial f^{\,1}}{\partial x^2} & \frac{\partial f^{\,1}}{\partial x^3} & \frac{\partial f^{\,1}}{\partial x^4} \\ \frac{\partial f^{\,2}}{\partial x^1} & \frac{\partial f^{\,2}}{\partial x^2} & \frac{\partial f^{\,2}}{\partial x^3} & \frac{\partial f^{\,2}}{\partial x^4} \end{pmatrix} = \begin{pmatrix} 2x^1 & 2x^2 & 0 & 0 \\ 0 & 0 & 2x^3 & 2x^4 \end{pmatrix}. \tag{25}$$

Clearly, rank$df \leq 1$ if only one of the rows of the Jacobi matrix is zero, which is impossible at the points representing solutions of system (24). Thus, the solutions of this system form a two-dimensional manifold of class C^∞. Since system (23) splits into two equations, each for its own group of variables, the set of the solutions of this system can also be expressed as the Cartesian product of the solutions of each equation, i.e. the solutions of system (23) are represented as the product of two copies of a circle. This manifold is called a (two-dimensional) *torus.*

1.8.2 Regular points

Definition 12 *Let $f : M_1 \to M_2$ be a smooth mapping. The point $P_0 \in M_1$ is called a regular point of f if the differential of the mapping $df_{P_0} : T_{P_0}(M_1) \to T_{Q_0}(M_2)$, $Q_0 = f(P_0)$, is an epimorphism, i.e. a mapping onto the entire space $T_{Q_0}(M_2)$. The point $Q_0 \in M_2$ is called a regular point of the mapping f if any point P_0 of the inverse image $f^{-1}(Q_0)$ is a regular point of f.*

This definition is, in fact, the condition of the implicit function theorem formulated in terms of the differential of a mapping. Indeed, in the local coordinate systems $(x^1, ..., x^n)$ and $(y^1, ..., y^m)$ in the neighbourhoods of points P_0 and Q_0, respectively, the mapping f is written as the following system of functions $y^k = f^{\,k}(x^1, ..., x^n)$:

$$\begin{cases} f^{\,1}(x^1, ..., x^n) = y_0^1 \\ \vdots \\ f^{\,m}(x^1, ..., x^n) = y_0^m. \end{cases} \tag{26}$$

Since $f\,(P_0) = Q_0$, $P_0 = (x_0^1, ..., x_0^n)$, $Q_0 = (y_0^1, ..., y_0^m)$, the point P_0 is a solution of system (26). The implicit function theorem (Theorem 3) gives the condition for the existence of a solution in the form: rank$df = m$. Hence, the rank of the Jacobi matrix df_{P_0} of f is equal to the dimension of the tangent space $T_{Q_0}(M_2)$. This means that the linear mapping df_{P_0} is an epimorphism. We come therefore to a generalization of Theorem 3.

Theorem 4 *Let $f : M_1 \to M_2$ be a smooth mapping of smooth manifolds, $Q_0 \in M_2$ a regular point of f. Then the inverse image $M_3 = f^{-1}(Q_0)$ is a smooth manifold,* $\dim M_3 = \dim M_1 - \dim M_2$*, and, moreover, some of the local coordinates in M_1 can be chosen as local coordinates in M_3*

1.9 Partition of the unit.

Theorem 5 (Urysohn's lemma) *Let X be a normal topological space, F be a closed subset, and $f : F \longrightarrow \mathbf{R}$ a continuous function on F. Then the function f is extendable to the continuous function $g : X \longrightarrow \mathbf{R}$. If the function f is bounded, $a \leq f(x) \leq b$, the function g can also be chosen bounded by the same constant, $a \leq g(x) \leq b$.*

The *support* of a continuous function f on a topological space X is the closure of the set of those points $x \in X$ for which $f(x) \neq 0$. The support of a function f is denoted by **supp** f. Thus, a function f is identically zero outside the support.

Theorem 6 *Let X be a normal space and $\{U_\alpha\}$ a finite open covering. Then there exist functions $\varphi_\alpha : X \longrightarrow \mathbf{R}$ such that*

$$\begin{aligned} &0 \leq \varphi_\alpha(x) \leq 1, \quad x \in X, \\ &\textbf{supp } \varphi_\alpha \subset U_\alpha, \\ &\textstyle\sum_\alpha \varphi_\alpha(x) \equiv 1. \end{aligned} \tag{27}$$

The system of functions $\{\varphi_\alpha\}$ is called a *partition of unity* subordinate to the covering $\{U_\alpha\}$. Here we do not assume that the covering is finite, but only require that any point $x \in X$ should have a neighbourhood $O(x)$ intersecting finitely many supports **supp** φ_α.

If a topological space X is actually a smooth manifolds we would like to formulate stronger result assuming that the partition of unity consists of smooth functions.

By definition, an atlas on a manifold M consists of open sets U_i homeomorphic to domains $V_i \subset R^n$. If M is a smooth manifold, the coordinate homeomorphisms $\varphi_i : U_i \to V_i$ are also smooth. It is convenient sometimes to simplify the form of domains V_i in R^n, though by the expense of an increased number of charts in the atlas.

Lemma 6 *In a smooth manifold M there exists an atlas $\{U_i\}$ such that any chart U_i is diffeomorphic to $\mathbf{R}^n$*

Proof.
First, we notice that there exists an atlas such that any chart is diffeomorphic to an open ball of radius ε in $\mathbf{R}^n$. The last we demonstrate that an ε-ball is diffeomorphic to $\mathbf{R}^n$. It is sufficient to consider the case $\varepsilon = 1$. Let $(x^1, ..., x^n)$ be a point in a unit ball, i.e. $(x^1)^2 + ... + (x^n)^2 < 1$. Put

$$y^k = \frac{x^k}{\sqrt{1 - (x^1)^2 - ... - (x^n)^2}}, \tag{28}$$

$$x^k = \frac{y^k}{\sqrt{1 + (y^1)^2 + ... + (y^n)^2}}. \tag{29}$$

Functions (28) and (29) are smooth and map a ball of radius 1 into R^n and vice versa.

Lemma 7 *Let M be a smooth compact manifold equipped with an atlas $\{U_\alpha\}$. Then there exists a smooth partition of unity ψ_α subordinate to the covering $\{U_\alpha\}$.*

Proof.
According to Lemma 6, it suffices to assume that all charts are homeomorphic to $\mathbf{R}^n$. Then for any point $x \in M$ here is an index $\alpha = \alpha(x)$ and a homeomorphism

$$\varphi_x : U_{\alpha(x)} \longrightarrow \mathbf{R}^n, \quad \varphi_x(x) = 0 \in \mathbf{R}^n.$$

Let $f : \mathbf{R}^n \longrightarrow \mathbf{R}^1$ be a smooth function such that

$$0 \le f(x) \le 1, \quad f(0) = 1, \quad \textbf{supp } f \text{ is compact.}$$

Put

$$\overline{\psi_x}(P) = \begin{cases} f(\varphi_x(P)), & P \in U_{\alpha(x)}, \\ 0, & P \notin U_{\alpha(x)}. \end{cases}$$

The open subsets $V_x = \{P : \overline{\psi_x}(P) > 0\} \subset U_{\alpha(x)}$ cover the manifold M. So due to compactness there is a finite family of points, x_β, such that

$$\bigcup_\beta V_{x_\beta} = M.$$

Hence the sum $\overline{\psi}(P) = \sum_\beta \overline{\psi_{x_\beta}}(P)$ is strictly positive at each point. We take then

$$\psi_\beta(P) = \frac{\overline{\psi_{x_\beta}}(P)}{\overline{\psi}(P)}.$$

The functions $\psi_\beta(P)$ form a smooth partition of unity subordinate to the covering $\{V_{x_\beta}\}$ which is a refinement of the atlas $\{U_\alpha\}$. Hence the partition of unity subordinate to $\{U_\alpha\}$ can be obtain from functions $\psi_\beta(P)$ by grouping of functions $\psi_\beta(P)$.

To finish the proof one can take a function f by the formula

$$f(x) = \begin{cases} e^{-\left(\langle x,x\rangle - 1)^2\right)^{-1}}, & \|x\| < 1, \\ 0, & \|x\| \geq 1. \end{cases}$$

1.10 The Whitney theorem.

Definition 13 *Let $f : M_1 \to M_2$ be a smooth mapping. The mapping f is called an immersion if at each point $P \in M_1$ the differential $df_P : T_p(M_1) \to T_{f(P)}(M_2)$ is a monomorphism, i.e. a one-to-one mapping onto its image. If moreover f maps bijectively M_1 onto its image $f(M_1)$ and this image is a closed subset, the mapping f is called an embedding. The image $f(M_1)$ (as well as (M_1)) is called in this case a submanifold in M_2.*

Example. The inverse image of a regular point of $f : M_1 \to M_2$ is, according to Theorem 4, a submanifold. Indeed, since some of the local coordinates of the enveloping manifold M_1 can be taken as a local coordinate system in M_3, the identity mapping $\varphi : M_3 \to M_1$ in the local coordinates takes the form

$$\begin{cases} x^1 = x^1, \\ \vdots \\ x^{n-m} = x^{n-m}, \\ x^{n-m+1} = x^{n-m+1}(x^1, ..., x^{n-m}), \\ \vdots \\ x^n = x^n(x^1, ..., x^{n-m}). \end{cases}$$

Therefore, the Jacobi matrix $d\varphi$ contains an identity square matrix. Hence, *rank* $d\varphi = n - m$, i.e. $d\varphi$ is a monomorphism.

Example. Let us consider a mapping $f : \mathbf{S}^1 \to \mathbf{R}^2$ given by $f(\varphi) = \{\cos\varphi, \sin 2\varphi\}$. The velocity vectors $\frac{df}{d\varphi} = \{-\sin\varphi, 2\cos 2\varphi\}$ does not vanish at any point, that is, the rank of the Jacobi matrix is equal to unity. Thus, f is an immersion (so called a Lissajous' figure).

Theorem 7 *Let M be a smooth compact manifold. Then for a proper dimension N there exists an embedding $\varphi : M \to R^N$.*

Proof.
Let $\{U_\alpha\}_{\alpha=1}^M$ be a finite atlas and $(x_\alpha^1, ..., x_\alpha^n)$ a local coordinate system in the chart U_α. We may assume without loss of generality that the U_α are homeomorphic to a ball $D^n \subset R^n$ of radius 1 and the coordinates $(x_\alpha^1, ..., x_\alpha^n)$ perform a homeomorphism φ_α of U_α onto the ball D^n. We may also assume that D^n lies in R^n and does not contain the origin (this can be achieved by a translation in R^n). Further, let $D_1^n \subset D^n$ be a ball of smaller radius with the same centre as D^n, let the manifold M be covered with open sets $U'_\alpha = \varphi_\alpha^{-1}(D_1^n) \subset U_\alpha$, and let f be a smooth function in R^n such that it is identically unity on D_1^n and $supp\ f \subset D^N$.. We put then

$$y_\alpha^k(P) = \begin{cases} f\ (\varphi_\alpha(P)) x_\alpha^k(P) & if \quad P \in U_\alpha, \\ 0 & if \quad P \notin U_\alpha. \end{cases}$$

Obviously, $y_\alpha^k(P) = x_\alpha^k(P)$ if $P \in U'_\alpha$. We have obtained a system of $N = n \cdot M$ smooth functions, $\left\{y_\alpha^k(P)\right\}$. This system defines the mapping g of M into a Euclidean space R^N , $g(P) = \left\{y_\alpha^k(P)\right\} \in R^N$. We now demonstrate that the differential of g is of rank n at each point. Let $P \in M$ be an arbitrary point, $U_\alpha \ni P$, and $(x_\alpha^1, ..., x_\alpha^n)$ a local coordinate system. The Jacobi matrix of g at point P in the local coordinate system $(x_\alpha^1, ..., x_\alpha^n)$ consists of partial derivatives $\left\{\frac{\partial y_\beta^k}{\partial x_\alpha^j}(P)\right\}$. In particular for $\beta = \alpha$,

$$\frac{\partial y_\beta^k}{\partial x_\alpha^j}(P) = \frac{\partial x_\alpha^k}{\partial x_\alpha^j}(P) = \delta_j^k,$$

i.e. the Jacobi matrix of g contains an identity matrix of order n. Hence, $rank\ dg = n$.

The mapping g is thus an immersion. In order that g be an embedding, it is necessary that distinct points P and Q be mapped into distinct points $g(P)$ and $g(Q)$. Construct a new mapping $\overline{g}(P) = \left\{y_\alpha^k(P), f\ (\varphi_\alpha(P))\right\} \in R^{N+M}$. Owing to the same arguments as for , this mapping is an immersion. Let $P \neq Q$ be two points on a manifold. Consider a number α such that $f\ (\varphi_\alpha(P)) = 1$. If $f\ (\varphi_\alpha(Q)) < 1$, then $\overline{g}(P) \neq \overline{g}(Q)$; if $f\ (\varphi_\alpha(Q)) = 1$, then $y_\alpha^k(P) = x_\alpha^k(P)$, $y_\alpha^k(Q) = x_\alpha^k(Q)$, and for a certain number k we have $x_\alpha^k(P) \neq x_\alpha^k(Q)$, i.e. $\overline{g}(P) \neq \overline{g}(Q)$. Thus, the mapping $g : M \to R^{N+M}$ is a one-to-one immersion, i.e. an embedding. Theorem 3 is proved.

Hence, any smooth compact manifold M may be considered as embedded (in the form of a submanifold) in an Euclidean space $\mathbf{R}^N$ of a rather large dimension N. In practice, however, the dimension of $\mathbf{R}^N$ can be reduced significantly. For example, a sphere $\mathbf{S}^n$ can be embedded in $\mathbf{R}^{n+1}$ and a torus $\mathbf{T}^n$ in $\mathbf{R}^{2n}$.

The projective plane $\mathbf{RP}^2$ cannot be embedded in $\mathbf{R}^3$, but it can be embedded in $\mathbf{R}^5$. Indeed, let $[x_1 : x_2 : x_3]$ be homogeneous coordinates of a point P in RP^2. Putting

$$\begin{array}{lll} y^1 = \frac{x_1^2}{x_1^2+x_2^2+x_3^2}, & y^2 = \frac{x_2^2}{x_1^2+x_2^2+x_3^2}, & y^3 = \frac{x_3^2}{x_1^2+x_2^2+x_3^2}, \\ y^4 = \frac{x_1 x_2}{x_1^2+x_2^2+x_3^2}, & y^5 = \frac{x_2 x_3}{x_1^2+x_2^2+x_3^2}, & y^6 = \frac{x_3 x_1}{x_1^2+x_2^2+x_3^2}, \end{array}$$

we obtain the mapping $g : \mathbf{RP}^2 \to \mathbf{R}^6$, $g(P) = g[x_1 : x_2 : x_3] = (y^1, y^2, y^3, y^4, y^5, y^6)$. It appears however that the image of g lies in the linear subspace $R^5 \subset R^6$ defined by the equation $y^1 + y^2 + y^3 = 1$. Verify that g is an immersion, i.e. the differential dg is a monomorphism at each point $P \in RP^2$. In other words, we have to prove that in any local coordinate system the rank of the Jacobi matrix of g is equal to 2. All the coordinate functions of g are symmetric with respect to the permutation of homogeneous coordinates $[x_1 : x_2 : x_3]$. Without loss of generality we can therefore assume that $x_1 \neq 0$ in a neighbourhood of point $P_0 \in RP^2$, so that the remaining coordinates x_2, x_3 (for $x_1 = 1$) can be chosen as a local coordinate system. The Jacobi matrix of g is

$$dg = \begin{pmatrix} -2x_2 & -2x_3 \\ 2x_2(1+x_3^2) & -2x_2^2x_3 \\ -2x_3^2x_2 & 2x_3(1+x_2^2) \\ (1-x_2^2+x_3^2) & -2x_2x_3 \\ x_3(1-x_2^2+x_3^2) & x_2(1+x_2^2-x_3^2) \\ -2x_2x_3 & (1+x_2^2-x_3^2) \end{pmatrix}$$

to within a proportionality factor. If $x_2x_3 \neq 0$, the minor of the first two rows does not vanish. If $x_2 = 0$, $x_3 = 0$, the minor of the fourth and sixth rows is not zero, and if $x_2 = 0$, $x_3 \neq 0$, the minor of the first and fourth rows is also non-zero. Thus, rank$dg = 2$, i.e. g is an immersion. Let demonstrate that g is a one-to-one mapping. Without loss of generality we may assume that the homogeneous coordinates are so chosen that $x_1^2 + x_2^2 + x_3^2 = 1$. Let $P = [x_1 : x_2 : x_3]$, $Q = [x_1' : x_2' : x_3']$. Then

$$g(P) = (x_1^2, x_2^2, x_3^2, x_1x_2, x_2x_3, x_3x_1),$$
$$g(Q) = (x_1'^2, x_2'^2, x_3'^2, x_1'x_2', x_2'x_3', x_3'x_1').$$

If $g(P) = g(Q)$, then $x_1^2 = x_1'^2$. If $x_1 \neq 0$, then $x_1 = \pm x_1'$ and we can take $x_1 = x_1'$, since homogeneous coordinates can be multiplied by ± 1. Since the fourth and sixth coordinates are equal, we obtain $x_2 = x_2'$, $x_3 = x_3'$, *i.e.*$P = Q$. Since all coordinates x_1, x_2 and x_3 are equivalent, we always have $P = Q$. Thus, g is an embedding of a projective plane RP^2 in a five-dimensional Euclidean space R^5.

1.11 Transversal mappings. The Sard lemma, the Abraham theorem.

We have shown how the local properties of a smooth manifold can be deduced from the properties of the differential. Conversely, in certain cases we can find the properties of a differential from the properties of the manifold itself. For example, if $f : M_1 \to M_2$ is a smooth homeomorphism, then the differential $df : T_p(M_1) \to T_{f(P)}(M_2)$ is, by Lemma 5, an isomorphism. Consider now a problem which is inverse, to a certain extent. Let $f : M_1 \to M_2$ be a smooth mapping of M_1 onto the entire manifold M_2 i.e. $f(M_1) = M_2$. Such a mapping can be considered as an analogue of an epimorphism for a linear mapping. A question

then arises: is the differential $df : T_p(M_1) \to T_{f(P)}(M_2)$ an epimorphism? Unfortunately, the answer is no. Consider the following example. Let $M_1 = M_2 = \mathbf{R}^1$, $f : \mathbf{R}^1 \to \mathbf{R}^1$, $f(x) = x^3$, $x \in \mathbf{R}^1$. In this case f is a smooth mapping and $f(\mathbf{R}^1) = \mathbf{R}^1$. But at point $x = 0$ the differential df equals zero and is not therefore an epimorphism. At other points the differential $df = 3x^2dx$ is an epimorphism. This example suggests the general answer to the question. We shall formulate it as the following statement.

Theorem 8 (Sard's theorem) *Let $f : M_1 \to M_2$ be a smooth mapping of compact manifolds. Then the set G of regular points $Q \in M_2$ of f is open and everywhere dense.*

Before proceeding with them proof of Theorem 8 we shall consider several examples.

Example. Let $f : \mathbf{R}^1 \to \mathbf{R}^1$, $f(x) = a = const$. In this case the differential df is not an epimorphism at any point, but the image $f(\mathbf{R}^1)$ consists of a single point a, i.e. by definition, any point $y \neq a$ is regular (since $f^{-1}(y) = \emptyset$). Hence, the set of regular points is open and everywhere dense.

Example. Let $f : \mathbf{R}^1 \to \mathbf{R}^1$ be a finite smooth function and $F = \{x : f'(x) = 0\}$. The set F is closed. The image $f(F)$ is a compact set and consists of all non-regular points. We shall demonstrate that $f(F)$ is nowhere dense. If this is not so, we can find an interior point $y \in f(F)$, i.e. y is contained in $f(F)$ together with a neighbourhood $y \in U \subset f(F)$. Since f is finite, the image $f(F)$ lies in the image of the interval $f([a, b])$. In other words, it is sufficient to prove that the image $f(F')$, $F' = F \bigcap [a, b]$ is nowhere dense.

Let $U \supset F'$ be a neighbourhood of the set F', $V \subset (-2a, 2a)$. Then $f(V)$ contains U and hence there exists a point $x \in V$ such that $f'(x)$ exceeds, by modulus, the number $\varepsilon = \mathbf{diam}U/4a$. Diminishing the neighbourhood V, we obtain a sequence of points $x_n \in F'$. We may assume, without loss of generality, that $x_n \to x_0 \in F'$. Then $f'(x_n) \to f'(x_0)$, i.e. $|f'(x_0)| \geq \varepsilon$, which contradicts the condition $f'(x_0) = 0$ at point $x_0 \in F' \subset F$.

It is more convenient to formulate Theorem 8 in more general terms:*if $F \subset M_1$ is a compact set consisting of non-regular points, then $f(F)$ is nowhere dense.*

Let demonstrate that it is sufficient to prove Theorem 8 for the case where M_1 is a neighbourhood of a closed disk in an Euclidean space. Indeed, cover M_1 with a finite atlas U_α and choose a covering $V_\alpha \subset \overline{V}_\alpha \subset U_\alpha$ such that V_α is homeomorphic to a disk in an Euclidean space. Let $G_\alpha \subset M_2$ be a set of regular points of the mapping f on V_α. Then the intersection $G = \bigcap G_\alpha$ consists of regular points of the entire mapping f. If G_α are open sets which are everywhere dense then G is also open and everywhere dense. Choose a sufficiently fine atlas U_α such that the image $f(U_\alpha)$ lies in a single chart W_β of M_2 .

Then Theorem 8 can be proved for regular points of the mapping $f|_{U_\alpha} : U_\alpha \to W_\beta$ on $\overline{V}_\alpha$. Indeed, if $G \in W_\beta$ is the set of regular points of $f|_{U_\alpha}$, then $G \bigcap (M_2 \setminus f(\overline{V}_\alpha))$ is the set of regular points of the mapping $f : U_\alpha \to M_2$ on $\overline{V}_\alpha$.

Thus, let U be a neighbourhood of a disk D^n in $\mathbf{R}^n$ and let $f : U \to \mathbf{R}^m$ be a smooth mapping. We shall demonstrate that the set of points $y \in \mathbf{R}^m$, for which $D^n \bigcap F^{-1}(y)$ consists of regular points, is open and everywhere dense.

Lemma 8 *Theorem 8 holds for $m = 1$.*

Proof.
Let $F \subset D^n$ be a set of non-regular points of a function f. Then $f(F)$ is compact and contains all non-regular points of f. Demonstrate that $\mathbf{R}^1 \setminus f(F)$ is everywhere dense. If this is not the case there exists an interval $V \subset f(F)$. Fix $k > n$ and consider the set F_k of those points for which all partial derivatives of f of order up to k inclusive are zero. Expanding f in a Taylor series in a neighbourhood of an arbitrary point $y \in F_k$ we obtain $|f(y) - f(x)| < C \ |x - y|^k$ where C does not depend on the choice of $y \in F_k$ and $x \in D^n$.

This means that if F_k is covered with cubes of side $1/N$ (the number of these cubes does not exceed N^n), the image $f(F_k)$ will be covered with intervals, the length of each interval not exceeding $2\sqrt{n^k}C/N^k$. Hence, the sum of the lengths of all the intervals does not exceed $2\sqrt{n^k}C/N^{k-n}$ and vanishes for $N \to \infty$, that is, $f(F_k)$ is nowhere dense.

The remaining part of the set F, i.e. $F \backslash F_k$, can be represented as a union of a finite family of subsets, each lying on a submanifold defined by one of the equations

$$\frac{\partial^l f}{\partial x_1^{l_1} \cdots \partial x_n^{l_n}} = 0, \qquad l_1 + ... + l_n = l < k.$$

Indeed, let $F_{l_1 \ldots l_n}$ be the set of those points in F at which

$$\begin{aligned} \frac{\partial^l f}{\partial x_1^{l_1} \cdots \partial x_n^{l_n}} &= 0, \\ \mathbf{grad} \frac{\partial^l f}{\partial x_1^{l_1} \cdots \partial x_n^{l_n}} &\neq 0. \end{aligned} \tag{30}$$

Obviously, $F \backslash F_k = \bigcup\limits_{l_1 + \ldots + l_n < k} F_{l_1 \ldots l_n}$. On the other hand, the set $F_{l_1 \ldots l_n}$ lies on the submanifold $M_{l_1 \ldots l_n}$ of those points where conditions (30) are satisfied. The dimension of $M_{l_1 \ldots l_n}$ is less than n, and we may conclude, by induction, that Lemma 8 is true for $M_{l_1 \ldots l_n}$.

Thus, $f(F_k)$ does not cover the interval V. Hence, there exists a neighbourhood $U_k \supset F_k$ such that $\overline{f(U_k)}$ does not cover V. Let $l_1 + ... + l_n = k - 1$. Then $F_{l_1 \ldots l_n} \setminus U_k$ is a compact set on $M_{l_1 \ldots l_n}$, so that $f(F_{l_1 \ldots l_n} \setminus U_k)$ does not cover $V \backslash \overline{f(U_k)}$ i.e. $f(F_k \bigcap F_{l_1 \ldots l_n})$ does not cover V. Hence, there exists a neighbourhood

$$U_s = F_k \cup \left(\bigcup_{s < l_1 + \ldots + l_n < k} F_{l_1 \ldots l_n} \right)$$

such that $\overline{f(U_s)}$ does not cover the interval V. Thus, for $l_1 + ... + l_n = s$ the sets $F_{l_1 \dots l_n} \setminus U_s$ do not cover the remainder

$$V \setminus f\left(F_k \cup \left(\bigcup_{s<l_1+\dots+l_n<k} F_{l_1\dots l_n}\right)\right)$$

under the mapping f, and therefore $f\left(F_k \bigcup \bigcup_{s \le l_1+\dots+l_n<k} F_{l_1\dots l_n}\right)$ does not cover V. After a finite number of steps we find that $f(F)$ does not cover V.

We now apply Lemma 8 to prove by induction on m the theorem 8 for a system of functions $f : U \to R^m$, $D^n \subset U$, $f(P) = (f^1(P), \dots, f^m(P))$. Since f^1 is a smooth function, we find by Lemma 8 that the set G_1 of regular values of f^1 is open and everywhere dense in $\mathbf{R}^1$. Let $y_0^1 \in G_1$,$N = (f^1)^{-1}(y_0^1)$. The set N is a smooth submanifold mapped by f into a hyperplane $\mathbf{R}^{m-1}$. Then the point $(y_0^2, \dots, y_0^m)$ is regular for the mapping $f|_N$ if and only if the point $(y_0^1, \dots, y_0^m)$ is regular for f. By induction, the set of points $(y_0^2, \dots, y_0^m)$ which are regular for the mapping $f\mid_N$ is everywhere dense in $\mathbf{R}^{m-1}$; hence, the set of regular points for f is also everywhere dense in R^m.

In order to demonstrate that the set of regular points is open we note that the inverse image of a regular point $D^n \bigcap f^{-1}(y_0^1, \dots, y_0^m)$ is a compact and the minor of the matrix of the differential df is non-zero at each point of the compact. Therefore, for any sufficiently close point $(y_1^1, \dots, y_1^m)$ the inverse image $D^n \bigcap f^{-1}(y_1^1, \dots, y_1^m)$ lies in a quite small neighbourhood of $D^n \bigcap f^{-1}(y_0^1, \dots, y_0^m)$, that is, the same minors are non-zero. This means that the set of regular points of f is open. Theorem 8 is proved.

As an application of Sard's theorem we shall consider a smooth mapping $f : M_1 \to M_2$ for $\dim M_1 < \dim M_2$. Then, none of the points $P \in M_1$ can be regular, and this means that the image $f(M_1)$ is nowhere dense in M_2. In particular, the image of f does not cover M_2.

Remark. Sard's theorem can be generalized to non-compact separable manifolds. In this case the set of regular points need not necessarily be an open set, but it should only be the intersection of a countable number of open sets which are everywhere dense. Such sets are called G_δ-sets. It is known from general topology that the intersection of a countable number of sets, which are open and everywhere dense in R^n, is always non-empty and everywhere dense. Hence, in the case of non-compact manifolds the set of regular points is non-empty and everywhere dense.

2 Tangent vectors and vector fields. 3 definitions of tangent vectors.

2.1 Linear approximation of a submanifold. Tangent subspace.

We have seen that the so-called infinitesimal properties of space are of great help in the study of metric properties of curves and surfaces and, generally, of metric properties of domains in a Euclidean space. These are the properties defined in a very small neighbourhood of a fixed point P by neglecting small quantities of an order higher than the distance from P. In mathematical analysis we use a similar procedure of neglecting infinitesimal quantities while studying the behaviour of a function in a neighbourhood of a point. In the case of smooth manifolds there is also a natural desire to neglect infinitesimal quantities. One of such methods is to introduce special concepts analogous to tangent vector to a curve and tangent plane to a surface.

2.1.1 Simple examples

Let us consider a smooth curve $x = x(t) = (x^1(t), x^2(t), x^3(t))$ in a three-dimensional space $\mathbf{R}^3$, where t is a parameter. Fix a value to and expand the vector function $x = x(t)$ in a Taylor series about the point t_0

$$x(t_0 + \triangle t) = x(t_0) + \frac{dx}{dt}(t_0)\triangle t + O(\triangle t^2). \tag{31}$$

The first two terms on the right-hand side of Eq.(**??**) may be considered, on the one hand, as an approximation of $x(t)$ in a neighbourhood of the point t_0 by a linear vector function. On the other hand, this linear function

$$y(\triangle t) = x(t_0) - \frac{dx(t_0)}{dt} \cdot \triangle t$$

defines in $\mathbf{R}^3$ a straight line through the point $P_0 = x(t_0)$. Furthermore, among all straight lines through P_0 the straight line $y(t)$ "approaches most closely" the initial curve $x(t)$. Let us first explain the term: a straight line "approaches closely" a curve $x(t)$.

We say that the straight line $y(t) = a + bt$ ($|b| = 1$) is *tangent* to a curve $x(t)$ at the point $x(t_0)$ if the distance from the point $x(t)$ to the straight line is an infinitesimal quantity in comparison with the distance from $P_0 = x(t_o)$ to $x(t)$. The point P_0 lies then on the straight line $y(t)$. We may assume $x(t_0) = y(t_0)$, so that $y(t_0 + \triangle t) = x(t_0) + b\triangle t$. The distance from the point $x(t_0 + \triangle t)$ to the line $y(t)$ is

$$|x(t_0 + \triangle t) - x(t_0) - b(x(t_0 + \triangle t) - x(t_0), b)| = O\left(|x(t_0 + \triangle t) - x(t_0)|\right). \tag{32}$$

Suppose $\frac{dx}{dt}(t_0) \neq 0$. Expanding $x(t)$ by formula (31), we obtain instead of Eq.(32)

$$\left|\frac{dx}{dt}(t_0)\triangle t - b(\frac{dx}{dt}(t_0)\triangle t, b)\right| = O(\triangle t^2)$$

for $\triangle t \to 0$ or, dividing by $\triangle t$,

$$\left| \frac{dx}{dt}(t_0) - b(\frac{dx}{dt}(t_0), b) \right| = O(\triangle t). \tag{33}$$

(3)

since the left-hand side of Eq.(33) does not depend on $\triangle t$, we obtain by a limiting process for $\triangle t \to 0$

$$\frac{dx}{dt}(t_0) = b(\frac{dx}{dt}(t_0), b), \tag{34}$$

(4)

which means that the vectors b and $\frac{dx}{dt}(t_0)$ are collinear. Thus, the linear part of the Taylor formula (31) for the vector function $x(t)$ defines a parametric representation of a tangent straight line at point P_0.

Consider now a surface M in a three-dimensional space $\mathbf{R}^3$ given in parametric form as a vector function $x = x(u, v)$ of two independent parameters u and v. The surface $x(u, v)$ is called *non-degenerate* if at each point the partial derivatives $\frac{\partial x}{\partial u}(u, v)$ and $\frac{\partial x}{\partial v}(u, v)$ are linearly independent as vectors in $\mathbf{R}^3$. Fix parameters (u_0, v_0) and a plane Π , $x = x(u_0, v_0) + au + bv$, through point $P_0 = x(u_0, v_0)$ on the surface. The plane Π is called *tangent* to the surface M at point P_0 if the distance from Π to the point $x(u, v)$ is an infinitesimal quantity in comparison with the distance from $x(u, v)$ to P_0. Expanding the function $x(u, v)$ in a Taylor series about the point (u_0, v_0)

$$\begin{aligned} &x(u_0 + \triangle u, v_0 + \triangle v_0) = x(u_0, v_0) + \\ &+ \tfrac{\partial x}{\partial u}(u_0, v_0)\triangle u + \tfrac{\partial x}{\partial v}(u_0, v_0)\triangle v + O(\triangle u^2 + \triangle v^2), \end{aligned} \tag{35}$$

(5)

we find that the linear part in (35) defines a two-parameter representation of the tangent plane Π to the surface M at point $P_0 = x(u_0, v_0)$. It is natural to call any vector emerging from P_0 and lying in the plane Π a *tangent vector* to the surface M at point P_0. It can be seen from Eq.(35) that parametric representation of a plane Π tangent to a surface M at a point P_0 is of the form

$$x(\triangle u, \triangle v) = x(u_0, v_0) + \frac{\partial x}{\partial u}(u_0, v_0)\triangle u + \frac{\partial x}{\partial v}(u_0, v_0)\triangle v. \tag{36}$$

(6)

Hence, any tangent vector ξ can be decomposed into a linear combination of the vectors $\frac{\partial x}{\partial u}(u_0, v_0)$ and $\frac{\partial x}{\partial v}(u_0, v_0)$

$$\xi = \frac{\partial x}{\partial u}(u_0, v_0)\triangle u + \frac{\partial x}{\partial v}(u_0, v_0)\triangle v \tag{37}$$

$$(7)$$

for an appropriate choice of the parameters $\triangle u$ and $\triangle v$. Thus, the vectors $\frac{\partial x}{\partial u}(u_0, v_0)$ and $\frac{\partial x}{\partial v}(u_0, v_0)$ form a basis in the tangent plane Π, and $\triangle u$ and $\triangle v$ are linear coordinates of the tangent vector ξ in this basis.

Let us draw a smooth curve $x = x(t)$ through point P_0 on a surface M. Since the curve $x = x(t)$ lies on the surface M, it can be represented parametrically as the composition

$$x(t) = x(u(t), v(t)) \tag{38}$$

of functions $u(t)$ and $v(t)$. In other words, the functions $u(t)$ and $v(t)$ define parametrically a curve in a local coordinate system (u, v) on the surface M. Then, the condition that the curve passes through point P_0 can be expressed as the condition for the coordinates: $u_0 = u(t_0)$, $v_0 = (v(t_0)$. Calculation of a tangent vector to a curve (or, as it is frequently called, the velocity vector of a curve) yields

$$\frac{dx}{dt}(t_0) = \frac{d}{dt}(x(u(t), v(t)))|_{t=t_0} =$$

$$= \frac{\partial x}{\partial u}(u(t_0), v(t_0))\frac{du}{dt}(t_o) + \frac{\partial x}{\partial v}(u(t_0), v(t_0))\frac{dv}{dt}(t_o) =$$

$$= \frac{\partial x}{\partial u}(u_0, v_0)\frac{du}{dt}(t_o) + \frac{\partial x}{\partial v}(u_0, v_o)\frac{dv}{dt}(t_o).$$

Hence, *a tangent vector to a curve on a surface M lies in the tangent plane.*

Definition. Let $\xi = \frac{\partial x}{\partial u}(u_0, v_0)\xi^1 + \frac{\partial x}{\partial v}(u_0, v_0)\xi^2$ be a tangent vector to a surface M at a point P_0. Then the numbers (ξ^1, ξ^2) are called *coordinates of the tangent vector ξ to M at point P_0 in a local coordinate system (u, v) on the surface M.*

This definition works not only for the coordinates (u, v) describing parametrically the surface M, but also for any coordinate system (u', v') in a neighbourhood of point P_0. Indeed, if (u', v') is some other coordinate system, the coordinates u and are expressed as smooth functions of u' and v' :

$$\begin{array}{ll} u = u(u', v'), & v = v(u', v') \\ u_0 = u(u'_0, v'_0), & v_0 = v(u'_0, v'_0). \end{array} \tag{39}$$

Considering then compositions of the functions, we obtain a new parametric definition of a surface M

$$x = x(u(u', v'), v(u', v')). \tag{40}$$

$$(9)$$

The parametric equation of a tangent plane Π at point P_0 for the new parameters (u', v') is

$$x = x(u(u_0', v_0'), v(u_0', v_0')) + \frac{\partial x}{\partial u_0'}\triangle u' + \frac{\partial x}{\partial u_0'}\triangle u' + \frac{\partial x}{\partial v_0'}\triangle v' =$$

$$= x(u_0, v_0) + \frac{\partial x}{\partial u}(u_0, v_0)\frac{\partial u}{\partial u'}\triangle u' + \frac{\partial x}{\partial v}(u_0, v_0)\frac{\partial v}{\partial u'}\triangle u' +$$

$$+\frac{\partial x}{\partial u}(u_0, v_0)\frac{\partial u}{\partial v'}\triangle v' + \frac{\partial x}{\partial v}(u_0, v_0)\frac{\partial v}{\partial v'}\triangle v' =$$

$$= x(u_0, v_0) + \frac{\partial x}{\partial u}(u_0, v_0)\left(\frac{\partial u}{\partial u'}(u_0', v_0')\triangle u' + \frac{\partial u}{\partial v'}(u_0', v_0')\triangle v'\right) +$$

$$+\frac{\partial x}{\partial v}(u_0, v_0)\left(\frac{\partial v}{\partial u'}(u_0', v_0')\triangle u' + \frac{\partial v}{\partial v'}(u_0', v_0')\triangle v'\right).$$

Assuming

$$\begin{aligned} \triangle u &= \frac{\partial u}{\partial u'}(u_0', v_0')\triangle u' + \frac{\partial u}{\partial v'}(u_0', v_0')\triangle v', \\ \triangle v &= \frac{\partial v}{\partial u'}(u_0', v_0')\triangle u' + \frac{\partial v}{\partial v'}(u_0', v_0')\triangle v' \end{aligned} \tag{41}$$

(10)

we come to the parametric definition of a tangent plane for the initial parameters (u, v)

$$x = x(u_0, v_0) + \frac{\partial x}{\partial u}(u_0, v_0)\triangle u + \frac{\partial x}{\partial v}(u_0, v_0)\triangle v.$$

Thus, formula (40) is another parametric definition of the surface M with the same tangent plane Π. The curve (38) can therefore be written in terms of the functions $u'(t)$ and $u'(t)$ in such a way that

$$x(t) = x(u(u'(t), v'(t)), v(u'(t), v'(t))).$$

Then, according to the definition of the coordinates of a tangent vector to a curve in a local coordinate system (u', v'), the numbers $\left(\frac{du'}{dt}(t_0), \frac{dv'}{dt}(t_0)\right)$ are the coordinates of a tangent vector to a curve. Differentiating the composite functions u and v, we obtain a relation between the coordinates of a tangent vector to a curve in different local coordinate systems

$$\begin{aligned} \frac{du}{dt}(t_0) &= \frac{\partial u}{\partial u'}(u_0', v_0')\frac{du'}{dt}(t_0) + \frac{\partial u}{\partial v'}(u_0', v_0')\frac{dv'}{dt}(t_0), \\ \frac{dv}{dt}(t_0) &= \frac{\partial v}{\partial u'}(u_0', v_0')\frac{du'}{dt}(t_0) + \frac{\partial v}{\partial v'}(u_0', v_0')\frac{dv'}{dt}(t_0). \end{aligned} \tag{42}$$

(11)

Comparison of (41) and (42) shows that this relation coincides with that for parameter transformation in the definition of a tangent plane.

2.1.2 General definition of a tangent vector

Definition 14 *Let M be a smooth n-dimensional manifold and $P_0 \in M$ an arbitrary point. A tangent vector at point P_0 to the manifold M is a correspondence which associates with any local coordinate system $(x_i^1, ..., x_i^n)$ a set of numbers $(\xi_i^1, ..., \xi_i^n)$ satisfying the following relation for each pair of local coordinate systems:*

$$\xi_i^k = \sum_{l=1}^{n} \frac{\partial x_i^k}{\partial x_j^l}(P_0)\xi_j^l. \tag{43}$$

The numbers $(\xi_i^1, ..., \xi_i^n)$ are called coordinates of the tangent vector ξ in the local coordinate system $(x_i^1, ..., x_i^n)$ and relation (12) is called a tensor law of coordinate transformation for the tangent vector ξ.

(12)

Definition 14 is a generalization of the concept of the coordinates of a tangent vector to a curve on a surface. The law (42) of the transformation of these coordinates is a particular case of the tensor law (43) of the coordinate transformation for a vector tangent to a manifold. Moreover, any smooth curve on a smooth manifold is endowed at each point with a tangent vector in the sense of Definition 14. This important property can be formulated as the following proposition.

Proposition 1 *Let M be a smooth manifold and $\gamma : (-1, 1) \to M$ a smooth mapping of the interval $(-1, 1)$ into M. Then the correspondence which associates with each local coordinate system $(x^1, \ldots, x^n)$ in a neighbourhood of point $P_0 = \gamma(0)$ the set of numbers $\left(\frac{dx^1}{dt}(\gamma(t)), \ldots, \frac{dx^n}{dt}(\gamma(t))\right)$, is a tangent vector in the sense of Definition 14.*

It is therefore natural to call the correspondence used in Proposition 1 a *tangent vector to a curve* γ or a *velocity vector of a curve* γ. The tangent vector to a curve γ will be denoted by $\frac{d\gamma}{dt}(t_0)$ or $\dot{\gamma}(t_0)$.

2.2 Tangent vector as a tensor.

The set of all tangent vectors to a manifold M at a fixed point P is called *a tangent space to the manifold M at point P*. This set is denoted by $T_P(M)$. Each tangent vector $\xi \in T_P(M)$ is uniquely defined by its coordinates in a fixed coordinate system. Indeed, suppose we are given a set of numbers $(\eta^1, \ldots, \eta^n)$ and assume this set to be the coordinates of a tangent vector in question in a fixed local coordinate system $\left(x_\alpha^1, \ldots, x_\alpha^n\right)$, i.e. $\eta^k = \xi_\alpha^k$. In order to define a tangent vector, we have to find its coordinates in any local coordinate system $\left(x_\beta^1, ..., x_\beta^n\right)$. Let us put

$$\xi^k_\beta = \sum_{l=1}^{n} \frac{\partial x^k_\beta}{\partial x^l_\alpha}(P)\eta^l$$

These coordinates must satisfy the tensor law of coordinate transformation (43). To verify this, we substitute ξ^k_β and ξ^l_γ into formula (43)

$$\sum_{l=1}^{n} \frac{\partial x^k_\beta}{\partial x^l_\alpha}(P)\eta^l = \sum_{s=1}^{n} \frac{\partial x^k_\beta}{\partial x^s_\gamma}(P) \sum_{l=1}^{n} \frac{\partial x^s_\beta}{\partial x^l_\alpha}(P)\eta^l =$$

$$= \sum_{l=1}^{n} \left(\sum_{s=1}^{n} \frac{\partial x^k_\beta}{\partial x^s_\gamma}(P) \frac{\partial x^s_\gamma}{\partial x^l_\alpha}(P) \right) \eta^l.$$

Since

$$\frac{\partial x^k_\beta}{\partial x^l_\alpha}(P) = \sum_{s=1}^{n} \frac{\partial x^k_\beta}{\partial x^s_\gamma}(P) \frac{\partial x^s_\gamma}{\partial x^l_\alpha}(P)$$

(the law of transformation of the Jacobi matrix under triple coordinate transformation), relation (43) is satisfied identically.

We have demonstrated that the set of all tangent vectors to a manifold M at point P is uniquely determined by the coordinates of these vectors in one fixed local coordinate system. Hence, the entire tangent space $T_P(M)$ can be identified with the arithmetic vector space $\mathbf{R}^n$. This means that $T_P(M)$ can be endowed with the structure of a linear space. Seemingly, the structure of a linear space in $T_P(M)$ should depend on the local coordinate system in a neighbourhood of point P. This is not the case, however.

Proposition 2 *Addition of vectors and multiplication of a vector by a number in a tangent space $T_P(M)$ do not depend on the local coordinate system on M in a neighbourhood of point P.*

The tensor law of coordinate transformation may be considered as a method of identifying arithmetic spaces of the coordinates of tangent vectors in any local coordinate system. The method lies in multiplying the coordinate column (ξ^k_γ) by the Jacobi matrix of the transformation from $(x^1_\gamma, \ldots, x^n_\gamma)$ to $(x^1_\beta, \ldots, x^n_\beta)$:

$$\begin{pmatrix} \xi^1_\beta \\ \vdots \\ \xi^n_\beta \end{pmatrix} = \begin{pmatrix} \frac{\partial x^1_\beta}{\partial x^1_\gamma} & \cdots & \frac{\partial x^1_\beta}{\partial x^n_\gamma} \\ \vdots & & \vdots \\ \frac{\partial x^n_\beta}{\partial x^1_\gamma} & \cdots & \frac{\partial x^n_\beta}{\partial x^n_\gamma} \end{pmatrix} \begin{pmatrix} \xi^1_\gamma \\ \vdots \\ \xi^n_\gamma \end{pmatrix}$$

or $\left(\xi^k_\beta\right) = \left(\frac{\partial x_\beta}{\partial x_\gamma}\right)\left(\xi^l_\gamma\right)$

Hence, the tangent space $T_P(M)$ is a space isomorphic to all arithmetic spaces of the coordinates of tangent vectors.

2.3 Tangent vector as a sheaf of osculating curves.

Definition 15 *Two curves γ_1 and γ_2 on a manifold M intersecting at a point P are called tangent if in each local coordinate system $(x^1, \ldots, x^n)$ in a neighbourhood of P we have*

$$\sum_{k=1}^{n} (x^k(\gamma_1(t)) - x^k(\gamma_2(t)))^2 = o(t - t_0)^2 \quad for \quad t \to t_0. \tag{44}$$

As before, it is sufficient to verify the tangency condition (44) only in one local coordinate system. The tangency condition is closely related to tangent vectors. In particular the following theorem justifies the term "tangent curves".

Theorem 9 *Two smooth curves γ_1 and γ_2 on a manifold M are tangent at a point P if and only if their tangent vectors at P_0 coincide.*

Theorem 9 gives an alternative definition of a tangent vector to a curve. The set of all smooth curves through a given point P_0 on a manifold M splits into disjoint classes of pairwise tangent curves. The class of tangent curves through a point $P \in M$ is called a tangent vector. We have then a one-to-one correspondence between tangent vectors in the sense of Definition 14 and in the sense of classes of tangent curves.

2.4 Tangent vector as a differentiation operator.

Definition 16 *Suppose $P_0 \in M$, $\xi \in T_{P_0}(M)$, $\gamma(t)$ is a smooth curve through P_0, $\gamma(t_0) = P_0$, ξ is its tangent vector at P_0, $\overset{\bullet}{\gamma}(t_0) = \xi$, and f is a smooth function on M. The derivative*

$$\frac{d}{dt} f(\gamma(t))|_{t=t_0} = \xi(f). \tag{45}$$

is called the derivative of the function f with respect to the tangent vector ξ. Calculation of the derivative is called the differentiation of the function with respect to the vector ξ.

Theorem 10 *Let $(x^1, \ldots, x^n)$ be a local coordinate system in a neighbourhood of a point $P = (x_0^1, ...x_0^n)$ of a manifold M, let $\xi = (\xi^1, \ldots, \xi^n)$ be a tangent vector to M at P, and let $f = f(x^1, \ldots, x^n)$ be a smooth function in the neighbourhood of P represented as a function of the local coordinates $(x^1, \ldots, x^n)$. Then*

$$\xi(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(x_0^1, \ldots, x_0^n)\xi^i. \tag{46}$$

Hence, the definition (45) of the derivative does not depend on the choice of a curve in the class of tangent curves, and the right-hand side of (46) does not depend on a local coordinate system. If g is another smooth function in the neighbourhood of P the product of f and g obeys the Leibniz formula

$$\xi(fg) = f(x_0^1, ...x_0^n)\xi(g) + g(x_0^1, ...x_0^n)\xi(f). \tag{47}$$

Definition 17 *The linear operation A which associates with any smooth function f of class C^∞ on a smooth manifold M the number $A(f)$ satisfying the Leibniz formula (47) is called differentiation at point $P \in M$*

Definition 4. In the Leibniz formula (20) the values of functions are calculated at a single point P, so that differentiation at distinct points P and P_1 need not coincide. Obviously, differentiation with respect to a tangent vector ξ is a particular case of differentiation in the sense of Definition 17, and there are no other differentiation operations. This means that for each differentiation in the sense of Definition 17 there exists a tangent vector with respect to which the function is differentiated.

Theorem 11 *Let M be a C^∞-manifold, $P \in M$ be an arbitrary point, and let A denote differentiation in the sense of Definition 17. Then there exists a unique tangent vector ξ at P such that $A(f) = \xi(f)$ for any smooth function f in a neighbourhood of P.*

Proof.
The tangent vector will be sought as a column of its coordinates in some local coordinate system $(x^1, \ldots, x^n)$ in a neighbourhood of P. Then any smooth function will be represented as a function of the variables $(x^1, ..., x^n)$.

Lemma 9 *Any C^∞-function $f(x^1, \ldots, x^n)$ can be represented in the form*

$$\begin{aligned} f(x^1, \ldots, x^n) = f(x_0^1, \ldots, x_0^n) + \sum_{i=1}^{n} \tfrac{\partial f}{\partial x^i}(x_0^1, \ldots, x_0^n)(x^i - x_0^i) + \\ + \sum_{i,j=1}^{n} h_{ij}(x^1, \ldots, x^n)(x^i - x_0^i)(x^j - x_0^j). \end{aligned} \tag{48}$$

where $h_{ij}(x^1, \ldots, x^n)$ are C^∞-functions.

Proof.
Write the identity

$$f(x^1, \ldots, x^n) \equiv f(x_0^1, \ldots, x_0^n) + \int_0^1 \frac{d}{dt} f(x_0^1 + t(x^1 - x_0^1), \ldots, x_0^n + t(x^n - x_0^n))dt$$

and differentiate it with respect to t under the integral sign

$$\begin{aligned} f(x^1, \ldots, x^n) = f(x_0^1, \ldots, x_0^n) + \\ + \int_0^1 \sum_{i=1}^{n} \tfrac{\partial f}{\partial x^i}(x_0^1 + t(x^1 - x_0^1), ..., x_0^n + t(x^n - x_0^n))(x^i - x_0^i)dt \end{aligned} \tag{49}$$

In the last equality the functions

$$h_i(x^1,\dots,x^n)=\int\limits_0^1 \frac{\partial f}{\partial x^i}(x_0^1+t(x^1-x_0^1),\dots,x_0^n+t(x^n-x_0^n))dt$$

are smooth of class C^∞. Substitution of $x^i=x_0^i$ yields

$$h_i(x_0^1,...x_0^n)=\int\limits_0^1 \frac{\partial f}{\partial x^i}(x_0^1,\dots,x_0^n)dt=\frac{\partial f}{\partial x^i}(x_0^1,\dots,x_0^n). \tag{50}$$

Applying formula (49) to the functions $h_i(x^1,\dots,x^n)$, we obtain

$$h_i(x^1,\dots,x^n)=h_i(x_0^1,\dots,x_0^n)+\sum_{j=1}^n (x^j-x_0^j)h_{ij}(x^1,\dots,x^n), \tag{51}$$

where $h_{ij}(x^1,\dots,x^n)$ are C^∞-functions. Substituting (51) into (49) and taking into account (50), we arrive at the initial representation (48).

Lemma 10 *Let f and g be smooth functions on a manifold M such that $f(P)=g(P)=0$. Then for any differentiation A at the point P the equality $A(fg)=0$ is satisfied.*

Lemma 3.

The *proof* of Lemma 10 directly follows from the Leibniz formula (47).

Let us now turn to the proof of Theorem 11. Represent f in the form of (48) and apply differentiation A to the left-hand and right-hand sides. Since the operation is linear, we have

$$\begin{aligned} A(f)=f(x_0^1,\dots,x_0^n)A(1)+\sum\limits_{i=1}^n \tfrac{\partial f}{\partial x^i}(x_0^1,\dots,x_0^n)A(x^i-x_0^i)+ \\ +\sum\limits_{i,j=1}^n A\left((x^i-x_o^i)(x^j-x_o^j)h_{ij}(x^1,\dots,x^n)\right). \end{aligned} \tag{52}$$

Note that for a constant unit function we have

$$A(1)=A(1\cdot 1)=1\cdot A(1)+A(1)\cdot 1=2A(1)=0.$$

In the last sum of (52) each term can be represented as the product of two functions $(x^i-x_0^i)$ and $(x^j-x_o^j)\cdot h_{ij}(x^1,\dots,x^n)$, each vanishing at $P\in M$. Thus, by Lemma 10, $A((x^i-x_0^i)(x^j-x_0^j)h_{ij}(x^1,\dots,x^n))=0$. Hence,

$$A(f)=\sum_{i=1}^n \frac{\partial f}{\partial x^i}(x_0^1,\dots,x_0^n)A(x^i-x_0^i). \tag{53}$$

Put $\xi^i=A(x^i-x_0^i)$. We obtain the vector $\xi=(\xi^1,\dots,\xi^n)$ such that

$$A(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(x_0^1, \ldots, x_0^n)\xi^i = \xi(f)$$

∎

The theorem establishes a one-to-one correspondence between tangent vectors to a manifold M at a point $P \in M$ and differentiations of a smooth functions at the point P. We can therefore formulate a third equivalent definition of a tangent vector: a *tangent vector* is a differential operator applied to a smooth function at point P of a manifold M.

2.5 Tangent bundle of smooth manifold.

We have already seen that the set of all tangent vectors $T_P(M)$ to a manifold M at a point P is a linear space of the same dimension as that of M. In geometry it is sometimes useful to study the whole set of tangent vectors to a manifold M, which can, apparently, be represented as the union $\bigcup\limits_{P \in M} T_P(M)$. This space (not yet topological) is denoted by $T(M)$ and called a tangent bundle of M. The term bundle means that $T(M)$ consists of fibers — tangent spaces $T_P(M)$ to distinct points P of the manifold M. A tangent bundle is by no means a vector space, for it is meaningless to add vectors belonging to different fibers. If, for example, a manifold M is a two-dimensional surface in $\mathbf{R}^3$, then $T(M)$ represents the union of all tangent planes to M. It should be noted that tangent planes to a surface usually intersect, that is, they have common points. According to the definition of $T(M)$, however, these points in each fiber define different vectors, since they originate at distinct points.

Definition 18 *A tangent bundle $T(M)$ is a manifold of dimension $\dim T(M) = 2\dim M = 2n$ with the atlas of charts $V_\alpha = U_\alpha \times \mathbf{R}^n$ and local system of coordinates*

$$\left(x_\alpha^1, \ldots, x_\alpha^n; \xi_\alpha^1, \ldots, \xi_\alpha^n\right).$$

The transition functions are defined by the following formula

$$\begin{cases} x_\beta^1 &= x_\beta^1\left(x_\alpha^1, \ldots, x_\alpha^n\right), \\ \ldots & \\ x_\beta^n &= x_\beta^n\left(x_\alpha^1, \ldots, x_\alpha^n\right), \\ & \\ \xi_\beta^1 &= \sum_k \xi_\alpha^k \frac{\partial x_\beta^1}{\partial \xi_\alpha^k}\left(x_\alpha^1, \ldots, x_\alpha^n\right), \\ \ldots & \\ \xi_\beta^n &= \sum_k \xi_\alpha^k \frac{\partial x_\beta^n}{\partial \xi_\alpha^k}\left(x_\alpha^1, \ldots, x_\alpha^n\right). \end{cases} \tag{54}$$

Here are some examples from mechanics which demonstrate that non-trivial manifolds and tangent bundles to them are convenient in the description of mechanical systems.

Example 1. Consider the motion of a plane pendulum, i.e. a rigid bar hinged at a point. The position of the bar is determined by one parameter, the angle φ between the bar axis and the vertical. Thus, the set of all positions of the bar is a circle $\mathbf{S}^1$. Such a set is called a *configuration space*.

Consider a two-link compound pendulum, i.e. two bars pivoted together. The position of this pendulum is determined by two angles φ_1 and φ_2, so the set of all positions represents a two-dimensional torus $\mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1$.

Example 2. In mechanics the motion of a mechanical system is usually described by the parameters characterizing the position of the system and by the velocities of its parts. The set of all these positions and velocities is called a *phase space*, which can naturally be identified with a tangent bundle to a configuration space. For instance, if a particle moves along a two-dimensional sphere at a constant velocity, the phase space is a subset in a tangent bundle consisting of tangent vectors of constant length.

Example 3. There exist more complicated configuration and phase spaces. Let us consider, for instance, a three-dimensional rigid body with a fixed point. Any position of this body in $\mathbf{R}^3$ can be described as follows. Choose in the body three orthogonal unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ emerging from the fixed point. Any position of the body is given then by the position of these three vectors in $\mathbf{R}^3$. Thus, the configuration space can be identified with the connected component of the set of all orthogonal unit bases in $\mathbf{R}^3$.

2.6 Weak and strong Whitney theorem

The theorem 7 is called a weak Whitney theorem while the following is so called strong Whitney theorem.

Theorem 12 *Let M be a smooth compact manifold of* $\dim M = n$. *Then the embedding $\varphi : M \to \mathbf{R}^{(2n+1)}$ does exist.*

Proof.
We shall use Theorem 7 and try to reduce the dimension of $\mathbf{R}^N$. Let $\mathbf{e} \in \mathbf{R}^N$ be a non-zero vector and $p_e : \mathbf{R}^N \to \mathbf{R}^{N-1}$ the orthogonal projection along $\mathbf{e}$ onto a subspace orthogonal to $\mathbf{e}$. Sometimes, the composition

$$M \xrightarrow{\varphi} \mathbf{R}^N \xrightarrow{p_e} R^{N-1}$$

remains an embedding. Analyze the conditions under which the composition $p_e\varphi$ is an embedding. We have to verify two conditions: (a) that the differential is a monomorphism and (b) that the mapping is one-to-one.

We first consider condition (a). Let $P \in M$, $V_P = d\varphi_P(T_PM) \subset \mathbf{R}^N$. In order that the differential of the composition $d(p_e\varphi)$ at a point P be a monomorphism, it is necessary and sufficient that the projection p_e should map the subspace V_P injectively into $\mathbf{R}^{N-1}$, which is equivalent to $\mathbf{e} \notin V_P$. Fix a local coordinate system $(x^1, \ldots, x^n)$ in a neighbourhood U of point P and

a basis in $\mathbf{R}^n$. Construct the mapping $h : U \times \mathbf{R}^n \to \mathbf{RP}^{N-1}$, assuming $h(x^1, \ldots, x^n, \xi^1, \ldots, \xi^n) = (\zeta^1 : \zeta^2 : \ldots : \zeta^N)$, where

$$\zeta^k = \sum_{\alpha=1}^{n} \frac{\partial y^k}{\partial x^\alpha} \xi^\alpha$$

and $(y^1, \ldots, y^N) = \varphi(x^1, \ldots, x^n)$. The mapping h is a smooth mapping of the $2n$-dimensional manifold $U \times R^n$ onto the projective space RP^{N-1}. Then the condition $\mathbf{e} \notin V_P$ exactly means that the straight line generated by $\mathbf{e}$, being a point in $\mathbf{RP}^{N-1}$, does not belong to the image of h. By Sard's theorem, for $2n < N-1$ the set of such points is open and everywhere dense in the entire space $\mathbf{RP}^{N-1}$. Considering a finite atlas in M, we find that for an open dense set G in $\mathbf{RP}^{N-1}$ the vector $\mathbf{e}$ generating points in G satisfy $\mathbf{e} \notin V_P$ for any point $P \in M$. This means that the set of $\mathbf{e}$ such that the projection p_e embeds $\varphi(M)$ in $\mathbf{R}^{N-1}$ is open and dense.

Let us now turn to condition (b). The absence of bijectivity means that $\mathbf{e}$ is parallel to a straight line through a pair of points $P \neq Q$ on $\varphi(M)$. Just like in the case of condition (a), we shall consider the mapping $h' : (M \times M \setminus \Delta) \to \mathbf{RP}^{N-1}$ which for a pair of points $P \neq Q$ assigns a straight line through $\varphi(P)$ and $\varphi(Q)$. According to Sard's theorem, for $2n < N-1$ the set of points not contained in the image of h' is an open dense set G'. The vector $\mathbf{e}$ should therefore be so chosen that the straight line generated by the point $\mathbf{e}$ lies in $G \bigcap G' \neq \emptyset$. We have demonstrated that if $2n < N-1$, there exists a projection p_e such that the composition $p_e\varphi$ is an embedding. Hence, step by step we can reduce the dimension of the enveloping Euclidean space $\mathbf{R}^N$ unless the equality $2n = N-1$ is satisfied, whence $N = 2n+1$. This is the minimal dimension of the Euclidean space $\mathbf{R}^{(2n+1)}$ which admits an embedding of any n-dimensional compact manifold . Theorem 12 is proved.

2.7 Vector fields and dynamic system on manifolds

3 Some application of the theory of manifolds.

3.1 The mapping degree of orientable manifolds, the main algebra theorem.

3.1.1 Examples

1. Let us consider a circle $\mathbf{S}^1$ realized as the set of complex numbers with modulus equal to unity, and a mapping

$$f : \mathbf{S}^1 \longrightarrow \mathbf{S}^1,$$

$f(z) = z^n$. This mapping is smooth. Any point of $\mathbf{S}^1$ is regular for the mapping f. Indeed, in the local parameter φ the mapping f is of the form $f(\varphi) = n\varphi$, $df(\varphi) = n$, $n \neq 0$. Hence the differential df is an isomorphism. The inverse image of any point $z_0 \in \mathbf{S}^1$ consists of exactly n points, the roots of order n of

the complex number z_0. Geometrically, the mapping f may be looked upon as an n-fold "winding" of $\mathbf{S}^1$ onto itself.

2. Consider a sphere $\mathbf{S}^2$ as a complex projective line $\mathbf{RP}^1$. Let

$$f : \mathbf{RP}^1 \longrightarrow \mathbf{RP}^1$$

be the mapping, defined in projective coordinates by formula

$$f([z_0 : z_1]) = [z_0^n : z_1^n]. \tag{55}$$

Similar to previous example one can calculate differential of the mapping f using complex coordinate $z = \frac{z_0}{z_1}$. Then $df(z) = nz^{n-1} \neq 0$ if $z \neq 0$. Hence all points $[z_0 : z_1]$, $z_0 \neq 0$, $z_1 \neq 0$ are regular and inverse image consists of exactly n points. The map f can be looked upon as an n-fold "winding" along parallels of the sphere $\mathbf{S}^2$.

3.1.2 Manifolds with boundary

Definition 19 *A metric space M is called a smooth manifold with boundary if there exists an atlas $\{U_\alpha\}$ and coordinate homeomorphisms $\varphi : U_\alpha \longrightarrow V_\alpha \subset \mathbf{R}^n_+$, where V_alpha is an open set in the halfspace $\mathbf{R}^n_+ \subset \mathbf{R}^N$ defined by the inequality $x^n \geq 0$. The transition function assume to be smooth function right up to boundary points of $\mathbf{R}^n_+$.*

The condition $x^n_\alpha(P) > 0$ does not depend of choice of the chart U_α. This means that all points are divided into two classes — interior points and boundary points. The family of boundary points denotes as ∂M and is called the boundary of M.

Theorem 13 *Let M be a smooth manifolds with boundary. Then the boundary ∂M is smooth closed manifold, $\dim \partial M = \dim -1$, atlas of charts V_α can be defined as*

$$V_\alpha = U_\alpha \cap \partial M,$$

as coordinate system can be taken all coordinates from M except the last coordinate x^n_α which actually vanishes on ∂M.

3.1.3 The mapping degree mod 2

Definition 20 *Let $f : M_1 \longrightarrow M_2$ be a smooth mapping of compact, connected closed manifold, $\dim M_1 = \dim M_2$, and let $P \in M_2$ be a regular point. The degree of the mapping f (relative to a regular point P) is the number*

$$\deg_P f = \# f^{-1}(P) \tag{56}$$

Theorem 14 *The mapping degree modulo 2 does not depend*

(a) on the choice of the regular point P,

(b) on the choice of the mapping f in the class of (smooth) homotopic mappings.

3.1.4 Oriented manifolds.

A manifold X is said to be *oriented* if there is an atlas $\{U_\alpha\}$ such that all the transition functions $\varphi_{\alpha\beta}$ have positive Jacobians at each point. The choice of a such atlas is called an *orientation* of the manifold X.

Proposition 3 *On a connected oriented manifold there exist exactly two distinct orientations, any chart defining a local orientation that coincides with one of the orientations of M.*

Theorem 15 *If a manifold with boundary M is oriented, then the boundary ∂M is also an oriented manifold with the atlas of charts, defined in the theorem 13.*

3.1.5 The mapping degree

Definition 21 *Let $f : M_1 \longrightarrow M_2$ be a smooth mapping of compact, connected, oriented, closed manifold,* $\dim M_1 = \dim M_2$*, and let $P \in M_2$ be a regular point. For $Q \in f^{-1}(P)$ we put $\epsilon(Q) = +1$ if the determinant of the Jacobi matrix of f at the point Q is positive, and $\epsilon(Q) = -1$ if the determinant is negative. The degree of the mapping f (relative to a regular point P) is the number*

$$\deg_P f = \sum_{Q \in f^{-1}(P)} \epsilon(Q). \tag{57}$$

Theorem 16 *Definition 21 of the mapping degree does not depend (a) on the choice of the regular point P,*

(b) on the choice of the mapping f in the class of (smooth) homotopic mappings.

3.1.6 The main algebra theorem

The fundamental theorem of algebra states that any polynomial $P(z)$ of the degree n over the field of complex numbers has at least one complex root.

There are many various proofs of this theorem. One of them rests on using the concept of the degree of a mapping and theorem 2. Let us consider a smooth mapping $P : \mathbf{C} \longrightarrow \mathbf{C}$ of the complex plane defined by

$$w = P(z) = z^n + a_{n-l} z^{n-1} + \cdots + a_1 z + a_0. \tag{58}$$

This mapping can be extended to the mapping of a two-dimensional sphere $\mathbf{S}^2$ into itself, assuming $\mathbf{S}^2$ to be a complex projective line $\mathbf{CP}^1$. To this end we assume that the complex parameter z is equal to the ratio of homogeneous coordinates on $\mathbf{CP}^l$: $z = \frac{z_0}{z_1}$ for $z_1 \neq 0$. Similarly, $w = \frac{w_0}{w_1}$ for $w_1 \neq 0$. Therefore, the formula

$$\left\{ \begin{array}{lcl} w_0 & = & z_0^n + a_{n-1} z_0^{n-1} z_1 + \cdots + a_1 z_0 z_1^{n-1} + a_0 z_1^n, \\ w_1 & = & z_1^n \end{array} \right. \tag{59}$$

correctly defines the mapping of $\mathbf{CP}^1$ into itself. Mapping (58) is apparently a smooth one. Indeed in the chart $z_1 \neq 0$ this follows from (58) in the chart $z_0 \neq 0$ as a complex coordinate we can take the function $z' = \frac{z_1}{z_0}$. By setting $w' = \frac{w_1}{w_0}$, we obtain

$$\left\{ \begin{array}{lcl} w' & = & \frac{z_1^n}{z_0^n + a_{n-1} z_0^{n-1} z_1 + \cdots + a_1 z_0 z_1^{n-1} + a_0 z_1^n} = \\ & = & \frac{(z')^n}{1 + a_{n-1} z' + \cdots + a_1 (z')^{n-1} + a_0 (z')^n} \end{array} \right. \tag{60}$$

Taking a sufficiently small $\varepsilon > 0$, we choose a chart containing the point $z_1 = 0$ and define this chart by the inequality $|z'| < \varepsilon$ such that the denominator in (60) be non-zero. Thus, the mapping $f : \mathbf{CP}^1 \longrightarrow \mathbf{CP}^1$ given by formula (59) is smooth. We now calculate the degree of f. According to theorem 2 the mapping f can be re- placed by a homotopic one. Let us consider the homotopy with respect to the parameter $t \quad 0 \leq t \leq 1$, defined by

$$\left\{ \begin{array}{lcl} w_0 & = & z_0^n + t \left(a_{n-1} z_0^{n-1} z_1 + \cdots + a_1 z_0 z_1^{n-1} + a_0 z_1^n \right), \\ w_1 & = & z_1^n \end{array} \right. \tag{61}$$

Just like in the case of (59), mappings (61) are smooth. At $t = 0$ we obtain a simple mapping

$$\left\{ \begin{array}{lcl} w_0 & = & z_0^n, \\ w_1 & = & z_1^n \end{array} \right. \tag{62}$$

In the local coordinates $w = \frac{w_0}{w_1}, \quad z = \frac{z_0}{z_1}$ this mapping takes the form $w = z^n$, and, say, the point $w = 1$ is regular. Indeed, calculating the Jacobi matrix of the mapping $u = \Re w = \Re z^n$, $v = \Im w = \Im z^n$, $z = x + iy$, we obtain

$$\frac{\partial(u, v)}{\partial(x, y)}|_{w=1} = n^2 > 0. \tag{63}$$

Since the equation $z^n = 1$ has exactly n solutions, the degree of mapping (62) and, therefore, of mapping (59) is equal to n, i.e. $\deg f = n$. If the polynomial P did not have roots, the point $w = 0$ would not belong to the image of f and, hence, the mapping $f : \mathbf{CP}^1 \longrightarrow \mathbf{CP}^1$ would have a regular point $w = 0$ with empty inverse image, i.e. the degree of mapping f would be zero. Contradiction proves the theorem.

3.2 Submersions and smooth bundles.

Consider a smooth mapping $f : X \longrightarrow Y$ of compact manifolds. Assume that the differential $Df : T_x(X) \longrightarrow T_{f(x)}(Y)$ is epimorphism at each point $x \in X$. Then we say that the mapping f is a submersion. In other words all points $x \in X$ are regular with respect to the mapping f. In particular due to the Sard theorem for each point $y \in X$ the inverse image $f^{-1}(y) \subset X$ is a submanifold. More of that the distribution of the manifold X into submanifolds $\{f^{-1}(y) : y \in Y\}$ form so called locally trivial bundle of manifolds.

Definition 22 *Let E and B be two topological spaces with a continuous map*

$$p : E \longrightarrow B.$$

The map p is said to define a locally trivial bundle with a fiber F if for any point $x \in B$ there is a neighborhood $U \ni x$ for which the inverse image $p^{-1}(U)$ is homeomorphic to the Cartesian product $U \times F$. Moreover, it is required that the homeomorphism

$$\varphi : U \times F \longrightarrow p^{-1}(U)$$

preserves fibers, it is a 'fiberwise' map, that is, the following equality holds:

$$p(\varphi(x, f)) = x, x \in U, f \in F.$$

The space E is called total space of the bundle or the fiberspace, the space B is called the base of the bundle, the space F is called the fiber of the bundle and the mapping p is called the projection. The requirement that the homeomorphism φ be fiberwise means in algebraic terms that the diagram

$$\begin{array}{ccc} U \times F & \xrightarrow{\varphi} & p^{-1}(U) \\ \downarrow \pi & & \downarrow p \\ U & = & U \end{array}$$

where

$$\pi : U \times F \longrightarrow U, \quad \pi(x, f) = x$$

is the projection onto the first factor is commutative.

There is a simple criterion describing when a smooth mapping of manifolds gives a locally trivial bundle.

Theorem 17 *Let*

$$f : X \longrightarrow Y$$

be a smooth mapping of compact manifolds such that the differential Df is epimorphism at each point $x \in X$. Then f is a locally trivial bundle with the fiber a smooth manifold.

Proof.
Without loss of generality one can consider a chart $U \subset Y$ diffeomorphic to $\mathbf{R}^n$ and the part of the manifold X, namely, $f^{-1}(U)$. Then the mapping f gives a vector valued function

$$f : X \longrightarrow \mathbf{R}^n.$$

Assume firstly that $n = 1$. From the condition of the theorem we know that the gradient of the function f never vanishes.

Consider the vector field $\mathbf{grad} f$ (with respect to some Riemannian metric on X). The integral curves $\gamma(x_0, t)$ are orthogonal to each hypersurface of the

level of the function f. Choose a new Riemannian metric such that $\mathbf{grad} f$ is a unit vector field. Indeed, consider the new metric

$$(\xi, \eta)_1 = (\xi, \eta)(\mathbf{grad} f, \mathbf{grad} f).$$

Then

$$(\mathbf{grad} f, \xi) = \xi(f) = (\mathbf{grad}_1 f, \xi)_1 = (\mathbf{grad}_1 f, \xi)(\mathbf{grad}_1 f, \mathbf{grad}_1 f).$$

Hence

$$\mathbf{grad}_1 f = \frac{\mathbf{grad} f}{(\mathbf{grad} f, \mathbf{grad} f)}.$$

Then

$$\begin{aligned}(\mathbf{grad}_1 f, \mathbf{grad}_1 f)_1 &= (\mathbf{grad}_1 f, \mathbf{grad}_1 f)(\mathbf{grad} f, \mathbf{grad} f) = \\ &= \frac{(\mathbf{grad} f, \mathbf{grad} f)}{(\mathbf{grad} f, \mathbf{grad} f)^2}(\mathbf{grad} f, \mathbf{grad} f) = 1.\end{aligned}$$

Thus the integral curves

$$\frac{d}{dt} f(\gamma(t)) = (\mathbf{grad} f, \mathbf{grad} f) \equiv 1.$$

Hence the function $f(\gamma(t))$ is linear. This means that if

$$f(x_0) = f(x_1),$$

then

$$f(\gamma(x_0, t)) = f(\gamma(x_1, t)) = f(x_0) + t.$$

Put

$$g : Z \times \mathbf{R}^1 \longrightarrow X, \ g(x, t) = \gamma(x, t).$$

The mapping g is a fiberwise smooth homeomorphism. Hence the mapping

$$f : \longrightarrow \mathbf{R}^1$$

gives a locally trivial bundle. Further, the proof will follow by induction with respect to n. Consider a vector valued function

$$f(x) = \{f_1(x), \ldots, f_n(x)\}$$

which satisfies the condition of the theorem. Choose a Riemannian metric on the manifold X such that gradients

$$\mathbf{grad} f_1, \ldots, \mathbf{grad} f_n$$

are orthonormal. Such a metric exists. Indeed, consider firstly an arbitrary metric. The using the linear independence of the differentials $\{df_i\}$ we know that the gradients are also independent. Hence the matrix

$$a_{ij} = \langle \mathbf{grad} f_i(x), \ \mathbf{grad} f_j(x) \rangle$$

is nondegenerate in each point. Let $\|b_{ij}(x)\|$ be the matrix inverse to the matrix $\|a_{ij}(x)\|$, that is,

$$\sum_\alpha b_{i\alpha}(x)a_{j\alpha}(x) \equiv \delta_{ij}.$$

Put

$$\xi_k = \sum_i b_{kj}(x)\mathbf{grad} f_i(x).$$

Then

$$\begin{aligned}\xi_k(f_j) &= \sum_i b_{ki}(x)\mathbf{grad} f_i(f_j) = \\ &= \sum_i b_{ki}(x)\langle \mathbf{grad} f_i,\ \mathbf{grad} g_j\rangle = \sum_i b_{ki}(x)a_{ij}(x) \equiv \delta_{kj}.\end{aligned}$$

Let U_α be a sufficiently small neighborhood of a point of the manifold X. The system of vector fields $\{\xi_1, \ldots, \xi_n\}$ can be supplemented by vector fields $\eta_{n+1}, \ldots, \eta_N$ to form a basis such that

$$\eta_k(f_i) \equiv 0.$$

Consider the new metric in the chart U_α given by

$$\begin{aligned}\langle \xi_i,\ \xi_j\rangle_\alpha &\equiv \delta_{ij}, \\ \langle \xi_k,\ \eta_j\rangle_\alpha &\equiv 0.\end{aligned}$$

Let φ_α be a partition of unity subordinate to the covering $\{U_\alpha\}$ and put

$$\langle \xi,\ \eta\rangle_0 = \sum_\alpha \varphi_\alpha(x)\langle \xi,\ \eta\rangle_\alpha.$$

Then

$$\begin{aligned}\langle \xi_i,\ \xi_j\rangle_0 &\equiv \delta_{ij}, \\ \langle \xi_k,\ \eta\rangle_0 &\equiv 0\end{aligned}$$

for any vector η for which $\eta(f_i) = 0$. Let $\mathbf{grad}_0 f_i$ be the gradients of the functions f_i with respect to the metric $\langle \xi,\ \eta\rangle_0$. This means that

$$\langle \mathbf{grad} f_i, \xi\rangle_0 = \xi(f_i)$$

for any vector ξ. In particular one has

$$\begin{aligned}\langle \mathbf{grad} f_i,\ \xi_j\rangle_0 &\equiv \delta_{ij}, \\ \langle \mathbf{grad} f_k,\ \eta\rangle_0 &\equiv 0\end{aligned}$$

for any vector η for which $\eta(f_i) = 0$.

Similar relations hold for the vector field ξ_i. Therefore

$$\xi_i = \mathbf{grad} f_i,$$

that is,

$$\langle \mathbf{grad} f_i,\ \mathbf{grad} f_j \rangle \delta_{ij},$$

the latter proves the existence of metric with the necessarily properties.

Let us pass now to the proof of the theorem. Consider the vector function

$$g(x) = \{f_1(x), \ldots, f_{n-1}(x)\}.$$

This function satisfies the conditions of the theorem and the inductive assumption. It follows that the manifold X is diffeomorphic to the Cartesian product $X = Z \times \mathbf{R}^{n-1}$ and the functions f_i are the coordinate functions for the second factor. Then $\mathbf{grad} f_n$ is tangent to the first factor Z and hence the function $f_n(x,t)$ does not depend on $t \in \mathbf{R}^{n-1}$. Therefore one can apply the first step of the induction to the manifold Z, that is,

$$Z = Z_1 \times \mathbf{R}^1.$$

Thus

$$X = Z_1 \times \mathbf{R}^n.$$

∎

3.3 The Pontryagin–Thom construction, the bordism theory.

3.4 The Morse functions on manifolds

Consider a smooth function f on a manifold X. A point p_0 is called a *critical point* if

$$df(x_0) = 0.$$

A critical point x_0 is said to be *nondegenerate* if the matrix of second derivatives is nondegenerate. This property does not depend on a choice of local coordinates. Let T^*X be the total space of the cotangent bundle of manifold X (that is, the vector bundle which is dual to the tangent bundle). Then for each function

$$f : X \longrightarrow \mathbf{R}^1$$

there is a mapping

$$df : X \longrightarrow T^*X \tag{64}$$

adjoint to Df which to each point $x \in X$ associates the linear form on T_xX given by the differential of the function f at the point x. Then in the manifold T^*X there are two submanifolds: the zero section X_0 of the bundle T^*X and the

image $df(X)$. The common points of these submanifolds correspond to critical points of the function f. Further, a critical point is nondegenerate if and only if the intersection of submanifolds X_0 and $df(X)$ at that point is transversal. If all the critical points of the function f are nondegenerate then f is called a *Morse function*. Thus the function f is a Morse function if and only if the mapping (64) is transversal along the zero section $X_0 \subset T^*X$.

3.5 Vector fields, the Lie brackets, the Lie algebra structure, integrable distributions, foliations.

3.5.1 Dynamical system

Let $\xi(P)$ be a smooth field on M . We recall that the trajectory $\gamma(t)$ is called an integral curve of the field $\xi(P)$ if $\frac{d}{dt}\gamma(t) = \xi(\gamma(t))$, i.e. if the tangent velocity vectors to $\gamma(t)$ coincide with the vector field ξ.

Theorem 18 *Let ξ be a smooth vector field on a smooth compact manifold M. Them there is unique smooth mapping*

$$\varphi : M \times \mathbf{R}^1 \longrightarrow M \tag{65}$$

which forms the collection of all integral curves of the field ξ that is

$$\begin{array}{rcl} \varphi(x_0, 0) & = & x_0, \\ \frac{d}{dt}\varphi(x_0, t) & = & \xi(\varphi(x_0, t)). \end{array} \tag{66}$$

The mapping (65) satisfies the property

$$\varphi(\varphi(x_0, t), s) = \varphi(x_0, t + s),\ t, s \in \mathbf{R}^1, \tag{67}$$

that is a one parametric group of diffeomorphisms of the manifold M.

4 Differential forms, calculus, de Rham complex, de Rham cohomology, the Hodge theory.

5 Integration of differential forms, the general Stokes formula. Special cases: Newton–Leibniz, Green, Gauss-Ostrogradsky, 3-dimensional Stokes formula.

6 Application to the mapping degree and the Gauss-Bonnet formula.

7 Locally trivial bundles

7.1 Definition

The definition of a locally trivial bundle was coined to capture an idea which recurs in a number of different geometric situations. We commence by giving a number of examples.

The surface of the cylinder can be seen as a disjoint union of a family of line segments continuously parametrized by points of a circle. The Möbius band can be presented in similar way. The two dimensional torus embedded in the three dimensional space can presented as a union of a family of circles (meridians) parametrized by points of another circle (a parallel).

Now, let M be a smooth manifold embedded in the Euclidean space $\mathbf{R^N}$ and TM the space embedded in $\mathbf{R^N} \times \mathbf{R^N}$, the points of which are the tangent vectors of the manifold M. This new space TM can be also be presented as a union of subspaces T_xM, where each T_xM consists of all the tangent vectors to the manifold M at the single point x. The point x of M can be considered as a parameter which parametrizes the family of subspaces T_xM. In all these cases the space may be partitioned into fibers parametrized by points of the base.

The examples considered above share two important properties: a) any two fibers are homeomorphic, b) despite the fact that the whole space cannot be presented as a Cartesian product of a fiber with the base (the parameter space), if we restrict our consideration to some small region of the base the part of the fiber space over this region is such a Cartesian product. The two properties above are the basis of the following definition.

Definition 23 *Let E and B be two topological spaces with a continuous map*

$$p : E \longrightarrow B.$$

The map p is said to define a locally trivial bundle if there is a topological space F such that for any point $x \in B$ there is a neighborhood $U \ni x$ for which

the inverse image $p^{-1}(U)$ is homeomorphic to the Cartesian product $U \times F$. Moreover, it is required that the homeomorphism

$$\varphi : U \times F \longrightarrow p^{-1}(U)$$

preserves fibers, it is a 'fiberwise' map, that is, the following equality holds:

$$p(\varphi(x, f)) = x, x \in U, f \in F.$$

The space E is called total space of the bundle or thefiberspace, the space B is called thebase of the bundle, the space F is called thefiber of the bundle and the mapping p is called theprojection. The requirement that the homeomorphism φ be fiberwise means in algebraic terms that the diagram

$$\begin{array}{ccc} U \times F & \xrightarrow{\varphi} & p^{-1}(U) \\ \downarrow \pi & & \downarrow p \\ U & = & U \end{array}$$

where

$$\pi : U \times F \longrightarrow U, \quad \pi(x, f) = x$$

is the projection onto the first factor is commutative.

One problem in the theory of fiber spaces is to classify the family of all locally trivial bundles with fixed base B and fiber F. Two locally trivial bundles $p : E \longrightarrow B$ and $p' : E' \longrightarrow B$ are considered to be isomorphic if there is a homeomorphism $\psi : E \longrightarrow E'$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\psi} & E' \\ \downarrow p & & \downarrow p' \\ B & = & B \end{array}$$

is commutative. It is clear that the homeomorphism ψ gives a homeomorphism of fibers $F \longrightarrow F'$. To specify a locally trivial bundle it is not necessary to be given the total space E explicitly. It is sufficient to have a base B, a fiber F and a family of mappings such that the total space E is determined 'uniquely' (up to isomorphisms of bundles). Then according to the definition of a locally trivial bundle, the base B can be covered by a family of open sets $\{U_\alpha\}$ such that each inverse image $p^{-1}(U_\alpha)$ is fiberwise homeomorphic to $U_\alpha \times F$. This gives a system of homeomorphisms

$$\varphi_\alpha : U_\alpha \times F \longrightarrow p^{-1}(U_\alpha).$$

Since the homeomorphisms φ_α preserve fibers it is clear that for any open subset $V \subset U_\alpha$ the restriction of φ_α to $V \times F$ establishes the fiberwise homeomorphism of $V \times F$ onto $p^{-1}(V)$. Hence on $U_\alpha \times U_\beta$ there are two fiberwise homeomorphisms

$$\varphi_\alpha : (U_\alpha \cap U_\beta) \times F \longrightarrow p^{-1}(U_\alpha \cap U_\beta),$$
$$\varphi_\beta : (U_\alpha \cap U_\beta) \times F \longrightarrow p^{-1}(U_\alpha \cap U_\beta).$$

Let $\varphi_{\alpha\beta}$ denote the homeomorphism $\varphi_\beta^{-1}\varphi_\alpha$ which maps $(U_\alpha \cap U_\beta) \times F$ onto itself. The locally trivial bundle is uniquely determined by the following collection: the base B, the fiber F, the covering U_α and the homeomorphisms

$$\varphi_{\alpha\beta} : (U_\alpha \cap U_\beta) \times F \longrightarrow (U_\alpha \cap U_\beta) \times F.$$

The total space E should be thought of as a union of the Cartesian products $U_\alpha \times F$ with some identifications induced by the homeomorphisms $\varphi_{\alpha\beta}$. By analogy with the terminology for smooth manifolds, the open sets U_α are called *charts*, the family $\{U_\alpha\}$ is called *the atlas of charts*, the homeomorphisms φ_α are called *the coordinate homeomorphisms* and the $\varphi_{\alpha\beta}$ are called *the transition functions*or *the sewing functions*. Sometimes the collection $\{U_\alpha, \varphi_\alpha\}$ is called the atlas. Thus any atlas determines a locally trivial bundle. Different atlases may define isomorphic bundles but, beware, not any collection of homeomorphisms φ_α forms an atlas. For the classification of locally trivial bundles, families of homeomorphisms $\varphi_{\alpha\beta}$ that actually determine bundles should be selected and then separated into classes which determine isomorphic bundles. For the homeomorphisms $\varphi_{\alpha\beta}$ to be transition functions for some locally trivial bundle:

$$\varphi_{\beta\alpha} = \varphi_\beta^{-1}\varphi_\alpha. \tag{68}$$

Then for any three indices α, β, γ on the intersection $(U_\alpha \cap U_\beta \cap U_\gamma) \times F$ the following relation holds:

$$\varphi_{\alpha\gamma}\varphi_{\gamma\beta}\varphi_{\beta\alpha} = \mathbf{id},$$

where $\mathbf{id}$ is the identity homeomorphism and for each α,

$$\varphi_{\alpha\alpha} = \mathbf{id}. \tag{69}$$

In particular

$$\varphi_{\alpha\beta}\varphi_{\beta\alpha} = \mathbf{id}, \tag{70}$$

thus

$$\varphi_{\alpha\beta} = \varphi_{\beta\alpha}^{-1}.$$

Hence for an atlas the $\varphi_{\alpha\beta}$ should satisfy

$$\varphi_{\alpha\alpha} = \mathbf{id}, \quad \varphi_{\alpha\gamma}\varphi_{\gamma\beta}\varphi_{\beta\alpha} = \mathbf{id}. \tag{71}$$

These conditions are sufficient for a locally trivial bundle to be reconstructed from the base B, fiber F, atlas $\{U_\alpha\}$ and homeomorphisms $\varphi_{\beta\alpha}$. To see this, let

$$E' = \cup_\alpha (U_\alpha \times F)$$

be the disjoint union of the spaces $U_\alpha \times F$. Introduce the following equivalence relation: the point $(x, f) \in U_\alpha \times F$ is related to the point $(y, g) \in U_\beta \times F$ iff

$$x = y \in U_\alpha \cap U_\beta$$

and

$$(y, g) = \varphi_{\beta\alpha}(x, f).$$

The conditions (69), (70) guarantee that this is an equivalence relation, that is, the space E' is partitioned into disjoint classes of equivalent points. Let E be the quotient space determined by this equivalence relation, that is, the set whose points are equivalence classes. Give E the quotient topology with respect to the projection

$$\pi : E' \longrightarrow E$$

which associates to a (x, f) its the equivalence class. In other words, the subset $G \subset E$ is called open iff $\pi^{-1}(G)$ is open set. There is the natural mapping p' from E' to B:

$$p'(x, f) = x.$$

Clearly the mapping p' is continuous and equivalent points maps to the same image. Hence the mapping p' induces a map

$$p : E \longrightarrow B$$

which associates to an equivalence class the point assigned to it by p'. The mapping p is continuous. It remains to construct the coordinate homeomorphisms. Put

$$\varphi_\alpha = \pi|_{U_\alpha \times F} : U_\alpha \times F \longrightarrow E.$$

Each class $z \in p^{-1}(U_\alpha)$ has a unique representative $(x, f) \in U_\alpha \times F$. Hence φ_α is a one to one mapping onto $p^{-1}(U_\alpha)$. By virtue of the quotient topology on E the mapping φ_α is a homeomorphism. It is easy to check that (compare with (68))

$$\varphi_\beta^{-1}\varphi_\alpha = \varphi_{\beta\alpha}.$$

So we have shown that locally trivial bundles may be defined by atlas of charts $\{U_\alpha\}$ and a family of homeomorphisms $\{\varphi_{\beta\alpha}\}$ satisfying the conditions (69), (70). Let us now determine when two atlases define isomorphic bundles. First of all notice that if two bundles $p : E \longrightarrow B$ and $p' : E' \longrightarrow B$ with the same fiber F have the same transition functions $\{\varphi_{\beta\alpha}\}$ then these two bundles are isomorphic. Indeed, let

$$\varphi_\alpha : U_\alpha \longrightarrow p^{-1}(U_\alpha).$$
$$\psi_\alpha : U_\alpha \longrightarrow p'^{-1}(U_\alpha).$$

be the corresponding coordinate homeomorphisms and assume that

$$\varphi_{\beta\alpha} = \varphi_\beta^{-1}\varphi_\alpha = \psi_\beta^{-1}\psi_\alpha = \psi_{\beta\alpha}.$$

Then

$$\varphi_\alpha\psi_\alpha^{-1} = \varphi_\beta\psi_\beta^{-1}.$$

We construct a homeomorphism

$$\psi : E' \longrightarrow E.$$

Let $x \in E'$. The atlas $\{U_\alpha\}$ covers the base B and hence there is an index α such that $x \in p'^{-1}(U_\alpha)$. Set

$$\psi(x) = \varphi_\alpha\psi_\alpha^{-1}(x).$$

It is necessary to establish that the value of $\psi(x)$ is independent of the choice of index α. If $x \in p'^{-1}(U_\beta)$ also then

$$\begin{aligned}\varphi_\beta \psi_\beta^{-1}(x) &= \varphi_\alpha \varphi_\alpha^{-1} \varphi_\beta \psi_\beta^{-1} \psi_\alpha \psi_\alpha^{-1}(x) = \\ &= \varphi_\alpha \varphi_{\alpha\beta} \varphi_{\beta\alpha} \psi_\alpha^{-1}(x) = \varphi_\alpha \psi_\alpha^{-1}(x).\end{aligned}$$

Hence the definition of $\psi(x)$ is independent of the choice of chart. Continuity and other necessary properties are evident. Further, given an atlas $\{U_\alpha\}$ and coordinate homomorphisms $\{\varphi_\alpha\}$, if $\{V_\beta\}$ is a finer atlas (that is, $V_\beta \subset U_\alpha$ for some $\alpha = \alpha(\beta)$) then for the atlas $\{V_\beta\}$, the coordinate homomorphisms are defined in a natural way

$$\varphi'_\beta = \varphi_{\alpha(\beta)}|_{(V_\beta \times F)} : V_\beta \times F \longrightarrow p^{-1}(V_\beta).$$

The transition functions $\varphi'_{\beta_1,\beta_2}$ for the new atlas $\{V_\beta\}$ are defined using restrictions

$$\varphi'_{\beta_1,\beta_2} = \varphi_{\alpha(\beta_1),\alpha(\beta_2)}|_{(V_{\beta_1} \cap V_{\beta_2}) \times F} : (V_{\beta_1} \cap V_{\beta_2}) \times F \longrightarrow (V_{\beta_1} \cap V_{\beta_2}) \times F.$$

Thus if there are two atlases and transition functions for two bundles, with a common refinement, that is, a finer atlas with transition functions given by restrictions, it can be assumed that the two bundles have the same atlas. If $\varphi_{\beta\alpha}$, $\varphi'_{\beta\alpha}$ are two systems of the transition functions (for the same atlas), giving isomorphic bundles then the transition functions $\varphi_{\beta\alpha}$, $\varphi'_{\beta\alpha}$ must be related.

Theorem 19 *Two systems of the transition functions $\varphi_{\beta\alpha}$, and $\varphi'_{\beta\alpha}$ define isomorphic locally trivial bundles iff there exist fiber preserving homeomorphisms*

$$h_\alpha : U_\alpha \times F \longrightarrow U_\alpha \times F$$

such that

$$\varphi_{\beta\alpha} = h_\beta^{-1} \varphi'_{\beta\alpha} h_\alpha. \tag{72}$$

7.1.1 Examples

1. Let $E = B \times F$ and $p : E \longrightarrow B$ be projections onto the first factors. Then the atlas consists of one chart $U_\alpha = B$ and only one the transition function $\varphi_{\alpha\alpha} = \mathbf{id}$ and the bundle is said to be *trivial*.

2. Let E be the Möbius band. One can think of this bundle as a square in the plane, $\{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$ with the points $(0, y)$ and $(1, 1 - y)$ identified for each $y \in [0, 1]$. The projection p maps the space E onto the segment $I_x = \{0 \le x \le 1\}$ with the endpoints $x = 0$ and $x = 1$ identified, that is, onto the circle S^1. Let us show that the map p defines a locally trivial bundle. The atlas consists of two intervals (recall 0 and 1 are identified)

$$U_\alpha = \{0 < x < 1\},\ U_\beta = \{0 \le x < \frac{1}{2}\} \cup \{\frac{1}{2} < x \le 1\}.$$

The coordinate homeomorphisms may be defined as following:

$$\begin{aligned}
\varphi_\alpha : U_\alpha \times I_y &\longrightarrow E, \varphi_\alpha(x,y) = (x,y), \\
\varphi_\beta : U_\beta \times I_y &\longrightarrow E \\
\varphi_\beta(x,y) = (x,y) &\quad \text{for} \quad 0 \le x < \frac{1}{2}, \\
\varphi_\beta(x,y) = (x,1-y) &\quad \text{for} \quad \frac{1}{2} < x \le 1.
\end{aligned}$$

The intersection of two charts $U_\alpha \cap U_\beta$ consists of union of two intervals $U_\alpha \cap U_\beta = (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. The transition function $\varphi_{\beta\alpha}$ have the following form

$$\begin{aligned}
\varphi_{\beta\alpha} &= (x,y) \text{ for } \quad 0 < x < \frac{1}{2}, \\
\varphi_{\beta\alpha} &= (x,1-y) \text{ for } \quad \frac{1}{2} < x < 1.
\end{aligned}$$

The Möbius band is not isomorphic to a trivial bundle. Indeed, for a trivial bundle all transition functions can be chosen equal to the identity. Then by Theorem **??** there exist fiber preserving homeomorphisms

$$\begin{aligned}
h_\alpha : U_\alpha \times I_y &\longrightarrow U_\alpha \times I_y, \\
h_\beta : U_\beta \times I_y &\longrightarrow U_\beta \times I_y,
\end{aligned}$$

such that

$$\varphi_{\beta\alpha} = h_\beta^{-1} h_\alpha$$

in its domain of definition $(U_\alpha \cap U_\beta) \times I_y$. Then $h_\alpha,\ h_\beta$ are fiberwise homeomorphisms for fixed value of the first argument x giving homeomorphisms of interval I_y to itself. Each homeomorphism of the interval to itself maps end points to end points. So the functions

$$h_\alpha(x,0),\ h_\alpha(x,1),\ h_\beta(x,0),\ h_\beta(x,1)$$

are constant functions, with values equal to zero or one. The same is true for the functions $h_\beta^{-1} h_\alpha(x,0)$. On the other hand the function $\varphi_{\beta\alpha}(x,0)$ is not constant because it equals zero for each $0 < x < \frac{1}{2}$ and equals one for each $\frac{1}{2} < x < 1$.This contradiction shows that the Möbius band is not isomorphic to a trivial bundle.

3. Let E be the space of tangent vectors to two dimensional sphere $\mathbf{S}^2$ embedded in three dimensional Euclidean space $\mathbf{R}^3$. Let

$$p : E \longrightarrow \mathbf{S}^2$$

be the map associating each vector to its initial point. Let us show that p is a locally trivial bundle with fiber $\mathbf{R}^2$. Fix a point $s_0 \in \mathbf{S}^2$. Choose a Cartesian system of coordinates in $\mathbf{R}^3$ such that the point s_0 is the North Pole on the

sphere (that is, the coordinates of s_0 equal $(0,0,1)$). Let U be the open subset of the sphere $\mathbf{S}^2$ defined by inequality $z > 0$. If $s \in U,\ s = (x, y, z)$, then

$$x^2 + y^2 + z^2 = 1,\ z > 0.$$

Let $\vec{e} = (\xi, \eta, \zeta)$ be a tangent vector to the sphere at the point s. Then

$$x\xi + y\eta + z\zeta = 0,$$

that is,

$$\zeta = -(x\xi + y\eta)/z.$$

Define the map

$$\varphi : U \times \mathbf{R}^2 \longrightarrow p^{-1}(U)$$

by the formula

$$\varphi(x, y, z, \xi, \eta) = (x, y, z, \xi, \eta, -(x\xi + y\eta)/z)$$

giving the coordinate homomorphism for the chart U containing the point $s_0 \in \mathbf{S}^2$. Thus the map p gives a locally trivial bundle. This bundle is called the *tangent bundle* of the sphere $\mathbf{S}^2$.

7.2 The structure groups

The relations (71,72) obtained in the previous section for the transition functions of a locally trivial bundle are similar to those involved in the calculation of one dimensional cohomology with coefficients in some algebraic sheaf. This analogy can be explain after a slight change of terminology and notation and the change will be useful for us for investigating the classification problem of locally trivial bundles. Notice that a fiberwise homeomorphism of the Cartesian product of the base U and the fiber F onto itself

$$\varphi : U \times F \longrightarrow U \times F, \tag{73}$$

can be represented as a family of homeomorphisms of the fiber F onto itself, parametrized by points of the base B. In other words, each fiberwise homeomorphism φ defines a map

$$\bar{\varphi} : U \longrightarrow \mathbf{Homeo}\ (F), \tag{74}$$

where $\mathbf{Homeo}\ (F)$ is the group of all homeomorphisms of the fiber F. Furthermore, if we choose the right topology on the group $\mathbf{Homeo}\ (F)$ the map $\bar{\varphi}$ becomes continuous. Sometimes the opposite is true: the map (74) generates the fiberwise homeomorphism (73)with respect to the formula

$$\varphi(x, f) = (x, \bar{\varphi}(x)f).$$

So instead of $\varphi_{\alpha\beta}$ a family of functions

$$\bar{\varphi}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathbf{Homeo}\ (F),$$

can be defined on the intersection $U_\alpha \cap U_\beta$ and having values in the group **Homeo** (F). In homological algebra the family of functions $\bar{\varphi}_{\alpha\beta}$ is called a *one dimensional cochain* with values in the sheaf of germs of functions with values in the group **Homeo** (F). The condition (71) from the section **??** means that

$$\bar{\varphi}_{\alpha\alpha}(x) = \mathbf{id},$$
$$\bar{\varphi}_{\alpha\gamma}(x)\bar{\varphi}_{\gamma\beta}(x)\bar{\varphi}_{\beta\alpha}(x) = \mathbf{id}.$$
$$x \in U_\alpha \cap U_\beta \cap U_\gamma.$$

and we say that the cochain $\{\bar{\varphi}_{\alpha\beta}\}$ is a *cocycle*. The condition (72) means that there is a zero dimensional cochain $h_\alpha : U_\alpha \longrightarrow$**Homeo** (F) such that

$$\bar{\varphi}_{\beta\alpha}(x) = h_\beta^{-1}(x)\bar{\varphi}'_{\beta\alpha}(x)h_\alpha(x),\ x \in U_\alpha \cap U_\beta.$$

Using the language of homological algebra the condition (72) means that cocycles $\{\bar{\varphi}_{\beta\alpha}\}$ and $\{\bar{\varphi}'_{\beta\alpha}\}$ are cohomologous. Thus the family of locally trivial bundles with fiber F and base B is in one to one correspondence with the one dimensional cohomology of the space B with coefficients in the sheaf of the germs of continuous **Homeo** (F)–valued functions for given open covering $\{U_\alpha\}$. Despite obtaining a simple description of the family of locally trivial bundles in terms of homological algebra, it is ineffective since there is no simple method of calculating cohomologies of this kind. Nevertheless, this representation of the transition functions as a cocycle turns out very useful because of the situation described below.

First of all notice that using the new interpretation a locally trivial bundle is determined by the base B, the atlas $\{U_\alpha\}$ and the functions $\{\varphi_{\alpha\beta}\}$ taking the value in the group $G =$ **Homeo** (F). The fiber F itself does not directly take part in the description of the bundle. Hence, one can at first describe a locally trivial bundle as a family of functions $\{\varphi_{\alpha\beta}\}$ with values in some topological group G, and after that construct the total space of the bundle with fiber F by additionally defining an action of the group G on the space F, that is, defining a continuous homomorphism of the group G into the group **Homeo** (F).

Secondly, the notion of locally trivial bundle can be generalized and the structure of bundle made richer by requiring that both the transition functions $\bar{\varphi}_{\alpha\beta}$ and the functions h_α are not arbitrary but take values in some subgroup of the homeomorphism group **Homeo** (F).

Thirdly, sometimes information about locally trivial bundle may be obtained by substituting some other fiber F' for the fiber F but using the 'same' transition functions. Thus we come to a new definition of a locally trivial bundle with additional structure — the group where the transition functions take their values.

Definition 24 *Let E,B,F be topological spaces and G be a topological group which acts continuously on the space F. A continuous map*

$$p : E \longrightarrow B$$

is said to be a locally trivial bundle with fiber F and the structure group G if there is an atlas $\{U_\alpha\}$ and the coordinate homeomorphisms

$$\varphi_\alpha : U_\alpha \times F \longrightarrow p^{-1}(U_\alpha)$$

such that the transition functions

$$\varphi_{\beta\alpha} = \varphi_\beta^{-1}\varphi_\alpha : (U_\alpha \cap U_\beta) \times F \longrightarrow (U_\alpha \cap U_\beta) \times F$$

have the form

$$\varphi_{\beta\alpha}(x, f) = (x, \bar{\varphi}_{\beta\alpha}(x)f),$$

where $\bar{\varphi}_{\beta\alpha} : (U_\alpha \cap U_\beta) \longrightarrow G$ are continuous functions satisfying the conditions

$$\begin{aligned} &\bar{\varphi}_{\alpha\alpha}(x) \equiv 1,\ x \in U_\alpha, \\ &\bar{\varphi}_{\alpha\beta}(x)\bar{\varphi}_{\beta\gamma}(x)\bar{\varphi}_{\gamma\alpha}(x) \equiv 1,\ x \in U_\alpha \cap U_\beta \cap U_\gamma. \end{aligned} \tag{75}$$

The functions $\bar{\varphi}_{\alpha\beta}$ are also called the transition functions

Let

$$\psi : E' \longrightarrow E$$

be an isomorphism of locally trivial bundles with the structure group G. Let φ_α and $\varphi_\alpha^{'}$ be the coordinate homeomorphisms of the bundles $p : E \longrightarrow B$ and $p' : E' \longrightarrow B$, respectively. One says that the isomorphism ψ is *compatible with the structure group* G if the homomorphisms

$$\varphi_\alpha^{-1}\psi\varphi'_\alpha : U_\alpha \times F \longrightarrow U_\alpha \times F$$

are determined by continuous functions

$$h_\alpha : U_\alpha \longrightarrow G,$$

defined by relation

$$\varphi_\alpha^{-1}\psi\varphi'_\alpha(x, f) = (x, h_\alpha(x)f). \tag{76}$$

Thus two bundles with the structure group G and transition functions $\bar{\varphi}_{\beta\alpha}$ and $\varphi'_{\beta\alpha}$ are isomorphic, the isomorphism being compatible with the structure group G, if

$$\bar{\varphi}_{\beta\alpha}(x) = h_\beta(x)\bar{\varphi}'_{\beta\alpha}(x)h_\alpha(x) \tag{77}$$

for some continuous functions $h_\alpha : U_\alpha \longrightarrow G$. So two bundles whose the transition functions satisfy the condition (77) are called *equivalent bundles*. It is sometimes useful to increase or decrease the structure group G. Two bundles which are not equivalent with respect of the structure group G may become equivalent with respect to a larger structure group G', $G \subset G'$. When a bundle with the structure group G admits transition functions with values in a subgroup H, it is said that the structure group G is *reduced to subgroup* H. It is clear that if the structure group of the bundle $p : E \longrightarrow B$ consists of only one element then the bundle is trivial. So to prove that the bundle is trivial, it is

sufficient to show that its the structure group G may be reduced to the trivial subgroup. More generally, if

$$\rho : G \longrightarrow G'$$

is a continuous homomorphism of topological groups and we are given a locally trivial bundle with the structure group G and the transition functions

$$\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G$$

then a new locally trivial bundle may be constructed with structure group G' for which the transition functions are defined by

$$\varphi'_{\alpha\beta}(x) = \rho(\varphi_{\alpha\beta}(x)).$$

This operation is called *a change of the structure group* (with respect to the homomorphism ρ).

7.2.1 Remark

Note that the fiberwise homeomorphism

$$\varphi : U \times F \longrightarrow U \times F$$

in general is not induced by continuous map

$$\bar{\varphi} : U \longrightarrow \mathbf{Homeo}\,(F). \tag{78}$$

Because of lack of space we will not analyze the problem and note only that later on in all our applications the fiberwise homeomorphisms will be induced by the continuous maps (78) into the structure group G.

Now we can return to the third situation, that is, to the possibility to choosing a space as a fiber of a locally trivial bundle with the structure group G. Let us consider the fiber

$$F = G$$

with the action of G on F being that of left translation, that is, the element $g \in G$ acts on the F by the homeomorphism

$$g(f) = gf,\ f \in F = G.$$

Definition 25 *A locally trivial bundle with the structure group G is called principal G–bundle if $F = G$ and action of the group G on F is defined by the left translations.*

An important property of principal G–bundles is the consistency of the homeomorphisms with the structure group G and it can be described not only in terms of the transition functions (the choice of which is not unique) but also in terms of equivariant properties of bundles.

Theorem 20 *Let*

$$p : E \longrightarrow B$$

be a principal G–bundle,

$$\varphi_\alpha : U_\alpha \times G \longrightarrow p^{-1}(U_\alpha)$$

be the coordinate homeomorphisms. Then there is a right action of the group G on the total space E such that:

1. *the right action of the group G is fiberwise, that is,*

$$p(x) = p(xg),\ x \in E,\ g \in G.$$

2. *the homeomorphism φ_α^{-1} transforms the right action of the group G on the total space into right translations on the second factor, that is,*

$$\varphi_\alpha(x, f)g = f_\alpha(x, fg),\ x \in U_\alpha,\ f, g \in G. \tag{79}$$

Theorem 20 allows us to consider principal G–bundles as having a right action on the total space.

Theorem 21 *Let*

$$\psi : E' \longrightarrow E \tag{80}$$

be a fiberwise map of principal G–bundles. The map (80) is the isomorphism of locally trivial bundles with the structure group G, that is, compatible with the structure group G iff this map is equivariant (with respect to right actions of the group G on the total spaces).

Thus by Theorem 21, to show that two locally trivial bundles with the structure group G (and the same base B) are isomorphic it necessary and sufficient to show that there exists an equivariant map of corresponding principal G–bundles (inducing the identity map on the base B). In particular, if one of the bundles is trivial, for instance, $E' = B \times G$, then to construct an equivariant map $\psi : E' \longrightarrow E$ it is sufficient to define a continuous map ψ on the subspace $\{(x, e) : x \in B, \} \subset E' = B \times G$ into E. Then using equivariance, the map ψ is extended by formula

$$\psi(x, g) = \psi(x, e)g.$$

The map$\{(x, e) : x \in B, \} \xrightarrow{\psi} E'$ can be considered as a map

$$s : B \longrightarrow E \tag{81}$$

satisfying the property

$$ps(x) = x,\ x \in B. \tag{82}$$

The map (81) with the property (82) is called a *cross–section* of the bundle. So each trivial principal bundle has cross–sections. For instance, the map $B \longrightarrow B \times G$ defined by $x \longrightarrow (x, e)$ is a cross–section. Conversely, if a principal bundle has

a cross–section s then this bundle is isomorphic to the trivial principal bundle. The corresponding isomorphism $\psi : B \times G \longrightarrow E$ is defined by formula

$$\psi(x, g) = s(x)g, \ x \in B, \ g \in G.$$

Let us relax our restrictions on equivariant mappings of principal bundles with the structure group G. Consider arbitrary equivariant mappings of total spaces of principal G–bundles with arbitrary bases. Each fiber of a principal G–bundle is an orbit of the right action of the group G on the total space and hence for each equivariant mapping

$$\psi : E' \longrightarrow E$$

of total spaces, each fiber of the bundle

$$p' : E' \longrightarrow B'$$

maps to a fiber of the bundle

$$p : E \longrightarrow B. \tag{83}$$

In other words, the mapping ψ induces a mapping of bases

$$\chi : B' \longrightarrow B \tag{84}$$

and the following diagram is commutative

$$\begin{array}{ccc} E' & \xrightarrow{\psi} & E \\ \downarrow p' & & \downarrow p \\ B' & \xrightarrow{\chi} & B \end{array} \tag{85}$$

Let $U_\alpha \subset B$ be a chart in the base B and let U'_β be a chart such that

$$\chi(U'_\beta) \subset U_\alpha.$$

The mapping $\varphi_\alpha^{-1}\psi\varphi'_\beta$ makes the following diagram commutative

$$\begin{array}{ccc} U'_\beta \times G & \xrightarrow{\varphi_\alpha^{-1}\psi\varphi'_\beta} & U_\alpha \times G \\ \downarrow p'\varphi'_\beta & & \downarrow p\varphi_\alpha \\ U'_\beta & \xrightarrow{\chi} & U_\alpha \end{array} \tag{86}$$

In diagram (86), the mappings $p'\varphi'_\beta$ and $p\varphi_\alpha$ are projections onto the first factors. So one has

$$\varphi_\alpha^{-1}\psi\varphi'_\beta(x', g) = (\chi(x'), h_\beta(x')g).$$

Hence the mapping (84) is continuous. Compare the transition functions of these two bundles. First we have

$$(x', \bar{\varphi}'_{\beta_1\beta_2}(x')g) = \varphi'_{\beta_1\beta_2}(x', g) = \varphi'^{-1}_{\beta_1}\varphi'_{\beta_2}(x', g).$$

Then

$$
\begin{aligned}
& (\chi(x'), h_{\beta_1}(x')\bar{\varphi}'_{\beta_1\beta_2}(x')g) = \\
\varphi^{-1}_{\alpha_1}\psi\varphi'_{\beta_1}\varphi'^{-1}_{\beta_1}\varphi'_{\beta_2}(x',g) &= \varphi^{-1}_{\alpha_1}\psi\varphi'_{\beta_2}(x',g) = \\
= \varphi^{-1}_{\alpha_1}\varphi_{\alpha_1}\varphi^{-1}_{\alpha_2}\psi\varphi'_{\beta_2}(x',g) &= (\chi(x'), \bar{\varphi}_{\alpha_1\alpha_2}(\chi(x'))h_{\beta_2}(x')g),
\end{aligned}
$$

that is,

$$h_{\beta_1}(x')\bar{\varphi}_{\beta_1\beta_2}(x') = \bar{\varphi}_{\alpha_1\alpha_2}(\chi(x'))h_{\beta_2}(x'),$$

or

$$h_{\beta_1}(x')\bar{\varphi}'_{\beta_1\beta_2}(x')h^{-1}_{\beta_2}(x') = \bar{\varphi}_{\alpha_1\alpha_2}(\chi(x')). \tag{87}$$

By Theorem **??** the left part of (87) are the transition functions of a bundle isomorphic to the bundle

$$p' : E' \longrightarrow B'. \tag{88}$$

Thus any equivariant mapping of total spaces induces a mapping of bases

$$\chi : B' \longrightarrow B.$$

Moreover, under a proper choice of the coordinate homeomorphisms the transition functions of the bundle (88) are inverse images of the transition functions of the bundle (83). The inverse is true as well: if

$$\chi : B' \longrightarrow B$$

is a continuous mapping and

$$p : E \longrightarrow B$$

is a principal G–bundle then one can put

$$U'_\alpha = \chi^{-1}(U_\alpha),\ \bar{\varphi}'_{\alpha\beta}(x') = \bar{\varphi}_{\alpha\beta}(\chi(x)). \tag{89}$$

Then the transition functions (89) define a principal G–bundle

$$p' : \longrightarrow B,$$

for which there exists an equivariant mapping

$$\psi : E' \longrightarrow E$$

with commutative diagram (85). The bundle defined by the transition functions (89) is called the *inverse image of the bundle* $p : E \longrightarrow B$ *with respect to the mapping* χ . In the special case when the mapping $\chi : B' \longrightarrow B$ is an inclusion then we say that the inverse image of the bundle with respect to the mapping χ is the *restriction of the bundle to the subspace* $B'' = \chi(B')$. In this case the total space of the restriction of the bundle to the subspace B'' coincides with the inverse image

$$E'' = p^{-1}(B'') \subset E.$$

Thus if

$$E' \xrightarrow{\psi} E$$

is an equivariant mapping of total spaces of principal G–bundles then the bundle $p' : E' \longrightarrow B'$ is an inverse image of the bundle $p : \longrightarrow B$ with respect to the mapping $\chi : B' \longrightarrow B$. Constructing of the inverse image is an important way of construction new locally trivial bundles. The following theorem shows that inverse images with respect to homotopic mappings are isomorphic bundles.

Theorem 22 *Let*

$$p : E \longrightarrow B \times I$$

be a principal G–bundle, where the base is a Cartesian product of the compact space B and the unit interval $I = [0, 1]$, and let G be a Lie group. Then restrictions of the bundle p to the subspaces $B \times \{0\}$ and $B \times \{1\}$ are isomorphic.

Corollary 1 *If the transition functions $\varphi_{\alpha\beta}(x)$ and $\psi_{\alpha\beta}(x)$ are homotopic within the class of the transition functions then corresponding bundles are isomorphic.*

7.2.2 Examples

1. Consider the Möbius band. The transition functions $\varphi_{\alpha\beta}$ take two values in the homeomorphism group of the fiber: the identity homeomorphism $e(y) \equiv y,\ y \in I$ and homeomorphism $j(y) \equiv 1 - y,\ y \in I$. The group generated by the two elements e and j has the order two since $j^2 = e$. So instead of the Möbius band we can consider corresponding principal bundle with the structure group $G = Z_2$. As a topological space the group G consists of two isolated points. So the fiber of the principal bundle is the discrete two-point space. This fiber space can be thought of as two segments with ends which are identified crosswise. Hence the total space is also a circle and the projection $p : \mathbf{S}^1 \longrightarrow \mathbf{S}^1$ is a two-sheeted covering. This bundle is nontrivial since the total space of a trivial bundle would have two connected components.

2. Consider the tangent bundle of two dimensional sphere. The coordinate homeomorphisms

$$\varphi : U \times \mathbf{R}^2 \longrightarrow \mathbf{R}^3 \times \mathbf{R}^3$$

were defined by formulas that were linear with respect to the second argument. Hence the transition functions also have values in the group of linear transformations of the fiber $F = \mathbf{R}^2$, that is, $G = \mathbf{GL}\,(2, \mathbf{R})$. It can be shown that the structure group G can be reduced to the subgroup $\mathbf{O}(n)$ of orthonormal transformations, induced by rotations and reflections of the plane. Let us explain these statements about the example of the tangent bundle of the two dimensional sphere $\mathbf{S}^2$. To define a coordinate homeomorphism means to define a basis of tangent vectors $e_1(x),\ e_2(x)$ at each point $x \in U_\alpha$ such that functions $e_1(x)$ and $e_2(x)$ are continuous.

Let us choose two charts $U_\alpha = \{(x, y, z) : z \neq 1\}$, $U_\beta = \{(x, y, z) : z \neq -1\}$. The south pole $P_0 = (0, 0, -1)$ belongs to the chart U_α. The north pole $P_1 = (0, 0, +1)$ belongs to the chart U_β. Consider the meridians. Choose an

orthonormal basis for the tangent space of the point P_0 and continue it along the meridians by parallel transfer with respect to the Riemannian metric of the sphere $\mathbf{S}^2$ to all of the chart U_α. Thus we obtain a continuous family of orthonormal bases $e_1(x),\ e_2(x)$ defined at each point of U_α. In a similar way we construct a continuous family of orthonormal bases $e_1'(x),\ e_2'(x)$ defined over U_β. Then the coordinate homeomorphisms are defined by the following formulas

$$\begin{aligned} \varphi_\alpha(x,\xi,\eta) &= \xi e_1(x) + \eta e_2(x) \\ \varphi_\beta(x,\xi',\eta') &= \xi' e_1'(x) + \eta' e_2'(x). \end{aligned}$$

The transition function $f_{\beta\alpha} = \varphi_\beta^{-1}\varphi_\alpha$ expresses the coordinates of a tangent vector at a point $x \in U_\alpha \cap U_\beta$ in terms of the basis $e_1'(x), e_2'(x)$ by the coordinates of the same vector with respect to the basis $e_1(x), e_2(x)$. As both bases are orthonormal, the change of coordinates (ξ', η') into coordinates (ξ, η) is realized by multiplication by an orthogonal matrix. Thus the structure group $\mathbf{GL}\ (2, \mathbf{R})$ of the tangent bundle of the sphere $\mathbf{S}^2$ is reducing to the subgroup $\mathbf{O}(2) \subset \mathbf{GL}\ (2, \mathbf{R})$.

3. Any trivial bundle with the base B can be constructed as the inverse image of the mapping of the base B into a one-point space $\{\mathrm{pt}\}$ which is the base of a trivial bundle.

7.3 Vector bundles

The most important special class of locally trivial bundles with given structure group is the class of bundles where the fiber is a vector space and the structure group is a group of linear automorphisms of the vector space. Such bundles are called *vector bundles*. So, for example, the tangent bundle of two-dimensional sphere $\mathbf{S}^2$ is a vector bundle. One can also consider locally trivial bundles where fiber is a infinite dimensional Banach space and the structure group is the group of invertible bounded operators of the Banach space. In the case when the fiber is $\mathbf{R}^n$, the vector bundle ξ is said to be *finite dimensional* and the dimension of the vector bundle is equal to n ($\dim \xi = n$). When the fiber is an infinite dimensional Banach space, the bundle is said to be *infinite dimensional*. Vector bundles possess some special features.

First of all notice that each fiber $p^{-1}(x),\ x \in B$ has the structure of vector space which does not depend on the choice of coordinate homeomorphism. In other words, the operations of addition and multiplication by scalars is independent of the choice of coordinate homeomorphism. Indeed, since the structure group G is $\mathbf{GL}\ (n, \mathbf{R})$ the transition functions

$$\varphi_{\alpha\beta} : (U_\alpha \cap U_\beta) \times \mathbf{R}^n \longrightarrow (U_\alpha \cap U_\beta) \times \mathbf{R}^n$$

are linear mappings with respect to the second factor. Hence a linear combination of vectors goes to the linear combination of images with the same coefficients.

Denote by $\Gamma(\xi)$ the set of all sections of the vector bundle ξ. Then the set $\Gamma(\xi)$ becomes an (infinite dimensional) vector space. To define the structure of

vector space on the $\Gamma(\xi)$ consider two sections s_1, s_2:

$$s_1, s_2 : B \longrightarrow E.$$

Put

$$(s_1 + s_2)(x) = s_1(x) + s_2(x),\ x \in B, \tag{90}$$

$$(\lambda s_1)(x) = \lambda(s_1(x)),\ \lambda \in R,\ x \in B. \tag{91}$$

The formulas (90) and (91) define on the set $\Gamma(\xi)$ the structure of vector space. Notice that an arbitrary section $s : B \longrightarrow E$ can be described in local terms. Let $\{U_\alpha\}$ be an atlas, $\varphi_\alpha : U_\alpha \times \mathbf{R}^n \longrightarrow p^{-1}(U_\alpha)$ be coordinate homeomorphisms, $\varphi_{\alpha\beta} = \varphi_\beta^{-1}\varphi_\alpha$. Then the compositions

$$\varphi_\alpha^{-1} s : U_\alpha \longrightarrow U_\alpha \times \mathbf{R}^n$$

are sections of trivial bundles over U_α and determine vector valued functions $s_\alpha : U_\alpha \longrightarrow \mathbf{R}^n$ by the formula

$$(\varphi_\alpha^{-1})(x) = (x, s_\alpha),\ x \in U_\alpha.$$

On the intersection of two charts $U_\alpha \cap U_\beta$ the functions $s_\alpha(x)$ satisfy the following compatibility condition

$$s_\beta(x) = \varphi_{\beta\alpha}(x)(s_\alpha(x)). \tag{92}$$

Conversely, if one has a family of vector valued functions $s_\alpha : U_\alpha \longrightarrow \mathbf{R}^n$ which satisfy the compatibility condition (92) then the formula

$$s(x) = \varphi_\alpha(x, s_\alpha(x))$$

determines the mapping $s : B \longrightarrow E$ uniquely (that is, independent of the choice of chart U_α).

The map s is a section of the bundle ξ.

7.3.1 Operations of direct sum and tensor product

There are natural operations induced by the direct sum and tensor product of vector spaces on the family of vector bundles over a common base B. Firstly, consider the operation of direct sum of vector bundles. Let ξ_1 and ξ_2 be two vector bundles with fibers V_1 and V_2, respectively. Denote the transition functions of these bundles in a common atlas of charts by $\varphi^1_{\alpha\beta}(x)$ and $\varphi^2_{\alpha\beta}(x)$. Notice that values of the transition function $\varphi^1_{\alpha\beta}(x)$ lie in the group $\mathbf{GL}\,(V_1)$ whereas the values of the transition function $\varphi^2_{\alpha\beta}(x)$ lie in the group $\mathbf{GL}\,(V_2)$. Hence the transition functions $\varphi^1_{\alpha\beta}(x)$ and $\varphi^2_{\alpha\beta}(x)$ can be considered as matrix–values functions of orders $n_1 = \dim V_1$ and $n_2 = \dim V_2$, respectively. Both of them should satisfy the conditions (75) from the section 2.

We form a new space $V = V_1 \oplus V_2$. The linear transformation group $\mathbf{GL}\,(V)$ is the group of matrices of order $n = n_1 + n_2$ which can be decomposed into

blocks with respect to decomposition of the space V into the direct sum $V_1 \oplus V_2$. Then the group $\mathbf{GL}\,(V)$ has the subgroup $\mathbf{GL}\,(V_1) \oplus \mathbf{GL}\,(V_2)$ of matrices which have the following form:

$$A = \left\| \begin{array}{cc} A_1 & 0 \\ 0 & A_2 \end{array} \right\| = A_1 \oplus A_2,\ A_1 = \mathbf{GL}\,(V_1),\ A_2 = \mathbf{GL}\,(V_2).$$

Then we can construct new the transition functions

$$\varphi_{\alpha\beta}(x) = \varphi^1_{\alpha\beta}(x) \oplus \varphi^2_{\alpha\beta}(x) = \left\| \begin{array}{cc} \varphi^1_{\alpha\beta}(x) & 0 \\ 0 & \varphi^2_{\alpha\beta}(x) \end{array} \right\|. \tag{93}$$

The transition functions (93) satisfy the conditions (75) from section 2, that is, they define a vector bundle with fiber $V = V_1 \oplus V_2$. The bundle constructed above is called the *direct sum of vector bundles* ξ_1 and ξ_2 and is denoted by $\xi = \xi_1 \oplus \xi_2$. The direct sum operation can be constructed in a geometric way. Namely, let $p_1 : E_1 \longrightarrow B$ be a vector bundle ξ_1 and let $p_2 : E_2 \longrightarrow B$ be a vector bundle ξ_2. Consider the Cartesian product of total spaces $E_1 \times E_2$ and the projection

$$p_3 = p_1 \times p_2 : E_1 \times E_2 \longrightarrow B \times B.$$

It is clear that p is vector bundle with the fiber $V = V_1 \oplus V_2$.

Consider the diagonal $\Delta \subset B \times B$, that is, the subset $\Delta = \{(x, x) : x \in B\}$. The diagonal Δ is canonically homeomorphic to the space B. The restriction of the bundle p_3 to $\Delta \approx B$ is a vector bundle over B. The total space E of this bundle is the subspace $E \subset E_1 \times E_2$ that consists of the vectors (y_1, y_2) such that

$$p_1(y_1) = p_2(y_2).$$

It is easy to check that $\{U_{\alpha_1} \times U_{\alpha_2}\}$ gives an atlas of charts for the bundle p_3.

The transition functions $\varphi_{(\beta_1\beta_2)(\alpha_1\alpha_2)}(x, y)$ on the intersection of two charts $(U_{\alpha_1} \times U_{\alpha_2}) \cap (U_{\beta_1} \times U_{\beta_2})$ have the following form:

$$\varphi_{(\beta_1\beta_2)(\alpha_1\alpha_2)}(x, y) = \left\| \begin{array}{cc} \varphi^1_{\beta_1\alpha_1}(x) & 0 \\ 0 & \varphi^2_{\beta_2\alpha_2}(y) \end{array} \right\|.$$

Hence on the diagonal $\Delta \approx B$ the atlas consists of sets $U_\alpha \approx \Delta \cap (U_\alpha \times U_\alpha)$.

Then the transition functions for the restriction of the bundle p_3 on the diagonal have the following form:

$$\varphi_{(\beta\beta)(\alpha\alpha)}(x, x) = \left\| \begin{array}{cc} \varphi^1_{\beta\alpha}(x) & 0 \\ 0 & \varphi^2_{\beta\alpha}(x) \end{array} \right\|. \tag{94}$$

So the transition functions (94) coincide with the transition functions defined for the direct sum of the bundles ξ_1 and ξ_2. Now let us proceed to the definition of tensor product of vector bundles. As before, let ξ_1 and ξ_2 be two vector bundles with fibers V_1 and V_2 and let $\varphi^1_{\alpha\beta}(x)$ and $\varphi^2_{\alpha\beta}(x)$ be the transition functions of the vector bundles ξ_1 and ξ_2,

$$\varphi^1_{\alpha\beta}(x) \in \mathbf{GL}\,(V_1),\ \varphi^2_{\alpha\beta}(x) \in \mathbf{GL}\,(V_2),\ x \in V_\alpha \cap V_\beta.$$

Let $V = V_1 \otimes V_2$. Then form the tensor product $A_1 \otimes A_2 \in \mathbf{GL}\,(V_1 \otimes V_2)$ of the two matrices $A_1 \in \mathbf{GL}\,(V_1)$, $A_2 \in \mathbf{GL}\,(V_2)$. Put

$$\varphi_{\alpha\beta}(x) = \varphi^1_{\alpha\beta}(x) \otimes \varphi^2_{\alpha\beta}(x).$$

Now we have obtained a family of the matrix value functions $\varphi_{\alpha\beta}(x)$ which satisfy the conditions (75) from the section 2. The corresponding vector bundle ξ with fiber $V = V_1 \otimes V_2$ and transition functions $\varphi_{\alpha\beta}(x)$ will be called *the tensor product* of bundles ξ_1 and ξ_2 and denoted by

$$\xi = \xi_1 \otimes \xi_2.$$

What is common in the construction of the operations of direct sum and operation of tensor product? Both operations can be described as the result of applying the following sequence of operations to the pair of vector bundles ξ_1 and ξ_2:

1. Pass to the principal $\mathbf{GL}\,(V_1)$– and $\mathbf{GL}\,(V_2)$– bundles;
2. Construct the principal ($\mathbf{GL}\,(V_1) \times \mathbf{GL}\,(V_2)$)– bundle over the Cartesian square $B \times B$;
3. Restrict to the diagonal Δ, homeomorphic to the space B.
4. Finally, form a new principal bundle by means of the relevant representations of the structure group $\mathbf{GL}\,(V_1) \times \mathbf{GL}\,(V_2)$ in the groups $\mathbf{GL}\,(V_1 \oplus V_2)$ and $\mathbf{GL}\,(V_1 \otimes V_2)$, respectively.

The difference between the operations of direct sum and tensor product lies in choice of the representation of the group $\mathbf{GL}\,(V_1) \times \mathbf{GL}\,(V_2)$.

By using different representations of the structure groups, further operations of vector bundles can be constructed, and algebraic relations holding for representations induce corresponding algebraic relations vector bundles.

In particular, for the operations of direct sum and tensor product the following well known relations hold:

1. Associativity of the direct sum

$$(\xi_1 \oplus \xi_2) \oplus \xi_3 = \xi_1 \oplus (\xi_2 \oplus \xi_3).$$

This relation is a consequence of the commutative diagram

$$\begin{array}{ccccc}
 & & \mathbf{GL}\,(V_1 \oplus V_2) \times \mathbf{GL}\,(V_3) & & \\
 & & \nearrow \rho_1 \quad \searrow \rho_2 & & \\
\mathbf{GL}\,(V_1) \times \mathbf{GL}\,(V_2) \times & & \mathbf{GL}\,(V_3) & & \mathbf{GL}\,(V_1 \oplus V_2 \oplus V_3) \\
 & & \searrow \rho_3 \quad \nearrow \rho_4 & & \\
 & & \mathbf{GL}\,(V_1) \times \mathbf{GL}\,(V_2 \oplus V_3) & &
\end{array}$$

where

$$\begin{aligned}
\rho_1(A_1, A_2, A_3) &= (A_1 \oplus A_2, A_3), \\
\rho_2(B, A_3) &= B \oplus A_3, \\
\rho_3(A_1, A_2, A_3) &= (A_1, A_2 \oplus A_3), \\
\rho_4(A_1, C) &= A_1 \oplus C.
\end{aligned}$$

Then

$$\begin{aligned}
\rho_2\rho_1(A_1, A_2, A_3) &= (A_1 \oplus A_2) \oplus A_3, \\
\rho_4\rho_3(A_1, A_2, A_3) &= A_1 \oplus (A_2 \oplus A_3).
\end{aligned}$$

It is clear that

$$\rho_2\rho_1 = \rho_4\rho_3$$

since the relation

$$(A_1 \oplus A_2) \oplus A_3 = A_1 \oplus (A_2 \oplus A_3)$$

is true for matrices.

2. Associativity for tensor products

$$(\xi_1 \otimes \xi_2) \otimes \xi_3 = \xi_1 \otimes (\xi_2 \otimes \xi_3).$$

This relation is a consequence of the commutative diagram

$$\begin{array}{ccccc}
 & & \mathbf{GL}\,(V_1 \otimes V_2) \times & \mathbf{GL}\,(V_3) & \\
 & & \nearrow \rho_1 & \searrow \rho_2 & \\
\mathbf{GL}\,(V_1) \times & \mathbf{GL}\,(V_2) \times & \mathbf{GL}\,(V_3) & & \mathbf{GL}\,(V_1 \otimes V_2 \otimes V_3) \\
 & & \searrow \rho_3 & \nearrow \rho_4 & \\
 & & \mathbf{GL}\,(V_1) \times & \mathbf{GL}\,(V_2 \otimes V_3) &
\end{array}$$

The commutativity is implied from the relation

$$(A_1 \otimes A_2) \otimes A_3 = A_1 \otimes (A_2 \otimes A_3)$$

for matrices.

3. Distributivity:

$$(\xi_1 \oplus \xi_2) \otimes \xi_3 = (\xi_1 \otimes \xi_3) \oplus (\xi_2 \otimes \xi_3).$$

This property is implied by the corresponding relation

$$(A_1 \oplus A_2) \otimes A_3 = (A_1 \otimes A_3) \oplus (A_2 \otimes A_3).$$

for matrices.

4. Denote the trivial vector bundle with the fiber $\mathbf{R}^n$ by $\bar{n}$. The total space of trivial bundle is homeomorphic to the Cartesian product $B \times R_n$ and it follows that

$$\bar{n} = \bar{1} \oplus \bar{1} \oplus \ldots \oplus \bar{1}(n \text{ times}).$$

and

$$\xi \otimes \bar{1} = \xi,$$

$$\xi \otimes \bar{n} = \xi \oplus \xi \oplus \ldots \oplus \xi(n \text{ times}).$$

7.3.2 Other operations with vector bundles

Let $V = \mathbf{Hom}\,(V_1, V_2)$ be the vector space of all linear mappings from the space V_1 to the space V_2. For infinite dimensional Banach spaces we will assume that all linear mappings considered are bounded. Then there is a natural representation of the group $\mathbf{GL}\,(V_1) \times \mathbf{GL}\,(V_2)$ into the group $\mathbf{GL}\,(V)$ which to any pair $A_1 \in \mathbf{GL}\,(V_1),\ A_2 \in \mathbf{GL}\,(V_2)$ associates the mapping

$$\rho(A_1, A_2) : \mathbf{Hom}\,(V_1, V_2) \longrightarrow \mathbf{Hom}\,(V_1, V_2)$$

by the formula

$$\rho(A_1, A_2)(f) = A_2 \circ f \circ A_1^{-1}. \tag{95}$$

Then following the general method of constructing operations for vector bundles one obtains for each pair of vector bundles ξ_1 and ξ_2 with fibers V_1 and V_2 and transition functions $\varphi^1_{\alpha\beta}(x)$ and $\varphi^2_{\alpha\beta}(x)$ a new vector bundle with fiber V and transition functions

$$\varphi_{\alpha\beta}(x) = \rho(\varphi^1_{\alpha\beta}(x), \varphi^2_{\alpha\beta}(x)) \tag{96}$$

This bundle is denoted by $\mathbf{HOM}\,(\xi_1, \xi_2)$.

When $V_2 = \mathbf{R}^1$, the space $\mathbf{Hom}\,(V_1, R^1)$ is denoted by V_1^*. Correspondingly, when $\xi_2 = \bar{1}$ the bundle $\mathbf{HOM}\,(\xi, \bar{1})$ will be denoted by ξ^* and called the *dual bundle*. It is easy to check that the bundle ξ^* can be constructed from ξ by means of the representation of the group $\mathbf{GL}\,(V)$ to itself by the formula

$$A \longrightarrow (A^t)^{-1},\ A \in \mathbf{GL}\,(V).$$

There is a bilinear mapping

$$V \times V^* \xrightarrow{\beta} \mathbf{R}^1,$$

which to each pair (x, h) associates the value $h(x)$.

Consider the representation of the group $\mathbf{GL}\,(V)$ on the space $V \times V^*$ defined by matrix

$$A \longrightarrow \begin{Vmatrix} A & 0 \\ 0 & \rho(A, 1) \end{Vmatrix}$$

(see (95)). Then the structure group $\mathbf{GL}\,(V \times V^*)$ of the bundle $\xi \oplus \xi^*$ is reduced to the subgroup $\mathbf{GL}\,(V)$. The action of the group $\mathbf{GL}\,(V)$ on the $V \times V^*$ has the property that the mapping β is equivariant with respect to

trivial action of the group $\mathbf{GL}\,(V)$ on $\mathbf{R}^1$. This fact means that the value of the form h on the vector x does not depend on the choice of the coordinate system in the space V. Hence there exists a continuous mapping

$$\bar{\beta} : \xi \oplus \beta^* \longrightarrow \bar{1},$$

which coincides with β on each fiber.

Let $\Lambda_k(V)$ be the k-th exterior power of the vector space V. Then to each transformation $A : V \longrightarrow V$ is associated the corresponding exterior power of the transformation

$$\Lambda_k(A) : \Lambda_k(V) \longrightarrow \Lambda_k(V),$$

that is, the representation

$$\Lambda_k : \ \mathbf{GL}\,(V) \longrightarrow \mathbf{GL}\,(\Lambda_k(V)).$$

The corresponding operation for vector bundles will be called *the operation of the k-th exterior power* and the result denoted by $\Lambda_k(\xi)$. Similar to vector spaces, for vector bundles one has

$$\Lambda_1(\xi) = \xi,$$

$$\Lambda_k(\xi) = 0 \text{ for } k > \dim \xi,$$

$$\Lambda_k(\xi_1 \oplus \xi_2) = \oplus_{\alpha=0}^{k} \Lambda_\alpha(\xi_1) \otimes \Lambda_{k-\alpha}(\xi_2), \tag{97}$$

where by definition

$$\Lambda_0(\xi) = \bar{1}.$$

It is convenient to write the relation (97) using the partition function. Let us introduce the polynomial

$$\Lambda_t(\xi) = \Lambda_0(\xi) + \Lambda_1(\xi)t + \Lambda_2(\xi)t^2 + \ldots + \Lambda_n(\xi)t^n.$$

Then

$$\Lambda_t(\xi_1 \oplus \xi_2) = \Lambda_t(\xi_1) \otimes \Lambda_t(\xi_2). \tag{98}$$

and the formula (98) should be interpreted as follows: the degrees of the formal variable are added and the coefficients are vector bundles formed using the operations of tensor product and direct sum.

7.3.3 Mappings of vector bundles

Consider two vector bundles ξ_1 and ξ_2 where

$$\xi_i = \{p_i : E_i \longrightarrow B, \ V_i \text{ is fiber}\}.$$

Consider a fiberwise continuous mapping

$$f : E_1 \longrightarrow E_2.$$

The map f will be called a *linear map of vector bundles* or *homomorphism of bundles* if f is linear on each fiber. The family of all such linear mappings will be denoted by $\mathbf{Hom}\ (\xi_1, \xi_2)$. Then the following relation holds:

$$\mathbf{Hom}\ (\xi_1, \xi_2) = \Gamma(\mathbf{HOM}\ (\xi_1, \xi_2)). \tag{99}$$

By intuition, the relation (99) is evident since elements from both the left-hand and right-hand sides are families of linear transformations from the fiber V_1 to the fiber V_2, parametrized by points of the base B.

To prove the relation (99), let us express elements from both the left-hand and right-hand sides of (99) in terms of local coordinates. Consider an atlas $\{U_\alpha\}$ and coordinate homeomorphisms $\varphi^1_\alpha,\ \varphi^2_\alpha$ for bundles $\xi_1,\ \xi_2$. By means of the mapping $f : E_1 \longrightarrow E_2$ we construct a family of mappings:

$$(\varphi^2_\alpha)^{-1} f \varphi^1_\alpha : U_\alpha \times V_1 \longrightarrow U_\alpha \times V_2,$$

defined by the formula:

$$[(\varphi^2_\alpha)^{-1} f \varphi^1_\alpha](x, h) = (x, f_\alpha(x)h),$$

for the continuous family of linear mappings

$$f_\alpha(x) : V_1 \longrightarrow V_2.$$

On the intersection of two charts $U_\alpha \cap U_\beta$ two functions $f_\alpha(x)$ and $f_\beta(x)$ satisfy the following condition

$$\varphi^2_{\beta\alpha}(x) f_\alpha(x) = f_\beta(x) \varphi^1_{\beta\alpha}(x),$$

or

$$f_\beta(x) = \varphi^2_{\beta\alpha}(x) f_\alpha(x) \varphi^1_{\alpha\beta}(x).$$

Taking into account the relations (95), (96) we have

$$f_\beta(x) = \varphi_{\beta\alpha}(x)(f_\alpha(x)). \tag{100}$$

In other words, the family of functions

$$f_\alpha(x) \in V = \mathbf{Hom}\ (V_1, V_2),\ x \in U_\alpha$$

satisfies the condition (100), that is, determines a section of the bundle $\mathbf{HOM}\ (\xi_1, \xi_2)$. Conversely, given a section of the bundle $\mathbf{HOM}\ (\xi_1, \xi_2)$, that is, a family of functions $f_\alpha(x)$ satisfying condition (100) defines a linear mapping from the bundle ξ_1 to the bundle ξ_2. In particular, if

$$\xi_1 = \bar{1},\ V_1 = \mathbf{R}^1$$

then

$$\mathbf{Hom}\ (V_1, V_2) = V_2.$$

Hence

$$\mathbf{HOM}\ (\bar{1}, \xi_2) = \xi_2.$$

Hence

$$\Gamma(\xi_2) = \mathbf{Hom}\ (\bar{1}, \xi_2),$$

that is, the space of all sections of vector bundle ξ_2 is identified with the space of all linear mappings from the one dimensional trivial bundle $\bar{1}$ to the bundle ξ_2.

The second example of mappings of vector bundles gives an analogue of bilinear form on vector bundle. A bilinear form is a mapping

$$V \times V \longrightarrow \mathbf{R}^1,$$

which is linear with respect to each argument. Consider a continuous family of bilinear forms parametrized by points of base. This gives us a definition of bilinear form on vector bundle, namely, a fiberwise continuous mapping

$$f : \xi \oplus \xi \longrightarrow \bar{1} \tag{101}$$

which is bilinear in each fiber and is called a *bilinear form on the bundle* ξ. Just as on a linear space, a bilinear form on a vector bundle (101) induces a linear mapping from the vector bundle ξ to its dual bundle ξ^*

$$\bar{f} : \xi \longrightarrow \xi^*,$$

such that f decomposes into the composition

$$\xi \oplus \xi \xrightarrow{\bar{f} \oplus \mathbf{id}} \xi^* \oplus \xi \xrightarrow{\beta} \bar{1},$$

where

$$\mathbf{id} : \xi \longrightarrow \xi$$

is the identity mapping and

$$\xi \oplus \xi \xrightarrow{\bar{f} \oplus \mathbf{id}} \xi^* \oplus \xi$$

is the direct sum of mappings $\bar{f}$ and **id** on each fiber. When the bilinear form f is symmetric, positive and nondegenerate we say that f is a *scalar product on the bundle* ξ.

Theorem 23 *Let ξ be a finite dimensional vector bundle over a compact base space B. Then there exists a scalar product on the bundle ξ, that is, a nondegenerate, positive, symmetric bilinear form on the ξ.*

Proof.
We must construct a fiberwise mapping (101) which is bilinear, symmetric, positive, nondegenerate form in each fiber. This means that if $x \in B$, $v_1,\ v_2 \in p^{-1}(x)$ then the value $f(v_1, v_2)$ can be identified with a real number such that

$$f(v_1, v_2) = f(v_2, v_1)$$

and

$$f(v,v) > 0 \text{ for any } v \in p^{-1}(x),\ v \neq 0.$$

Consider the weaker condition

$$f(v,v) \geq 0.$$

Then we obtain a nonnegative bilinear form on the bundle ξ. If $f_1,\ f_2$ are two nonnegative bilinear forms on the bundle ξ then the sum $f_1 + f_2$ and a linear combination $\varphi_1 f_1 + \varphi_2 f_2$ for any two nonnegative continuous functions φ_1 and φ_2 on the base B gives a nonnegative bilinear form as well.

Let $\{U_\alpha\}$ be an atlas for the bundle ξ. The restriction $\xi|_{U_\alpha}$ is a trivial bundle and is therefore isomorphic to a Cartesian product $U_\alpha \times V$ where V is fiber of ξ. Therefore the bundle $\xi|_{U_\alpha}$ has a nondegenerate positive definite bilinear form

$$f_\alpha : \xi|_{U_\alpha} \oplus \xi|_{U_\alpha} \longrightarrow \bar{1}.$$

In particular, if $v \in p^{-1}(x),\ x \in U_\alpha$ and $v \neq 0$ then

$$f_\alpha(v,v) > 0.$$

Consider a partition of unity $\{g_\alpha\}$ subordinate to the atlas $\{U_\alpha\}$. Then

$$0 \leq g_\alpha(x) \leq 1,$$

$$\sum_\alpha g_\alpha(x) \equiv 1,$$

$$\textbf{supp } g_\alpha \subset U_\alpha.$$

We extend the form f_α by formula

$$\bar{f}_\alpha(v_1,v_2) = \begin{cases} g_\alpha(x) f_{\alpha(v_1,v_2)} & v_1, v_2 \in p^{-1}(x) \quad x \in U_\alpha, \\ 0 & v_1, v_2 \in p^{-1}(x) \quad x \notin U_\alpha. \end{cases} \tag{102}$$

It is clear that the form (102) defines a continuous nonnegative form on the bundle ξ. Put

$$f(v_1,v_2) = \sum_\alpha f_\alpha(v_1,v_2). \tag{103}$$

The form (103) is then positive definite. Actually, let $0 \neq v \in p^{-1}(x)$. Then there is an index α such that

$$g_\alpha(x) > 0.$$

This means that

$$x \in U_\alpha \text{ and } f_\alpha(v,v) > 0.$$

Hence

$$\bar{f}_\alpha(v,v) > 0$$

and

$$f(v,v) > 0.$$

∎

Theorem 24 *For any vector bundle ξ over a compact base space B with* $\dim \xi = n$*, the structure group* $\mathbf{GL}\,(n, \mathbf{R})$ *reduces to subgroup* $\mathbf{O}(n)$*.*

Proof.
Let us give another geometric interpretation of the property that the bundle ξ is locally trivial. Let U_α be a chart and let

$$\varphi_\alpha : U_\alpha \times V \longrightarrow p^{-1}(U_\alpha)$$

be a trivializing coordinate homeomorphism. Then any vector $v \in V$ defines a section of the bundle ξ over the chart U_α

$$\begin{aligned} \sigma : U_\alpha \longrightarrow p^{-1}(U_\alpha), & \\ \sigma(x) &= \varphi_\alpha(x, v) \in p^{-1}(U_\alpha). \end{aligned}$$

If $v_1, \ldots, v_n$ is a basis for the space V then corresponding sections

$$\sigma_k^\alpha(x) = \varphi_\alpha(x, v_k)$$

form a system of sections such that for each point $x \in U_\alpha$ the family of vectors $\sigma_1^\alpha(x), \ldots, \sigma_n^\alpha(x) \in p^{-1}(x)$ is a basis in the fiber $p^{-1}(x)$.

Conversely, if the system of sections

$$\sigma_1^\alpha, \ldots, \sigma_n^\alpha : U_\alpha \longrightarrow p^{-1}(U_\alpha)$$

forms basis in each fiber then we can recover a trivializing coordinate homeomorphism

$$\varphi_\alpha(x, \sum_i \lambda_i v_i) = \sum_i \lambda_i \sigma_i^\alpha(x) \in p^{-1}(U_\alpha).$$

From this point of view, the transition function $\varphi^{\beta\alpha} = \varphi_\beta^{-1}\varphi_\alpha$ has an interpretation as a change of basis matrix from the basis $\{\sigma_1^\alpha(x), \ldots, \sigma_n^\alpha(x)\}$ to $\{\sigma_1^\beta(x), \ldots, \sigma_n^\beta(x)\}$ in the fiber $p^{-1}(x)$, $x \in U_\alpha \cap U_\beta$. Thus Theorem 24 will be proved if we construct in each chart U_α a system of sections $\{\sigma_1^\alpha, \ldots, \sigma_n^\alpha\}$ which form an orthonormal basis in each fiber with respect to a inner product in the bundle ξ. Then the transition matrices from one basis $\{\sigma_1^\alpha(x), \ldots, \sigma_n^\alpha(x)\}$ to another basis $\{\sigma_1^\beta(x), \ldots, \sigma_n^\beta(x)\}$ will be orthonormal, that is, $\varphi_{\beta\alpha}(x) \in \mathbf{O}(n)$. The proof of Theorem 24 will be completed by the following lemma.

Lemma 11 *Let ξ be a vector bundle, f a scalar product in the bundle ξ and $\{U_\alpha\}$ an atlas for the bundle ξ. Then for any chart U_α there is a system of sections $\{\sigma_1^\alpha, \ldots, \sigma_n^\alpha\}$ orthonormal in each fiber $p^{-1}(x)$, $x \in U_\alpha$.*

Proof.
The proof of the lemma simply repeats the Gramm-Schmidt method of construction of orthonormal basis. Let

$$\tau_1, \ldots, \tau_n : U_\alpha \longrightarrow p^{-1}(U_\alpha)$$

be an arbitrary system of sections forming a basis in each fiber $p^{-1}(x),\ x \in U_\alpha$. Since for any $x \in U_\alpha$,

$$\tau_1(x) \neq 0$$

one has

$$f(\tau_1(x), \tau_1(x)) > 0.$$

Put

$$\tau_1'(x) = \frac{\tau_1(x)}{\sqrt{f(\tau_1(x), \tau_1(x))}}.$$

The new system of sections $\tau_1', \tau_2, \ldots, \tau_n$ forms a basis in each fiber. Put

$$\tau_2''(x) = \tau_2(x) - f(\tau_2(x), \tau_1'(x))\tau_1'(x).$$

The new system of sections $\tau_1', \tau_2'', \tau_3(x), \ldots, \tau_n$ forms a basis in each fiber. The vectors $\tau_1'(x)$ have unit length and are orthogonal to the vectors $\tau_2''(x)$ at each point $x \in U_\alpha$. Put

$$\tau_2'(x) = \frac{\tau_2''(x)}{\sqrt{f(\tau_2''(x), \tau_2''(x))}}.$$

Again, the system of sections $\tau_1', \tau_2', \tau_3(x), \ldots, \tau_n$ forms a basis in each fiber and, moreover, the vectors τ_1', τ_2' are orthonormal.

Then we rebuild the system of sections by induction. Let the sections $\tau_1', \ldots, \tau_k'$,
$\tau_{k+1}(x), \ldots, \tau_n$ form a basis in each fiber and suppose that the sections $\tau_1', \ldots, \tau_k'$ be are orthonormal in each fiber. Put

$$\tau_{k+1}''(x) = \tau_{k+1}(x) - \sum_{i=1}^{k} f(\tau_{k+1}(x), \tau_i'(x))\tau_i'(x),$$

$$\tau_{k+1}'(x) = \frac{\tau_{k+1}''(x)}{\sqrt{f(\tau_{k+1}''(x), \tau_{k+1}''(x))}}.$$

It is easy to check that the system $\tau_1', \ldots, \tau_{k+1}', \tau_{k+2}(x), \ldots, \tau_n$ forms a basis in each fiber and the sections $\tau_1', \ldots, \tau_{k+1}'$ are orthonormal. The lemma is proved by induction. Thus the proof of the Theorem 24 is finished. ∎

7.3.4 Remarks

1. In Lemma 11 we proved a stronger statement: if $\{\tau_1, \ldots, \tau_n\}$ is a system of sections of the bundle ξ in the chart U_α which is a basis in each fiber $p^{-1}(x)$ and if in addition vectors $\{\tau_1, \ldots, \tau_k\}$ are orthonormal then there are sections $\{\tau_{k+1}', \ldots, \tau_n'\}$ such that the system

$$\{\tau_1, \ldots, \tau_k, \tau_{k+1}', \ldots, \tau_n'\}$$

is orthonormal in each fiber. In other words, if a system of orthonormal sections can be extended to basis then it can be extended to orthonormal basis.

2. In theorems 23 and 24 the condition of compactness of the base B can be replaced by the condition of paracompactness. In the latter case we should first choose a locally finite atlas of charts.

7.3.5 Linear transformations of vector bundles

Many properties of linear mappings between vector spaces can be extended to linear mappings or homomorphisms between vector bundles. We shall consider some of them in this section. Fix a vector bundle ξ with the base B equipped with a scalar product. The value of the scalar product on a pair of vectors $v_1,\ v_2 \in p^{-1}(x)$ will be denoted by $\langle v_1, v_2\rangle$.

Consider a homomorphism $f : \xi \longrightarrow \xi$ of the vector bundle ξ to itself. The space of all such homomorphisms $\mathbf{Hom}\ (\xi, \xi)$ has the natural operations of summation and multiplication by continuous functions. Thus the space $\mathbf{Hom}\ (\xi, \xi)$ is a module over the algebra $C(B)$ of continuous functions on the base B. Further, the operation of composition equips the space $\mathbf{Hom}\ (\xi, \xi)$ with the structure of algebra. Using a scalar product in the bundle ξ one can introduce a norm in the algebra $\mathbf{Hom}\ (\xi, \xi)$ and hence equip it with a structure of a Banach algebra: for each homomorphism $f : \xi \longrightarrow \xi$ put

$$\|f\| = \sup_{v \neq 0} \frac{\|f(v)\|}{\|v\|}, \tag{104}$$

where

$$\|v\| = \sqrt{\langle v, v\rangle}.$$

The space $\mathbf{Hom}\ (\xi, \xi)$ is complete with respect to the norm (104). Indeed, if a sequence of homomorphisms $f_n : \xi \longrightarrow \xi$ is a Cauchy sequence with respect to this norm, that is,

$$\lim_{n,m \longrightarrow \infty} \|f_n - f_m\| = 0$$

then for any fixed vector $v \in p^{-1}(x)$ the sequence $f_n(v) \in p^{-1}(x)$ is a Cauchy sequence as well since

$$\|f_n(v) - f_m(v)\| \leq \|f_n - f_m\| \cdot \|v\|.$$

Hence there exists a limit

$$f(v) = \lim_{n \longrightarrow \infty} f_n(v).$$

The mapping $f : \xi \longrightarrow \xi$ is evidently linear. To show its continuity one should consider the mappings

$$h_{n,\alpha} = \varphi_\alpha^{-1} f_n \varphi_\alpha : U_\alpha \times V \longrightarrow U_\alpha \times V,$$

which are defined by the matrix valued functions on the U_α. The coefficients of these matrices give Cauchy sequences in the uniform norm and therefore the limit values are continuous functions.

This means that f is continuous. The scalar product in the vector bundle ξ defines an adjoint linear mapping f^* by the formula

$$\langle f^*(v_1), v_2\rangle = \langle v_1, f(v_2)\rangle,\ v_1, v_2 \in p^{-1}(x).$$

The proof of existence and continuity of the homomorphism f^* is left to reader as an exercise.

7.4 Complex bundles

The identity homomorphism of the bundle ξ to itself will be denoted by 1. Consider a homomorphism

$$I : \xi \longrightarrow \xi$$

which satisfies the condition

$$I^2 = -1.$$

The restriction

$$I_x = I_{|p^{-1}(x)}$$

is an automorphism of the fiber $p^{-1}(x)$ with the property

$$I_x^2 = -1.$$

Hence the transformation I_x defines a structure of a complex vector space on $p^{-1}(x)$.

In particular, one has

$$\dim V = 2n.$$

Let us show that in this case the structure group $\mathbf{GL}\,(2n, \mathbf{R})$ is reduced to the subgroup of complex transformations $\mathbf{GL}\,(n, \mathbf{C})$.

First of all notice that if the system of vectors $\{v_1, \ldots, v_n\}$ has the property that the system $\{v_1, \ldots, v_n, Iv_1, \ldots, Iv_n\}$ is a real basis in the space V, $\dim V = 2n$, then the system $\{v_1, \ldots, v_n\}$ is a complex basis of V. Fix a point x_0. The space $p^{-1}(x)$ is a complex vector space with respect to the operator I_{x_0}, and hence there is a complex basis $\{v_1, \ldots, v_n\} \subset p^{-1}(x)$.

Let $U_\alpha \ni x_0$ be a chart for the bundle ξ. There are sections $\{\tau_1, \ldots, \tau_n\}$ in the chart U_α such that

$$\tau_k(x_0) = v_k, 1 \leq k \leq n.$$

Then the system of sections $\{\tau_1, \ldots, \tau_n, I\tau_1, \ldots, I\tau_n\}$ forms a basis in the fiber $p^{-1}(x_0)$ and therefore forms basis in each fiber $p^{-1}(x)$ in sufficiently small neighborhood $U \ni x_0$. Hence the system $\{\tau_1(x), \ldots, \tau_n(x)\}$ forms a complex basis in each fiber $p^{-1}(x)$ in the neighborhood $U \ni x_0$. This means that there is a sufficient fine atlas $\{U_\alpha\}$ and a system of sections $\{\tau_1^\alpha(x), \ldots, \tau_n^\alpha(x)\}$ on each chart U_α giving a complex basis in each fiber $p^{-1}(x)$ in the neighborhood $U_\alpha \ni x_0$. Fix a complex basis $\{e_1, \ldots, e_n\}$ in the complex vector space $\mathbf{C}^n$. Put

$$\begin{aligned}
\varphi_\alpha : U_\alpha \times \mathbf{C}^n &\longrightarrow p^{-1}(U_\alpha), \\
\varphi_\alpha(x, \sum_{k=1}^n z_k e_k) &= \sum_{k=1}^n u_k \tau_k^\alpha(x) + \sum_{k=1}^n v_k I_x(\tau_k^\alpha(x)) \\
&= \sum_{k=1}^n z_k \tau_k^\alpha(x), \\
z_k &= u_k + iv_k,\ 1 \leq k \leq n.
\end{aligned}$$

Then the transition functions $\varphi_{\beta\alpha} = \varphi_\beta^{-1}\varphi_\alpha$ are determined by the transition matrices from the complex basis $\{\tau_1^\alpha(x), \ldots, \tau_n^\alpha(x)\}$ to the complex basis

$\{\tau_1^\beta(x), \ldots, \tau_n^\beta(x)\}$. These matrices are complex, that is, belong to the group $\mathbf{GL}\,(n, \mathbf{C})$. A vector bundle with the structure group $\mathbf{GL}\,(n, \mathbf{C})$ is called a *complex vector bundle*.

Let ξ be a real vector bundle. In the vector bundle $\xi \oplus \xi$, introduce the structure of a complex vector bundle by means the homomorphism

$$I : \xi \oplus \xi \longrightarrow \xi \oplus \xi$$

given by

$$I(v_1, v_2) = (-v_2, v_1),\ v_1, v_2 \in p^{-1}(x). \tag{105}$$

The complex vector bundle defined by (105) is called the *complexification of the bundle* ξ and is denoted by $c\xi$. Conversely, forgetting of the structure of complex bundle on the complex vector bundle ξ turns it into a real vector bundle. This operation is called the *realification of a complex vector bundle* ξ and is denoted by $r\xi$. It is clear that

$$rc\xi = \xi \oplus \xi.$$

The operations described above correspond to natural representations of groups:

$$c:\ \mathbf{GL}\,(n, \mathbf{R}) \longrightarrow \mathbf{GL}\,(n, \mathbf{C}),$$

$$r:\ \mathbf{GL}\,(n, \mathbf{C}) \longrightarrow \mathbf{GL}\,(2n, \mathbf{R}).$$

Let us clarify the structure of the bundle $cr\xi$. If ξ is a complex bundle, that is, ξ is a real vector bundle with a homomorphism $I : \xi \longrightarrow \xi$ giving the structure of complex bundle on it. By definition the vector bundle $cr\xi$ is a new real vector bundle $\eta = \xi \oplus \xi$ with the structure of complex vector bundle defined by a homomorphism

$$\begin{aligned} I_1 : \xi \oplus \xi &\longrightarrow \xi \oplus \xi, \\ I_1(v_1, v_2) &= (-v_2, v_1). \end{aligned} \tag{106}$$

The mapping (106) defines a new complex structure in the vector bundle η which is, in general, different from the complex structure defined by the mapping I.

Let us split the bundle η in another way:

$$\begin{aligned} f : \xi \oplus \xi &\longrightarrow \xi \oplus \xi, \\ f(v_1, v_2) &= (I(v_1 + v_2), v_1 - v_2). \end{aligned}$$

and define in the inverse image a new homomorphism

$$I_2(v_1, v_2) = (Iv_1, -Iv_2).$$

Then

$$fI_2 = I_1 f \tag{107}$$

because

$$fI_2(v_1, v_2) = f(Iv_1, -Iv_2) = (v_2 - v_1, I(v_1 + v_2)), \tag{108}$$

$$I_1 f(v_1, v_2) = I_1(I(v_1 + v_2), v_1 - v_2) = (v_2 - v_1, I(v_1 + v_2)). \tag{109}$$

Comparing (109) and (108) we obtain (107). Thus the mapping f gives an isomorphism of the bundle $cr\xi$ (in the image) with the sum of two complex vector bundles: the first is ξ and the second summand is homeomorphic to ξ but with different complex structure defined by the mapping I',

$$I'(v) = -I(v).$$

This new complex structure on the bundle ξ is denoted by $\bar{\xi}$. The vector bundle $\bar{\xi}$ is called the *complex conjugate of the complex bundle* ξ. Note that the vector bundles ξ and $\bar{\xi}$ are isomorphic as real vector bundles, that is, the isomorphism is compatible with respect to the large structure group $\mathbf{GL}\,(2n, \mathbf{R})$ but not isomorphic with respect to the structure group $\mathbf{GL}\,(n, \mathbf{C})$.

Thus we have the formula

$$cr\xi = \xi \oplus \bar{\xi}.$$

The next proposition gives a description of the complex conjugate vector bundle in term of the transition functions.

Proposition 4 *Let the*

$$\varphi_{\beta\alpha} : U_{\alpha\beta} \longrightarrow \mathbf{GL}\,(n, \mathbf{C})$$

be transition functions of a complex vector bundle ξ. Then the complex conjugate vector bundle $\bar{\xi}$ is defined by the complex conjugate matrices $\bar{\varphi}_{\beta\alpha}$.

Proposition 5 *A complex vector bundle ξ has the form $\xi = c\eta$ if and only if there is a real linear mapping*

$$* : \xi \longrightarrow \xi \tag{110}$$

such that

$$*^2 = 1, \ *I = -I*, \tag{111}$$

where I is the multiplication by the imaginary unit.

7.5 Subbundles

Let $f : \xi_1 \longrightarrow \xi_2$ be a homomorphism of vector bundles with a common base B and assume that the fiberwise mappings $f_x : (\xi_1)_x \longrightarrow (\xi_2)_x$ have constant rank. Let

$$p_1 : E_1 \longrightarrow B$$
$$p_2 : E_2 \longrightarrow B$$

be the projections of the vector bundles ξ_1 and ξ_2. Put

$$\begin{aligned} E_0 &= \{y \in E_1 : f(y) = 0 \in p_2^{-1}(x), \ x = p_1(y)\} \\ E &= f(E_1). \end{aligned}$$

Theorem 25 *1. The mapping*

$$p_0 = p_1|E_0 : E_0 \longrightarrow B$$

is a locally trivial bundle which admits a unique vector bundle structure ξ_0 such that natural inclusion $E_0 \subset E_1$ is a homomorphism of vector bundles.

2. The mapping

$$p = p_2|E : E \longrightarrow B \tag{112}$$

is a locally trivial bundle which admits a unique vector bundle structure ξ such that the inclusion $E \subset E_2$ and the mapping $f : E_1 \longrightarrow E$ are homomorphisms of vector bundles.

3. There exist isomorphisms

$$\begin{array}{lcl} \varphi : \xi_1 & \longrightarrow & \xi_0 \oplus \xi, \\ \psi : \xi_2 & \longrightarrow & \xi \oplus \eta, \end{array}$$

such that composition

$$\psi \circ f \circ \varphi^{-1} : \xi_0 \oplus \xi \longrightarrow \xi \oplus \eta$$

has the matrix form

$$\psi \circ f \circ \varphi^{-1} = \begin{pmatrix} 0 & \mathbf{id} \\ 0 & 0 \end{pmatrix}.$$

The bundle ξ_0 is called the *kernel* of the mapping f and denoted $\mathbf{Ker}\ f$, the bundle ξ is called the *image* of the mapping f and denoted $\mathbf{Im}\ f$. So we have

$$\dim \xi_1 = \dim \mathbf{Ker}\ f + \dim \mathbf{Im}\ f.$$

Theorem 26 *Let ξ be a vector bundle over a compact base B. Then there is a vector bundle η over B such that*

$$\xi \oplus \eta = \bar{N} = \text{a trivial bundle.}$$

Proof.
Let us use Theorem 25. It is sufficient to construct a homomorphism

$$f : \xi \longrightarrow \bar{N},$$

where the rank of f equals $\dim \xi$ in each fiber. Notice that if ξ is trivial then such an f exists. Hence for any chart U_α there is a homomorphism

$$f_\alpha : \xi|U_\alpha \longrightarrow \bar{N}_\alpha,\ \mathrm{rank} f_\alpha = \dim \xi.$$

Let $\{\varphi_\alpha\}$ be a partition of unity subordinate to the atlas $\{U_\alpha\}$. Then each mapping $\varphi_\alpha f_\alpha$ can be extended by the zero trivial mapping to a mapping

$$g_\alpha : \xi \longrightarrow \bar{N}_\alpha,\ g_\alpha|U_\alpha = \varphi_\alpha f_\alpha.$$

The mapping g_α has the following property: if $\varphi_\alpha(x) \neq 0$ then $\mathrm{rank} g_\alpha|_x = \dim \xi$. Let

$$\begin{array}{rcl} g : \xi & \longrightarrow & \oplus_\alpha \bar{N}_\alpha = \bar{N}, \\ g & = & \oplus_\alpha g_\alpha \text{ that is} \\ g(y) & = & (g_1(y), \ldots, g_\alpha(y), \ldots). \end{array}$$

It is clear that the rank of g at each point satisfies the relation

$$\mathrm{rank} g_\alpha \leq \mathrm{rank} g \leq \dim \xi.$$

Further, for each point $x \in B$ there is such an α that $\varphi_\alpha(x) \neq 0$. Hence,

$$\mathrm{rank} g \equiv \dim \xi. \tag{113}$$

Therefore we can apply Theorem 25. By (113) we get

$$\mathbf{Ker}\ g = 0.$$

Hence the bundle ξ is isomorphic to $\mathbf{Im}\ g$ and $\bar{N} = \mathbf{Im}\ g \oplus \eta$. ∎

Theorem 27 *Let ξ_1, ξ_2 be two vector bundles over a base B and let $B_0 \subset B$ a closed subspace. Let*

$$f_0 : \xi_1|B_0 \longrightarrow \xi_2|B_0$$

be a homomorphism of the restrictions of the bundles to the subspace B_0. Then the mapping f_0 can be extended to a homomorphism

$$f : \xi_1 \longrightarrow \xi_2,\ f|B_0 = f_0.$$

7.6 Vector bundles related to manifolds

The most natural vector bundles arise from the theory of smooth manifolds. Recall that by an *n–dimensional manifold* one means a metrizable space X such that for each point $x \in X$ there is an open neighborhood $U \ni x$ homeomorphic to an open subset V of n–dimensional linear space $\mathbf{R}^n$. A homeomorphism

$$\varphi : U \longrightarrow V \subset \mathbf{R}^n$$

is called a *coordinate homeomorphism*. The coordinate functions on the linear space $\mathbf{R}^n$ pulled back to points of the neighborhood U, that is, the compositions

$$x^j = x^j \circ \varphi : U \longrightarrow \mathbf{R}^1$$

are called *coordinate functions on the manifold X in the neighborhood U*. This system of functions $\{x^1, \ldots, x^n\}$ defined on the neighborhood U is called a *local system of coordinates of the manifold X*. The open set U equipped with the

local system of coordinates $\{x^1, \ldots, x^n\}$ is called a *chart*. The system of charts $\{U_\alpha, \{x^1_\alpha, \ldots, x^n_\alpha\}\}$ is called an *atlas* if $\{U_\alpha\}$ covers the manifold X, that is,

$$X = \cup_\alpha U_\alpha.$$

So each n–dimensional manifold has an atlas. If a point $x \in X$ belongs to two charts,

$$x \in U_\alpha \cap U_\beta,$$

then in a neighborhood of x there are two local systems of coordinates. In this case, the local coordinates x^j_α can expressed as functions of values of the local coordinates $\{x^1_\beta, \ldots, x^n_\beta\}$, that is, there are functions $f^k_{\alpha\beta}$ such that

$$x^k_\alpha = \varphi^k_{\alpha\beta}(x^1_\beta, \ldots, x^n_\beta). \tag{114}$$

The system of functions (114) are called a *change of coordinates* or *transition functions from one local coordinate system to another*. For brevity (114) will be written as

$$x^k_\alpha = x^k_\alpha(x^1_\beta, \ldots, x^n_\beta).$$

If an atlas $\{U_\alpha, \{x^1_\alpha, \ldots, x^n_\alpha\}\}$ is taken such that all the transition functions are differentiable functions of the class $\mathcal{C}^k$, $1 \leq k \leq \infty$ then one says X has the *structure of differentiable manifold of the class* $\mathcal{C}^k$. If all the transition functions are analytic functions then one says that X has the *structure of an analytic manifold*. In the case $n = 2m$ one has

$$\begin{cases} u^k_\alpha &= x^k_\alpha,\ 1 \leq k \leq m, \\ v^k_\alpha &= x^{m+k}_\alpha,\ 1 \leq k \leq m, \\ z^k_\alpha &= u^k_\alpha + iv^k_\alpha,\ 1 \leq k \leq m, \end{cases} \tag{115}$$

and the functions

$$z^k_\alpha = \varphi^k_{\alpha\beta}(z^1_\beta, \ldots, z^m_\beta) + i\varphi^{m+k}_{\alpha\beta}(z^1_\beta, \ldots, z^m_\beta)$$

are complex analytic functions in their domain of definition then X has the *structure of a complex analytic manifold*. Usually we shall consider infinitely smooth manifolds, that is, differentiable manifolds of the class $\mathcal{C}^\infty$.

A mapping $f : X \longrightarrow Y$ of differentiable manifolds is called *differentiable of class* $\mathcal{C}^k$ if in any neighborhood of the point $x \in X$ the functions which express the coordinates of the image $f(x)$ in terms of coordinates of the point x are differentiable functions of the class $\mathcal{C}^k$. It is clear that the class of differentiability k makes sense provided that k does not exceed the differentiability classes of the manifolds X and Y. Similarly, one can define analytic and complex analytic mappings.

Let X be a n–dimensional manifold, $\{U_\alpha, \{x^1_\alpha, \ldots, x^n_\alpha\}\}$ be an atlas. Fix a point $x_0 \in X$. A *tangent vector* ξ to the manifold X at the point x_0 is a system of numbers $(\xi^1_\alpha, \ldots, \xi^n_\alpha)$ satisfying the relations:

$$\xi^k_\alpha = \sum_{j=1}^n \xi^j_\beta \frac{\partial x^k_\alpha}{\partial x^j_\beta}(x_0). \tag{116}$$

The numbers $(\xi_\alpha^1, \ldots, \xi_\alpha^n)$ are called the *coordinates* or *components of the vector*ξ *with respect to the chart* $\{U_\alpha, \{x_\alpha^1, \ldots, x_\alpha^n\}\}$. The formula (116) gives the transformation law of the components of the tangent vector ξ under the transition from one chart to another. In differential geometry such a law is called a *tensor law of transformation of components of a tensor of the valency* $(1,0)$. So in terms of differential geometry, a tangent vector is a tensor of the valency $(1,0)$. Consider a smooth curve γ passing through a point x_0, that is, a smooth mapping of the interval

$$\gamma : (-1,1) \longrightarrow X,\ \gamma(0) = x_0.$$

In terms of a local coordinate system $\{x_\alpha^1, \ldots, x_\alpha^n\}$ the curve γ is determined by a family of smooth functions

$$x_\alpha^k(t) = x_\alpha^k(\gamma(t)),\ t \in (-1,1).$$

Let

$$\xi_\alpha^k = \frac{\partial}{\partial t}(x_\alpha^k(t))|_{t=0}. \tag{117}$$

Clearly the numbers (117) satisfy a tensor law (116), that is, they define a tangent vector ξ at the point x_0 to manifold X. This vector is called the *tangent vector to the curve* γ and is denoted by $\frac{d\gamma}{dt}(0)$, that is,

$$\xi = \frac{d\gamma}{dt}(0).$$

The family of all tangent vectors to manifold X is a denoted by TX. The set TX is endowed with a natural topology. Namely, a neighborhood V of the vector ξ_0 at the point x_0 contains all vectors η in points x such that $x \in U_\alpha$ for some chart U_α and for some ε,

$$\begin{aligned} \rho(x, x_0) &< \varepsilon, \\ \sum_{k=1}^n (\xi_{0\alpha}^k - \eta_\alpha^k)^2 &< \varepsilon. \end{aligned}$$

The verification that the system of the neighborhoods V forms a base of a topology is left to the reader.

Let

$$\pi : TX \longrightarrow X \tag{118}$$

be the mapping which to any vector ξ associates its point x of tangency. Clearly, the mapping π is continuous. Moreover, the mapping (118) defines a locally trivial vector bundle with the base X, total space TX and fiber isomorphic to the linear space $\mathbf{R}^n$. If U_α is a chart on the manifold X, we define a homeomorphism

$$\varphi_\alpha : U_\alpha \times \mathbf{R}^n \longrightarrow \pi^{-1}(U_\alpha)$$

which to the system $(x_0, \xi^1, \ldots, \xi^n)$ associates the tangent vector ξ whose components are defined by the formula

$$\xi^k_\beta = \sum_{j=1}^n \xi^j \frac{\partial x^k_\beta}{\partial x^j_\alpha}(x_0). \tag{119}$$

It is easy to check that the definition (119) gives the components of a vector ξ, that is, they satisfy the tensor law for the transformation of components of a tangent vector (116). The inverse mapping is defined by the following formula:

$$\varphi_\alpha^{-1}(\xi) = (\pi(\xi), \xi^1_\alpha, \ldots, \xi^n_\alpha),$$

where ξ^k_α are components of the vector ξ. Therefore the transition functions $\varphi_{\beta\alpha} = \varphi_\beta^{-1}\varphi_\alpha$ are determined by the formula

$$\varphi_{\beta\alpha}(x_0, \xi^1, \ldots, \xi^n) = \left(x_0, \sum \xi^j \frac{\partial x^1_\beta}{\partial x^j_\alpha}(x_0), \ldots, \sum \xi^j \frac{\partial x^n_\beta}{\partial x^j_\alpha}(x_0)\right). \tag{120}$$

Formula (120) shows that the transition functions are fiberwise linear mappings. Hence the mapping π defines a vector bundle. The vector bundle $\pi : TX \longrightarrow X$ is called the *tangent bundle of the manifold* X. The fiber T_xX is called the *tangent space at the point* x *to manifold* X.

The terminology described above is justified by the following. Let $f : X \longrightarrow \mathbf{R}^N$ be an inclusion of the manifold X in the Euclidean space $\mathbf{R}^N$. In a local coordinate system $(x^1_\alpha, \ldots, x^n_\alpha)$ in a neighborhood of the point $x_{0\in X}$, the inclusion f is determined as a vector valued function of the variables $(x^1_\alpha, \ldots, x^n_\alpha)$:

$$f(x) = f(x^1_\alpha, \ldots, x^n_\alpha). \tag{121}$$

If we expand the function (121) by a Taylor expansion at the point $x_0 = (x^1_{0\alpha}, \ldots, x^n_{0\alpha})$:

$$\begin{aligned} f(x^1_\alpha, \ldots, x^n_\alpha) &= f(x^1_{0\alpha}, \ldots, x^n_{0\alpha}) + \\ &+ \sum_{j=1}^n \frac{\partial f}{\partial x^j_\alpha}(x^1_{0\alpha}, \ldots, x^n_{0\alpha})\Delta x^j_\alpha + o(\Delta x^k_\alpha). \end{aligned}$$

Ignoring the remainder term $o(\Delta x^k_\alpha)$ we obtain a function g which is close to f :

$$\begin{aligned} g(x^1_\alpha, \ldots, x^n_\alpha) &= f(x^1_{0\alpha}, \ldots, x^n_{0\alpha}) + \\ &+ \sum_{j=1}^n \frac{\partial f}{\partial x^j_\alpha}(x^1_{0\alpha}, \ldots, x^n_{0\alpha})\Delta x^j_\alpha. \end{aligned}$$

Then if the vectors

$$\left\{\frac{\partial f}{\partial x^j_\alpha}(x^1_{0\alpha}, \ldots, x^n_{0\alpha})\Delta x^j_\alpha\right\},\ 1 \le k \le n$$

are linearly independent the function g defines a linear n–dimensional subspace in $\mathbf{R}^n$. It is natural to call this space the *tangent space to the manifold* X. Any vector ξ which lies in the tangent space to manifold X (having the initial point at x_0) can be uniquely decomposed into a linear combination of the basis vectors:

$$\xi = \sum_{j=1}^{n} \frac{\partial f}{\partial x_\alpha^j} \xi_\alpha^j. \tag{122}$$

The coordinate s $\{\xi_\alpha^k\}$ under a change of coordinate system change with respect to the law (116), that is, with respect to the tensor law. Thus the abstract definition of tangent vector as a system of components $\{\xi_\alpha^k\}$ determines by the formula (122) a tangent vector to the submanifold X in $\mathbf{R}^N$. Let

$$f : X \longrightarrow Y$$

be a differentiable mapping of manifolds. Let us construct the corresponding homomorphism of tangent bundles,

$$Df : TX \longrightarrow TY.$$

Let $\xi \in TX$ be a tangent vector at the point x_0 and let γ be a smooth curve which goes through the point x_0,

$$\gamma(0) = x_0,$$

and which has tangent vector ξ, that is,

$$\xi = \frac{d\gamma}{dt}(0).$$

Then the curve $f(\gamma(t))$ in the manifold Y goes through the point $y_0 = f(x_0)$. Put

$$Df(\xi) = \frac{d(f(\gamma))}{dt}(0). \tag{123}$$

This formula (123) defines a mapping of tangent spaces. It remains to prove that this mapping is fiberwise linear. For this, it is sufficient to describe the mapping Df in terms of coordinates of the spaces $T_{x_0}X$ and $T_{y_0}Y$. Let $\{x_\alpha^1, \ldots, x_\alpha^n\}$ and $\{y_\beta^1, \ldots, y_\beta^m\}$ be local systems of coordinates in neighborhoods of points the x_0 and y_0, respectively . Then the mapping f is defined as a family of functions

$$y_\beta^j = y_\beta^j(x_\alpha^1, \ldots, x_\alpha^n).$$

If

$$x_\alpha^k = x_\alpha^k(t)$$

are the functions defining the curve $\gamma(t)$ then

$$\xi_\alpha^k = \frac{dx_\alpha^k}{dt}(0).$$

Hence the curve $f(\gamma(t))$ is defined by the functions

$$y^j_\beta = y^j_\beta(x^1_\alpha(t), \dots, x^n_\alpha(t))$$

and the vector $Df(\xi)$ is defined by the components

$$\eta^j = \frac{dy^j_\beta}{dt}(0) = \sum_{k=1}^n \frac{\partial y^j_\beta}{\partial x^k_\alpha}(x_0)\frac{dx^k_\alpha}{dt}(0) = \sum_{k=1}^n \frac{\partial y^j_\beta}{\partial x^k_\alpha}(x_0)\xi^k. \tag{124}$$

Formula (124) shows firstly that the mapping Df is well- defined since the definition does not depend on the choice of curve γ but only on the tangent vector at the point x_0. Secondly, the mapping Df is fiberwise linear. The mapping Df is called the *differential of the mapping* f.

7.6.1 Examples

1. Let us show that the definition of differential Df of the mapping f is a generalization of the notion of the classical differential of function. A differentiable function of one variable may be considered as a mapping of the space $\mathbf{R}^1$ into itself:

$$f : \mathbf{R}^1 \longrightarrow \mathbf{R}^1.$$

The tangent bundle of the manifold $\mathbf{R}^1$ is isomorphic to the Cartesian product $\mathbf{R}^1 \times \mathbf{R}^1 = \mathbf{R}^2$. Hence the differential

$$Df : \mathbf{R}^1 \times \mathbf{R}^1 \longrightarrow \mathbf{R}^1 \times \mathbf{R}^1$$

in the coordinates (x, ξ) is defined by the formula

$$Df(x, \xi) = (x, f'(x)\xi).$$

The classical differential has the form

$$df = f'(x)dx.$$

So

$$Df(x, dx) = (x, df).$$

Consider a smooth manifold Y and a submanifold $X \subset Y$. The inclusion

$$j : X \subset Y$$

is a smooth mapping of manifolds such that the differential

$$Dj : TX \longrightarrow TY$$

is a fiberwise monomorphism. Then over the manifold X there are two vector bundles: the first is $j^*(TY)$, the restriction of the tangent bundle of the manifold Y to the submanifold X, the second is its subbundle TX.

According to Theorem 25, the bundle $j^*(TY)$ splits into a direct sum of two summands:

$$j^*(TY) = TX \oplus \eta.$$

The complement η is called the *normal bundle to the submanifold X of the manifold Y* [1]. Each fiber of the bundle η over the point x_0 consists of those tangent vectors to the manifold Y which are orthogonal to the tangent space $T_{x_0}(X)$. The normal bundle will be denoted by $\nu(X \subset Y)$ or more briefly by $\nu(X)$.

It is clear that the notion of a normal bundle can be defined not only for submanifolds but for any immersion $j : X \longrightarrow Y$ of the manifold X into the manifold Y. It is known that any compact manifold X has an inclusion in a Euclidean space $\mathbf{R}^N$ for some sufficiently large number N. Let $j : X \longrightarrow \mathbf{R}^N$ be such an inclusion. Then

$$j^*(T\mathbf{R}^N) = TX \oplus \nu(X \subset \mathbf{R}^N).$$

The bundle $T\mathbf{R}^N$ is trivial and so

$$TX \oplus \nu(X) = \bar{N}. \tag{125}$$

In this case the bundle $\nu(X)$ is called *the normal bundle for manifold X* (irrespective of the inclusion). Note, the normal bundle $\nu(X)$ of the manifold X is not uniquely defined. It depends on inclusion into the space $\mathbf{R}^N$ and on the dimension N. But the equation (125) shows that the bundle is not very far from being unique.

Let $\nu_1(X)$ be another such normal bundle, that is,

$$TX \oplus \nu_1(X) = \bar{N}_1.$$

Then

$$\nu(X) \oplus T(X) \oplus \nu_1(X) = \nu(X) \oplus \bar{N}_1 = \bar{N} \oplus \nu_1(X).$$

The last relation means that bundles $\nu(X)$ and $\nu_1(X)$ became isomorphic after the addition of trivial summands.

2. Let us study the tangent bundle of the one dimensional manifold $\mathbf{S}^1$, the circle. Define two charts on the $\mathbf{S}^1$:

$$U_1 = \{-\pi < \varphi < \pi\},$$
$$U_2 = \{0 < \varphi < 2\pi\},$$

where φ is angular parameter in the polar system of coordinates on the plane. On the U_1 take the function

$$x_1 = \varphi, \ -\pi < x_1 < \pi,$$

[1] By construction the bundle η depends on the choice of metric on TY. However, different metrics yield isomorphic complementary summands

as coordinate, whereas on the U_2 take the function

$$x_2 = \varphi,\ 0 < x_2 < 2\pi.$$

The intersection $U_1 \cap U_2$ consists of the two connected components

$$V_1 = \{0 < \varphi < \pi\},$$

$$V_2 = \{\pi < \varphi < 2\pi\}.$$

Then the transition function has the form

$$x_1 = x_1(x_2) = \begin{cases} x_2, & 0 < x_2 < \pi, \\ x_2 - 2\pi, & \pi < x_2 < 2\pi. \end{cases}$$

Then by (120), the transition function φ_{12} for the tangent bundle has the form

$$\varphi_{12}(x, \xi) = \xi \frac{\partial x_1}{\partial x_2} = \xi.$$

This means that the transition function is the identity. Hence the tangent bundle $T\mathbf{S}^1$ is isomorphic to Cartesian product

$$T\mathbf{S}^1 = \mathbf{S}^1 \times \mathbf{R}^1,$$

in other words, it is the trivial one dimensional bundle.

3. Consider the two-dimensional sphere $\mathbf{S}^2$. It is convenient to consider it as the extended complex plane

$$\mathbf{S}^2 = \mathbf{C}^1 \cup \{\infty\}.$$

We define two charts on the $\mathbf{S}^2$

$$\begin{aligned} U_1 &= \mathbf{C}^1 \\ U_2 &= (\mathbf{C}^1 \backslash \{0\}) \cup \{\infty\}. \end{aligned}$$

Define the complex coordinate $z_1 = z$ on the chart U_1 and $z_2 = \frac{1}{z}$ on the chart U_2 extended by zero at the infinity ∞. Then the transition function on the intersection $U_1 \cap U_2$ has the form

$$z_1 \equiv \frac{1}{z_2}$$

and the tangent bundle has the corresponding transition function of the form

$$\varphi_{12}(z, \xi) = \xi \frac{\partial z_1}{\partial z_2} = -\xi \frac{1}{z_2^2} = -\xi z^2.$$

The real form of the matrix φ_{12} is given by

$$\varphi_{12}(x, y) = \left\| \begin{array}{rr} -\Re z^2 & -\Im z_2 \\ \Im x_2 & -\Re z_2 \end{array} \right\| = \left\| \begin{array}{rr} y_2 - x_2 & -2xy \\ 2xy & y_2 - x_2 \end{array} \right\|,$$

where $z = x + iy$. In polar coordinates $z = \rho e^{i\alpha}$ this becomes

$$\varphi_{12}(\rho, \alpha) = \rho^2 \left\| \begin{array}{cc} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{array} \right\|.$$

Let us show that the tangent bundle $T\mathbf{S}^2$ is not isomorphic to a trivial bundle. If the bundle $T\mathbf{S}^2$ were trivial then there would be matrix valued functions

$$\begin{array}{rcl} h_1 : U_1 & \longrightarrow & \mathbf{GL}\ (2, \mathbf{R}) \\ h_2 : U_2 & \longrightarrow & \mathbf{GL}\ (2, \mathbf{R}), \end{array} \tag{126}$$

such that

$$\varphi_{12}(\rho, \alpha) = h_1(\rho, \alpha) h_2^{-1}(\rho, \alpha).$$

The charts U_1, U_2 are contractible and so the functions h_1, h_2 are homotopic to constant mappings.

Hence the transition function $\varphi_{12}(\rho, \alpha)$ must be homotopic to a constant function. On the other hand, for fixed ρ the function φ_{12} defines a mapping of the circle $\mathbf{S}^1$ with the parameter α into the group $\mathbf{SO}(2) = \mathbf{S}^1$ and this mapping has the degree 2. Therefore this mapping cannot be homotopic to a constant mapping.

4. Consider a vector bundle $p : E \longrightarrow X$ where the base X is a smooth manifold. Assume that the transition functions

$$\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathbf{GL}\ (n, \mathbf{R})$$

are smooth mappings. Then the total space E is a smooth manifold and also

$$\dim E = \dim X + n.$$

For if $\{U_\alpha\}$ is an atlas of charts for the manifold X then an atlas of the manifold E can be defined by

$$V_\alpha = p^{-1}(U_\alpha) = U_\alpha \times \mathbf{R}^n.$$

The local coordinates on V_α can be defined as the family of local coordinates on the chart U_α with Cartesian coordinates on the fiber. The smoothness of the functions $\varphi_{\alpha\beta}$ implies smoothness of the change of coordinates.

There is the natural question whether for any vector bundle over smooth manifold X there exists an atlas on the total space with smooth the transition functions $\varphi_{\alpha\beta}$. The answer lies in the following theorem.

Theorem 28 *Let $p : E \longrightarrow X$ be an n–dimensional vector bundle and X a compact smooth manifold. Then there exists an atlas $\{U_\alpha\}$ on X and coordinate homeomorphisms*

$$\varphi_\alpha : U_\alpha \times \mathbf{R}^n \longrightarrow p^{-1}(U_\alpha)$$

such that the transition functions

$$\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathbf{GL}\ (n, \mathbf{R})$$

are smooth.

5. Let us describe the structure of the tangent and normal bundles of a smooth manifold.

Theorem 29 *Let $j : X \subset Y$ be a smooth submanifold X of a manifold Y. Then there exists a neighborhood $V \supset X$ which is diffeomorphic to the total space of the normal bundle $\nu(X \subset Y)$.*

Proof.
Fix a Riemannian metric on the manifold Y (which exists by Theorem 23 and Remark 2 from the section 3). We construct a mapping

$$f : \nu(X) \longrightarrow Y.$$

Consider a normal vector $\xi \in \nu(X)$ at the point $x \in X \subset Y$. Notice that the vector ξ is orthogonal to the subspace $T_x(X) \subset T_x(Y)$. Let $\gamma(t)$ be the geodesic curve such that

$$\gamma(0) = x, \ \frac{d\gamma}{dt}(0) = \xi.$$

Put $f(\xi) = \gamma(1)$. The mapping f has nondegenerate Jacobian for each point of the zero section of the bundle $\nu(X)$. Indeed, notice that

1. if $\xi = 0$, $\xi \in T_x(Y)$ then $f(\xi) = x$,
2. $f(\lambda\xi) = \gamma(\lambda)$.

Therefore the Jacobian matrix of the mapping f at a point of the zero section of $\nu(X)$ that maps the tangent spaces

$$Df : T_x(\nu(X)) = T_x(X) \oplus \nu_x(X) \longrightarrow T_x(Y) = T_x(X) \oplus \nu_x(X)$$

is the identity. By the implicit function theorem there is a neighborhood V of zero section o f the bundle $\nu(X)$ which is mapped by F diffeomorphically onto a neighborhood $f(V)$ of the submanifold X. Since there is a sufficiently small neighborhood V which is diffeomorphic to the total space of the bundle $\nu(X)$ the proof of theorem is finished. ∎

6. There is a simple criterion describing when a smooth mapping of manifolds gives a locally trivial bundle.

Theorem 30 *Let*

$$f : X \longrightarrow Y$$

be a smooth mapping of compact manifolds such that the differential Df is epimorphism at each point $x \in X$. Then f is a locally trivial bundle with the fiber a smooth manifold.

Proof.
Without loss of generality one can consider a chart $U \subset Y$ diffeomorphic to $\mathbf{R}^n$ and part of the manifold X, namely, $f^{-1}(U)$. Then the mapping f gives a vector valued function

$$f : X \longrightarrow \mathbf{R}^n.$$

Assume firstly that $n = 1$. From the condition of the theorem we know that the gradient of the function f never vanishes. Consider the vector field $\mathbf{grad} f$ (with respect to some Riemannian metric on X). The integral curves $\gamma(x_0, t)$ are orthogonal to each hypersurface of the level of the function f. Choose a new Riemannian metric such that $\mathbf{grad} f$ is a unit vector field. Indeed, consider the new metric

$$(\xi, \eta)_1 = (\xi, \eta)(\mathbf{grad} f, \mathbf{grad} f).$$

Then

$$(\mathbf{grad} f, \xi) = \xi(f) = (\mathbf{grad}_1 f, \xi)_1 = (\mathbf{grad}_1 f, \xi)(\mathbf{grad}_1 f, \mathbf{grad}_1 f).$$

Hence

$$\mathbf{grad}_1 f = \frac{\mathbf{grad} f}{(\mathbf{grad} f, \mathbf{grad} f)}.$$

Then

$$\begin{aligned} (\mathbf{grad}_1 f, \mathbf{grad}_1 f)_1 &= (\mathbf{grad}_1 f, \mathbf{grad}_1 f)(\mathbf{grad} f, \mathbf{grad} f) = \\ &= \frac{(\mathbf{grad} f, \mathbf{grad} f)}{(\mathbf{grad} f, \mathbf{grad} f)^2}(\mathbf{grad} f, \mathbf{grad} f) = 1. \end{aligned}$$

Thus the integral curves

$$\frac{d}{dt} f(\gamma(t)) = (\mathbf{grad} f, \mathbf{grad} f) \equiv 1.$$

Hence the function $f(\gamma(t))$ is linear. This means that if

$$f(x_0) = f(x_1),$$

then

$$f(\gamma(x_0, t)) = f(\gamma(x_1, t)) = f(x_0) + t.$$

Put

$$g : Z \times \mathbf{R}^1 \longrightarrow X, \ g(x, t) = \gamma(x, t).$$

The mapping g is a fiberwise smooth homeomorphism. Hence the mapping

$$f : \longrightarrow \mathbf{R}^1$$

gives a locally trivial bundle. Further, the proof will follow by induction with respect to n. Consider a vector valued function

$$f(x) = \{f_1(x), \ldots, f_n(x)\}$$

which satisfies the condition of the theorem. Choose a Riemannian metric on the manifold X such that gradients

$$\mathbf{grad} f_1, \ldots, \mathbf{grad} f_n$$

are orthonormal. Such a metric exists. Indeed, consider firstly an arbitrary metric. The using the linear independence of the differentials $\{df_i\}$ we know that the gradients are also independent. Hence the matrix

$$a_{ij} = \langle \mathbf{grad} f_i(x),\ \mathbf{grad} f_j(x) \rangle$$

is nondegenerate in each point. Let $\|b_{ij}(x)\|$ be the matrix inverse to the matrix $\|a_{ij}(x)\|$, that is,

$$\sum_\alpha b_{i\alpha}(x) a_{j\alpha}(x) \equiv \delta_{ij}.$$

Put

$$\xi_k = \sum_i b_{kj}(x) \mathbf{grad} f_i(x).$$

Then

$$\begin{aligned} \xi_k(f_j) = \sum_i b_{ki}(x) \mathbf{grad} f_i(f_j) = \\ = \quad \sum_i b_{ki}(x) \langle \mathbf{grad} f_i,\ \mathbf{grad} g_j \rangle = \sum_i b_{ki}(x) a_{ij}(x) \equiv \delta_{kj}. \end{aligned}$$

Let U_α be a sufficiently small neighborhood of a point of the manifold X. The system of vector fields $\{\xi_1, \ldots, \xi_n\}$ can be supplemented by vector fields $\eta_{n+1}, \ldots, \eta_N$ to form a basis such that

$$\eta_k(f_i) \equiv 0.$$

Consider the new metric in the chart U_α given by

$$\begin{aligned} \langle \xi_i,\ \xi_j \rangle_\alpha &\equiv \delta_{ij}, \\ \langle \xi_k,\ \eta_j \rangle_\alpha &\equiv 0. \end{aligned}$$

Let φ_α be a partition of unity subordinate to the covering $\{U_\alpha\}$ and put

$$\langle \xi,\ \eta \rangle_0 = \sum_\alpha \varphi_\alpha(x) \langle \xi,\ \eta \rangle_\alpha.$$

Then

$$\begin{aligned} \langle \xi_i,\ \xi_j \rangle_0 &\equiv \delta_{ij}, \\ \langle \xi_k,\ \eta \rangle_0 &\equiv 0 \end{aligned}$$

for any vector η for which $\eta(f_i) = 0$.Let $\mathbf{grad}_0 f_i$ be the gradients of the functions f_i with respect to the metric $\langle \xi,\ \eta \rangle_0$. This means that

$$\langle \mathbf{grad} f_i, \xi \rangle_0 = \xi(f_i)$$

for any vector ξ. In particular one has

$$\begin{aligned} \langle \mathbf{grad} f_i,\ \xi_j \rangle_0 &\equiv \delta_{ij}, \\ \langle \mathbf{grad} f_k,\ \eta \rangle_0 &\equiv 0 \end{aligned}$$

for any vector η for which $\eta(f_i) = 0$. Similar relations hold for the vector field ξ_i. Therefore

$$\xi_i = \mathbf{grad} f_i,$$

that is,

$$\langle \mathbf{grad} f_i,\ \mathbf{grad} f_j \rangle \delta_{ij},$$

the latter proves the existence of metric with the necessarily properties.

Let us pass now to the proof of the theorem. Consider the vector function

$$g(x) = \{f_1(x), \ldots, f_{n-1}(x)\}.$$

This function satisfies the conditions of the theorem and the inductive assumption. It follows that the manifold X is diffeomorphic to the Cartesian product $X = Z \times \mathbf{R}^{n-1}$ and the functions f_i are the coordinate functions for the second factor. Then $\mathbf{grad} f_n$ is tangent to the first factor Z and hence the function $f_n(x,t)$ does not depend on $t \in \mathbf{R}^{n-1}$. Therefore one can apply the first step of the induction to the manifold Z, that is,

$$Z = Z_1 \times \mathbf{R}^1.$$

Thus

$$X = Z_1 \times \mathbf{R}^n.$$

∎

7.7 Linear groups and related bundles

This section consists of examples of vector bundles which arise naturally in connection with linear groups and their homogeneous spaces.

7.7.1 The Hopf bundle

The set of all one dimensional subspaces or lines of $\mathbf{R}^{n+1}$ is called the *real (n dimensional) projective space* and denoted by $\mathbf{RP}^n$. It has a natural topology given by the metric which measures the smaller angle between two lines. The projective space $\mathbf{RP}^n$ has the structure of smooth (and even real analytical) manifold. To construct the smooth manifold structure on $\mathbf{RP}^n$ one should first notice that each line l in $\mathbf{R}^{n+1}$ is uniquely determined by any nonzero vector x belonging to the line. Let $\{x_0, \ldots, x_n\}$ be the Cartesian coordinates of such a vector x, not all vanishing. Then the line l is defined by the coordinates $\{x_0, \ldots, x_n\}$ and any $\{\lambda x_0, \ldots, \lambda x_n\}$, $\lambda \neq 0$. Thus the point of $\mathbf{RP}^n$ is given by a class $[x_0 : x_1 : \ldots : x_n]$ of coordinates $\{x_0, \ldots, x_n\}$ (not all vanishing) determined up to multiplication by a nonzero real number λ. The class $[x_0 : x_1 : \ldots : x_n]$ gives the *projective coordinates* of the point of $\mathbf{RP}^n$. We define an atlas $\{U_k\}_{k=0}^n$ on $\mathbf{RP}^n$ as follows. Put

$$U_k = \{[x_0 : x_1 : \ldots : x_n] : x_k \neq 0\},$$

and define coordinates on U_k by the following functions:

$$y_k^\alpha = \frac{x_\alpha}{x_k},\ 0 \le \alpha \le n,\ \alpha \ne k.$$

where in the numbering of the coordinates by index α there is a gap when $\alpha = k$. The change of variables on the intersection $U_k \cap U_j,\ (k \ne j)$ of two charts has the following form:

$$y_k^\alpha = \begin{cases} \frac{y_j^\alpha}{y_j^k} & \text{when} \quad \alpha \ne j, \\ \frac{1}{y_j^k} & \text{when} \quad \alpha = j. \end{cases} \tag{127}$$

The formula (127) is well defined because $k \ne j$ and on $U_k \cap U_j$

$$y_j^k = \frac{x_k}{x_j} \ne 0.$$

All the functions in (127) are smooth functions making $\mathbf{RP}^n$ into a smooth manifold of dimension n.

Consider now the space E in which points have the form (l, x), where l is a one dimensional subspace of $\mathbf{R}^{n+1}$ and x is a point on l. The space E differs from the space $\mathbf{R}^{n+1}$ in that instead of zero vector of $\mathbf{R}^{n+1}$ in the space E there are many points of the type $(l, 0)$. The mapping

$$p : E \longrightarrow \mathbf{RP}^n, \tag{128}$$

which associates with each pair (l, x) its the first component, gives a locally trivial vector bundle. Indeed, the space E can be represented as a subset of the Cartesian product $\mathbf{RP}^n \times \mathbf{R}^{n+1}$ defined by the following system of equations in each local coordinate system:

$$\operatorname{rank} \left\| \begin{matrix} x_0 & x_1 & \dots & x_n \\ y_0 & y_1 & \dots & y_n \end{matrix} \right\| = 1,$$

where $(x_0, x_1, \dots, x_n)$ are projective coordinates of a point of $\mathbf{RP}^n$, and $(y_0, y_1, \dots, y_n)$ are coordinates of the point of $\mathbf{R}^{n+1}$. For example, in the case when $U_0 = \{x_0 \ne 0\}$ we put $x_0 = 1$. Then

$$\operatorname{rank} \left\| \begin{matrix} 1 & x_1 & \dots & x_n \\ y_0 & y_1 & \dots & y_n \end{matrix} \right\| = 1,$$

that is,

$$\begin{cases} \det \left\| \begin{matrix} 1 & x_k \\ y_0 & y_k \end{matrix} \right\| = 0, \\ \det \left\| \begin{matrix} x_k & x_j \\ y_k & y_j \end{matrix} \right\| = 0, \end{cases}$$

Hence the set E is defined in the $\mathbf{RP}^n \times \mathbf{R}^{n+1}$ by the following system of n equations:

$$f_k(x_1, \dots, x_n, y_0, \dots, y_n) = y_k - x_k y_0 = 0,\ k = 1, \dots, n.$$

The Jacobian matrix of the functions f_k is

$$\left\| \begin{matrix} -y_0 & 0 & \dots & 0 & -x_1 & 1 & 0 & \dots & 0 \\ 0 & -y_0 & \dots & 0 & -x_2 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -y_0 & -x_n & 0 & 0 & \dots & 1 \end{matrix} \right\| \tag{129}$$

Clearly the rank of this matrix (129) is maximal. By the implicit function theorem the space E is a submanifold of dimension $n+1$. As coordinates one can take $(y_0, x_1, \dots, x_n)$. The projection (128) consists of forgetting the first coordinate y_0. So the inverse image $p^{-1}(U_0)$ is homeomorphic to the Cartesian product $U_0 \times \mathbf{R}^1$. Changing to another chart U_k, one takes another coordinate y_k as the coordinate in the fiber which depends linearly on y_0. Thus the mapping (128) gives a one dimensional vector bundle.

7.7.2 The complex Hopf bundle

This bundle is constructed similarly to the previous example as a one dimensional complex vector bundle over the base $\mathbf{CP}^n$. In the both cases, real and complex, the corresponding principal bundles with the structure groups $\mathbf{O}(1) = \mathbf{Z}_2$ and $\mathbf{U}(1) = \mathbf{S}^1$ can be identified with subbundles of the Hopf bundle. The point is that the structure group $\mathbf{O}(1)$ can be included in the fiber $\mathbf{R}^1$, $\mathbf{O}(1) = \{-1, 1\} \subset \mathbf{R}^1$, in a such way that the linear action of $\mathbf{O}(1)$ on $\mathbf{R}^1$ coincides on the subset $\{-1, 1\}$ with the left multiplication. Similarly, the group $\mathbf{U}(1) = \mathbf{S}^1$ can be included in $\mathbf{C}^1$:

$$\mathbf{S}^1 = \{z : |z| = 1\} \subset \mathbf{C}^1,$$

such that linear action of $\mathbf{U}(1)$ on $\mathbf{C}^1$ coincides on $\mathbf{S}^1$ with left multiplication. Hence the Hopf bundle has the principal subbundle consisting of vectors of unit length. Let

$$p : E_S \longrightarrow \mathbf{RP}^n$$

be principal bundle associated with the Hopf vector bundle. The points of the total space E_S are pairs (l, x), where l is a line in $\mathbf{R}_{n+1}$ and $x \in l, |x| = 1$. Since $x \neq 0$, the pair (l, x) is uniquely determined by the vector x. Hence the total space E_S is homeomorphic to the sphere $\mathbf{S}^n$ of unit radius a nd the principal bundle

$$p : \mathbf{S}^n \longrightarrow \mathbf{RP}^n$$

is the two-sheeted covering. In the case of the complex Hopf bundle, the associated principal bundle

$$p : E_S \longrightarrow \mathbf{CP}^n$$

is

$$E_S = \mathbf{S}^{2n+1}$$

and the fiber is a circle $\mathbf{S}^1$.

Both these two principal bundles are also called *Hopf bundles*. When $n = 1$, we have $\mathbf{S}^3 \longrightarrow \mathbf{P}^1$, that is, the classical Hopf bundle

$$\mathbf{S}^3 \longrightarrow \mathbf{S}^2. \tag{130}$$

In this last case it is useful to describe the transition functions for the intersection of charts. Let us consider the sphere $\mathbf{S}^3$ as defined by the equation

$$|z_0|^2 + |z_1|^2 = 1,$$

in the two dimensional complex vector space $\mathbf{C}^2$. The map (130) associates the point (z_0, z_1) to the point in the $\mathbf{CP}^1$ with the projective coordinates $[z_0 : z_1]$. So over the base $\mathbf{CP}^1$ we have the atlas consisting of two charts:

$$\begin{aligned} U_0 &= \{[z_0, z_1] : z_0 \neq 0\}, \\ U_1 &= \{[z_0, z_1] : z_1 \neq 0\}, \end{aligned}$$

The points of the chart U_0 are parametrized by the complex parameter

$$w_0 = \frac{z_1}{z_0} \in \mathbf{C}^1$$

whereas points of the chart U_1 are parametrized by the complex parameter

$$w_1 = \frac{z_0}{z_1} \in \mathbf{C}^1.$$

The homeomorphisms

$$\begin{aligned} \varphi_0 : p^{-1}(U_0) &\longrightarrow \mathbf{S}^1 \times \mathbf{C}^1 = \mathbf{S}^1 \times U_0, \\ \varphi_1 : p^{-1}(U_1) &\longrightarrow \mathbf{S}^1 \times \mathbf{C}^1 = \mathbf{S}^1 \times U_1, \end{aligned}$$

have the form

$$\begin{aligned} \varphi_0(z_0, z_1) &= \left(\frac{z_0}{|z_0|}, \frac{z_1}{z_0}\right) = \left(\frac{z_0}{|z_0|}, [z_0 : z_1]\right), \\ \varphi_1(z_0, z_1) &= \left(\frac{z_1}{|z_1|}, \frac{z_0}{z_1}\right) = \left(\frac{z_1}{|z_1|}, [z_0 : z_1]\right). \end{aligned}$$

The mappings φ_0 and φ_1 clearly are invertible:

$$\begin{aligned} \varphi_0^{-1}(\lambda, w_0) &= \left(\frac{\lambda}{\sqrt{1 + |w_0|^2}}, \frac{\lambda w_0}{\sqrt{1 + |w_0|^2}}\right), \\ \varphi_1^{-1}(\lambda, w_1) &= \left(\frac{\lambda}{\sqrt{1 + |w_1|^2}}, \frac{\lambda w_1}{\sqrt{1 + |w_1|^2}}\right). \end{aligned}$$

Hence the transition function

$$\varphi_{01} : \mathbf{S}_1 \times (U_0 \cap U_1) \longrightarrow \mathbf{S}_1 \times (U_0 \cap U_1)$$

is defined by the formula:

$$\varphi_{01}(\lambda, [z_{0:z_1}]) = \left(\frac{z_1|z_0|}{z_0|z_1|}, [z_0 : z_1], \right),$$

that is, multiplication by the number

$$\frac{z_1|z_0|}{z_0|z_1|} = \frac{w_1}{|w_1|}.$$

7.7.3 The tangent bundle of the Hopf bundle

Let ξ be the complex Hopf bundle over $\mathbf{CP}^n$ and let $T\mathbf{CP}^n$ be the tangent bundle. Then

$$T\mathbf{CP}^n \oplus \overline{\mathbf{1}} = \xi^* \oplus \xi^* \oplus \ldots \oplus \xi^* = (n+1)\xi^*. \tag{131}$$

When $n = 1$ this gives

$$\mathbf{TCP}^1 \oplus \overline{\mathbf{1}} = \xi^* \oplus \xi^*.$$

Indeed, using the coordinate transition function

$$w_2 = \frac{1}{w_1},$$

the transition function for the tangent bundle $\mathbf{TCP}^1$ has the following form:

$$\psi_{01}(w_1) = -\frac{1}{w_1^2} = -\frac{\bar{w}_1^2}{|w_1|^4}. \tag{132}$$

On the other hand, the transition function for the Hopf bundle is

$$\varphi_{01}(w_1) = \frac{w_1}{|w_1|}. \tag{133}$$

Using homotopies the formulas (132) and (133) can be simplified to

$$\begin{aligned} \psi_{01}(w_1) &= \bar{w}_1^2, \\ \varphi_{01}(w_1) = w_1. \end{aligned}$$

This shows that the matrix functions

$$\begin{Vmatrix} \bar{w}_1^2 & 0 \\ 0 & 1 \end{Vmatrix} \text{ and } \begin{Vmatrix} \bar{w}_1 & 0 \\ 0 & \bar{w}_1 \end{Vmatrix}$$

are homotopic in the class of invertible matrices when $w_1 \neq 0, w_1 \in \mathbf{C}^1$. In general, let us consider the manifold $\mathbf{CP}^n$ as the quotient space of the unit sphere $\mathbf{S}^{2n+1}$ in $\mathbf{C}^{n+1}$ by the action of the group $\mathbf{S}^1 \subset \mathbf{C}^1$ of complex numbers with unit norm. The total space of the tangent bundle $T\mathbf{CP}^n$ is the quotient

space of the family of all tangent vectors of the sphere which are orthogonal to the orbits of the action of the group $\mathbf{S}^1$. In other words, one has

$$T\mathbf{CP}^n = \frac{\{(u,v) : u,v \in \mathbf{C}^{n+1}, |u| = 1, \langle u,v\rangle = 0\}}{\{(\lambda u, \lambda v) \sim (u,v), \lambda \in \mathbf{S}^1\}},$$

where $\langle u,v\rangle$ is the Hermitian inner product in $\mathbf{C}^{n+1}$ Consider the quotient space

$$\mathbf{A} = \frac{\{(\bar{u},s) : u \in \mathbf{C}^{n+1}, |\bar{u}| = 1,\ s \in \mathbf{C}^1\}}{\{(\lambda\bar{u}, \lambda s) \sim (\bar{u},s), \lambda \in \mathbf{S}^1\}}.$$

Associate to each pair $(\bar{u},s) \in \mathbf{A}$ the line l which passes through vector $\bar{u} \in \mathbf{C}^{n+1}$ and the vector $s\bar{u} \in l$. If $(\lambda\bar{u}, \lambda s)$ is an equivalent pair (which passes through vector $\lambda\bar{u}$) then it corresponds the same line l and the same vector $s\bar{u} = (\lambda s)(\overline{\lambda u})$. This means that the space $\mathbf{A}$ is homeomorphic to the total space of vector bundle ξ^*. Hence the total space of the vector bundle $(n+1)\xi^*$ is homeomorphic to the space

$$\mathbf{B} = \frac{\{(u,s) : u,v \in \mathbf{C}^{n+1}, |u| = 1,\ \}}{\{(\lambda u, \lambda v) \sim (u,v), \lambda \in \mathbf{S}^1\}}.$$

The space $T\mathbf{CP}^n =$ is clearly a subspace of $\mathbf{B}$. A complementary subbundle can be defined as a quotient space

$$\mathbf{D} = \frac{\{(u,v) : u,v \in \mathbf{C}^{n+1}, |u| = 1,\ v = su,\ s \in \mathbf{C}^1\}}{\{(\lambda u, \lambda v) \sim (u,v), \lambda \in \mathbf{S}^1\}}.$$

The latter is homeomorphic to the space

$$\mathbf{D} = \frac{\{(u,s) : u \in \mathbf{C}^{n+1}, |u| = 1,\ s \in \mathbf{C}^1\}}{\{\lambda u \sim u, \lambda \in \mathbf{S}^1\}},$$

which is homeomorphic to the Cartesian product $T\mathbf{CP}^n \times \mathbf{C}^1$. Thus one has the isomorphism (131).

7.7.4 Bundles of classical manifolds

Denote by $\mathbf{V}_{n,k}$ the space where the points are orthonormal sets of k vectors of n–dimensional euclidean space ($\mathbf{R}^n$ or $\mathbf{C}^n$). If we need we will write $\mathbf{V}^R_{n,k}$ or $\mathbf{V}^C_{n,k}$. Correspondingly, let us denote by $\mathbf{G}_{n,k}$ the space in which points are k–dimensional subspaces of n–dimensional euclidean space ($\mathbf{R}^n$ or $\mathbf{C}^n$). By expanding a k–frame to a basis in $\mathbf{C}^n$ we obtain

$$\mathbf{V}^C_{n,k} = \mathbf{U}(n)/\mathbf{U}(n-k),$$

where $\mathbf{U}(n-k) \subset \mathbf{U}(n)$ is a natural inclusion of unitary matrices:

$$A_{n-k} \longrightarrow \left\| \begin{matrix} 1_k & 0 \\ 0 & A_{n-k} \end{matrix} \right\|.$$

Similarly, the space $\mathbf{G}_{n,k}$ is homeomorphic to the homogeneous space

$$\mathbf{U}(n)/\left(\mathbf{U}(k)\oplus\mathbf{U}(n-k)\right),$$

where $\mathbf{U}(k)\oplus\mathbf{U}(n-k)\subset\mathbf{U}(n)$ is natural inclusion

$$(A_k,B_{n-k})\longrightarrow\left\|\begin{matrix}A_k & 0\\ 0 & B_{n-k}\end{matrix}\right\|.$$

Generally speaking, if G is a Lie group, and $H\subset G$ is a subgroup then the projection

$$p:G\longrightarrow G/H$$

is a locally trivial bundle (a principal H–bundle) since the rank of Jacobian matrix is maximal and consequently constant. Hence the following mappings give locally trivial bundles

$$\mathbf{V}_{n,k}\overset{\mathbf{V}_{n-k_1,k-k_1}}{\longrightarrow}\mathbf{V}_{n,k_1},$$
$$\mathbf{V}^C_{n,k}\overset{\mathbf{U}(k)}{\longrightarrow}\mathbf{G}^C_{n,k}.$$

(The fibers are shown over the arrows.) In particular,

$$\begin{aligned}\mathbf{V}^R_{n,n} &= \mathbf{O}(n),\\ \mathbf{V}^C_{n,n} &= \mathbf{U}(n),\\ \mathbf{V}^R_{n,1} &= \mathbf{S}^{n-1},\\ \mathbf{V}^C_{n,1} &= \mathbf{S}^{2n-1}.\end{aligned}$$

Hence we have the following locally trivial bundles

$$\mathbf{U}(n)\overset{\mathbf{U}(n-1)}{\longrightarrow}\mathbf{S}^{2n-1},$$
$$\mathbf{O}(n)\overset{\mathbf{O}(n-1)}{\longrightarrow}\mathbf{S}^{n-1},$$

All the mappings above are defined by forgetting some of the vectors from the frame. The manifolds $\mathbf{V}_{n,k}$ are called the *Stieffel manifolds*, and the $\mathbf{G}_{n,k}$ called *Grassmann manifolds*.

7.8 Classifying theorems

7.8.1 Exact homotopy sequence

7.8.2 Constructions of classifying spaces

8 Elliptic operators on manifolds

8.1 Calculus of operators on manifolds

We describe some of the most fruitful applications of vector bundles, namely, in elliptic operator theory. We study some of the geometrical constructions which appear naturally in the analysis of differential and pseudodifferential operators on smooth manifolds.

8.1.1 Symbols of pseudodifferential operators

Consider a linear differential operator A which acts on the space of smooth functions of n real variables:

$$A : C^\infty(\mathbf{R}^n) \longrightarrow C^\infty(\mathbf{R}^n).$$

The operator A is a finite linear combination of partial derivatives

$$A = \sum_\alpha a_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}, \tag{134}$$

where the $\alpha = (\alpha_1, \ldots, \alpha_n)$ is multi-index, $a_\alpha(x)$ are smooth functions and

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} = \frac{\partial |\alpha|}{(\partial x^1)^{\alpha_1} (\partial x^2)^{\alpha_2} \ldots (\partial x^n)^{\alpha_n}},$$

$$|\alpha| = \alpha_1 + \ldots + \alpha_n$$

is the operator given by the partial derivatives.

The maximal value of $|\alpha|$ is called the *order of the differential operator*differential operator, so the formula (134) can be written

$$A = \sum_{|\alpha| \le m} a_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha},$$

Let us introduce a new set of variables $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$. Put

$$a(x, \xi) = \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha i^{|\alpha|},$$

where $\xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_n^{\alpha_n}$. The function $a(x, \xi)$ is called the *symbol of a differential operatorA*. The operator A can be reconstructed from its symbol by substitution of the operators $\frac{1}{i} \frac{\partial}{\partial x^k}$ for the variables ξ_k, that is,

$$A = a\left(x, \frac{1}{i} \frac{\partial}{\partial x}\right).$$

Since the symbol is polynomial with respect to variables ξ, it can be split into homogeneous summands

$$a(x, \xi) = a_m(x, \xi) + a_{m-1}(x, \xi) + \ldots + a_0(x.\xi).$$

The highest term $a_m(x, x)$ is called the *principal symbol* of the operator A while whole symbol sometimes is called the *full symbol*. The reason for singling out the principal symbol is as follows:

Proposition 6 *Let*

$$y^k = y^k\left(x^1, \ldots, x^n\right) \ \ (\ \textit{or}\ y = y(x))$$

be a smooth change of variables. Then in the new coordinate system the operator B defined by the formula

$$(Bu)(y) = \left(Au\left(y(x)\right)\right)_{x=x(y)}$$

is again a differential operator of order m for which the principal symbol is

$$b_m(y,\eta) = a_m\left(x(y), \eta\frac{\partial y(x(y))}{\partial x}\right). \tag{135}$$

The formula (135) shows that variables ξ change as a tensor of valency $(0,1)$, that is, as components of a cotangent vector.

The concept of a differential operator exists on an arbitrary smooth manifold M. The concept of a whole symbol is not well defined but the principal symbol can be defined as a function on the total space of the cotangent bundle T^*M. It is clear that the differential operator A does not depend on the principal symbol alone but only up to the addition of an operator of smaller order.

The notion of a differential operator can be generalized in various directions. First of all notice that if

$$\mathbf{F}_{x\longrightarrow\xi}(u)(\xi) = \frac{1}{(2\pi)^{n/2}}\int\limits_{\mathbf{R}^n} e^{-i(x,\xi)}u(x)dx$$

and

$$\mathbf{F}_{\xi\longrightarrow x}(v)(x) = \frac{1}{(2\pi)^{n/2}}\int\limits_{\mathbf{R}^n} e^{i(x,\xi)}v(\xi)d\xi$$

are the direct and inverse Fourier transformations then

$$\left(Au\right)(x) = \mathbf{F}_{\xi\longrightarrow x}\left(a(x,\xi)\left(\mathbf{F}_{x\longrightarrow\xi}(u)(\xi)\right)\right) \tag{136}$$

Hence we can enlarge the family of symbols to include some functions which are not polynomials. Namely, suppose a function a defined on the cotangent bundle T^*M satisfies the condition

$$\left|\frac{\partial^{|\alpha|}}{\partial\xi^\alpha}\frac{\partial^{|\beta|}}{\partial x^\beta}a(x,\xi)\right| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|} \tag{137}$$

for some constants $C_{\alpha,\beta}$. Denote by $\mathbf{S}$ the Schwartz space of functions on $\mathbf{R}^n$ which satisfy the condition

$$\left|x^\beta\frac{\partial^{|\alpha|}}{\partial x^\alpha}u(x)\right| \le C_{\alpha,\beta}$$

for any multiindexes α and β. Then the operator A defined by formula (136) is called a *pseudodifferential operator of order m* (more exactly, not greater than m). The pseudodifferential operator A acts in the Schwartz space $\mathbf{S}$.

This definition of a pseudodifferential operator can be extended to the Schwartz space of functions on an arbitrary compact manifold M. Let $\{U_\alpha\}$ be an atlas of charts with a local coordinate system $x_\alpha = \left(x_\alpha^1, \ldots, x_\alpha^n\right)$. Without loss of generality we can assume that the local coordinate system x_α maps the chart U_α onto the space $\mathbf{R}^n$. Let $\xi_\alpha = (\xi_{1\alpha}, \ldots, \xi_{n\alpha})$ be the corresponding components of a cotangent vector. Let $\{\varphi_\alpha\}$ be a partition of unity subordinate to the atlas of charts, that is,

$$0 \leq \varphi_\alpha(x) \leq 1, \ \sum_\alpha \varphi_\alpha(x) \equiv 1, \ \mathbf{supp} \ \varphi_\alpha \subset U_\alpha.$$

Finally, let $\psi_\alpha(x)$ be functions such that

$$\mathbf{supp} \ \psi_\alpha \subset U_\alpha, \ \varphi_\alpha(x)\psi_\alpha(x) \equiv \varphi_\alpha(x).$$

Then we can define an operator A by the formula

$$A(u)(x) = \sum_\alpha \psi_\alpha(x) A_\alpha \left(\varphi_\alpha(x) u(x)\right), \tag{138}$$

where A_α is a pseudodifferential operator on the chart U_α (which is diffeomorphic to $\mathbf{R}^n$) with principal symbol

$$a_\alpha(x_\alpha, \xi_\alpha) = a(x, \xi).$$

When the function $a(x, \xi)$ is polynomial (of order m) the operator A defined by formula (138) is a differential operator not depending on the choice of functions ψ_α. In general, the operator A depends on the choice of functions ψ_α, φ_α and the local coordinate system x_α, uniquely up to the addition of a pseudodifferential operator of order strictly less than m.

The next useful generalization consists of a change from functions on the manifold M to smooth sections of vector bundles. Let ξ_1 and ξ_2 be two vector bundles over the manifold M. Consider a linear mapping

$$a : \pi^*(\xi_1) \longrightarrow \pi^* \xi_2, \tag{139}$$

where $\pi : T^*M \longrightarrow M$ is the natural projection. Then in any local coordinate system (x_α, ξ_α) the mapping (139) defines a matrix valued function which we require to satisfy the condition (137). Then the mapping (139) defines a pseudodifferential operator

$$A = a(D) : \Gamma^\infty(\xi_1) \longrightarrow \Gamma^\infty(\xi_2)$$

by formulas similar to (138), again uniquely up to the addition of a pseudodifferential operator of the order less than m. The crucial property of the definition (138) is the following

Proposition 7 *Let*

$$a : \pi^*(\xi_1) \longrightarrow \pi^*(\xi_2), ; \ b : \pi^*(\xi_a) \longrightarrow \pi^*(\xi_3), ;$$

be two symbols of orders m_1, m_2. Let $c = ba$ be the composition of the symbols. Then the operator

$$b(D)a(D) - c(D) : \Gamma^\infty(\xi_1) \longrightarrow \Gamma^\infty(\xi_3)$$

is a pseudodifferential operator of order $m_1 + m_2 - 1$.

Proposition 7 leads to a way of solving equations of the form

$$Au = f \tag{140}$$

for certain pseudodifferential operators A. To find a solution of (140), it sufficient to construct a left inverse operator B, that is, $BA = 1$. If B is the pseudodifferential operator $B = b(D)$ then

$$1 = b(D)a(D) = c(D) + (b(D)a(A) - c(D)).$$

Then by 7, the operator $c(D)$ differs from identity by an operator of order -1. Hence the symbol c has the form

$$c(x, \xi) = 1 + \text{ symbol of order } (-1).$$

Hence for existence of the left inverse operator B, it is necessary that symbol b satisfies the condition

$$a(x, \xi)b(x, \xi) = 1 + \text{ symbol of order } (-1). \tag{141}$$

In particular, the condition (141) holds if

Condition 1 *$a(x, \xi)$ is invertible for sufficiently large $|\xi| \geq C$.*

In fact, if the condition (141) holds then we could put

$$b(x, \xi) = a^{-1}(x, \xi)\chi(x, \xi),$$

where $\chi(x, \xi)$ is a function such that

$$\begin{aligned} \chi(x, \xi) &\equiv 1 \text{ for } |\xi| \geq 2C, \\ \chi(x, \xi) &\equiv 0 \text{ for } |\xi| \leq C. \end{aligned}$$

Then the pseudodifferential operator $a(D)$ is called an *elliptic* if Condition 1 holds.

The final generalization for elliptic operators is the substitution of a sequence of pseudodifferential operators for a single elliptic operator. Let $\xi_1, \xi_2, \ldots, \xi_k$ be a sequence of vector bundles over the manifold M and let

$$0 \longrightarrow \pi^*(\xi_1) \xrightarrow{a_1} \pi^*(\xi_2) \xrightarrow{a_2} \ldots \xrightarrow{a_{k-1}} \pi^*(\xi_k) \longrightarrow 0 \tag{142}$$

be a sequence of symbols of order $(m_1, \ldots, m_{k-1})$. Suppose the sequence (142) forms a complex, that is, $a_s a_{s-1} = 0$. Then the sequence of operators

$$0 \longrightarrow \Gamma^\infty(\xi_1) \xrightarrow{a_1(D)} \Gamma^\infty(\xi_2) \longrightarrow \ldots \longrightarrow \Gamma^\infty(\xi_k) \longrightarrow 0 \tag{143}$$

in general, does not form a complex because we can only know that the composition $a_k(D)a_{k-1}(D)$ is a pseudodifferential operator of the order less then $m_s + m_{s-1}$.

If the sequence of pseudodifferential operators forms a complex and the sequence of symbols (142) is exact away from a neighborhood of zero section in T^*M then the sequence (143) is called an *elliptic complex* of pseudodifferential operators.

8.1.2 Fredholm operators

A bounded linear operator

$$F : H \longrightarrow H$$

on a Hilbert space H is called a *Fredholm operator* if

$$\dim \mathbf{Ker}\, F < \infty, \ \dim \mathbf{Coker}\, F < \infty$$

and the image, $\mathbf{Im}\, F$, is closed. The number

$$\mathbf{index}\, F = \dim \mathbf{Ker}\, F - \dim \mathbf{Coker}\, F$$

is called *the index of the Fredholm operator* F. The index can be obtained as

$$\mathbf{index}\, F = \dim \mathbf{Ker}\, F - \dim \mathbf{Ker}\, F^*,$$

where F^* is the adjoint operator.

The bounded operator $K : H \longrightarrow H$ is said to be a *compact* if any bounded subset $X \subset H$ is mapped to a precompact set, that is, the set $\overline{F(X)}$ is compact.

Theorem 31 *Let F be a Fredholm operator. Then*

1. *there exists $\varepsilon > 0$ such that if $\|F - G\| < \varepsilon$ then G is a Fredholm operator and*

$$\mathbf{index}\, F = \mathbf{index}\, G,$$

2. *if K is compact then $F + K$ is also Fredholm and*

$$\mathbf{index}\, (F + K) = \mathbf{index}\, F. \tag{144}$$

The operator F is Fredholm if and only if there is an operator G such that both $K = FG - 1$ and $K' = GF - 1$ are compact. If F and G are Fredholm operators then the composition FG is Fredholm and

$$\mathbf{index}\, (FG) = \mathbf{index}\, F + \mathbf{index}\, G.$$

The notion of a Fredholm operator has an interpretation in terms of the finite dimensional homology groups of a complex of Hilbert spaces. In general, consider a sequence of Hilbert spaces and bounded operators

$$0 \longrightarrow C_0 \xrightarrow{d_0} C_1 \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} C_n \longrightarrow 0. \tag{145}$$

We say that the sequence (145) is *Fredholm complex* if $d_k d_{k-1} 0$, $\mathbf{Im}\, d_k$ is a closed subspace and

$$\dim\left(\mathbf{Ker}\ d_k/\mathbf{Coker}\ d_{k-1}\right) = \dim H\left(C_k, d_k\right) < \infty.$$

Then the *index of Fredholm complex* (145) is defined by the following formula:

$$\mathbf{index}\ (C,d) = \sum_k (-1)^k \dim H(C_k, d_k).$$

Theorem 32 *Let*

$$0 \longrightarrow C_0 \xrightarrow{d_0} C_1 \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} C_n \longrightarrow 0 \tag{146}$$

be a sequence satisfying the condition that each $d_k d_{k-1}$ is compact. Then the following conditions are equivalent:

1. *There exist operators $f_k : C_k \longrightarrow C_{k-1}$ such that $f_{k+1} d_k + d_{k-1} f_k = 1 + r_k$ where each r_k is compact.*
2. *There exist compact operators s_k such that the sequence of operators $d'_k = d_k + s_k$ forms a Fredholm complex. The index of this Fredholm complex is independent of the operators s_k.*

We leave the proof to the reader. Theorem 32 allows us to generalize the notion of a Fredholm complex using one of the equivalent conditions from Theorem 32.

8.1.3 The Sobolev norms

Consider the Schwartz space $\mathbf{S}$. Define the *Sobolev norm* by the formula

$$\|u\|_s^2 = \int\limits_{\mathbf{R}^n} \bar{u}(x)(1+\Delta)^s u(x) dx,$$

where

$$\Delta = \sum_{k=1}^n \left(i \frac{\partial}{\partial x^k} \right)^2$$

is the Laplace operator. Using the Fourier transformation this becomes

$$\|u\|_s^2 = \int\limits_{\mathbf{R}^n} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \tag{147}$$

where the index s need not be an integer.

The Sobolev space $H_s(\mathbf{R}^n)$ is the completion of $\mathbf{S}$ with respect to the Sobolev norm (147).

Proposition 8 *The Sobolev norms (147) for different coordinate systems are all equivalent on any compact $K \subset \mathbf{R}^n$, that is, there are two constants $C_1 > 0$ and $C_2 > 0$ such that for* $\mathbf{supp}\ u \subset K$

$$\|u\|_{1,s} \le C_1 \|u\|_{2,s} \le C_2 \|u\|_{1,s}.$$

Proposition 8 can easily be checked when s is an integer. For arbitrary s we leave it to the reader. This proposition allows us to generalize the Sobolev spaces to an arbitrary compact manifold M and vector bundle ξ. Let $\{U_\alpha\}$ be an atlas of charts where the vector bundle is trivial over each U_α. Let $\{\varphi_\alpha\}$ be a partition of unity subordinate to the atlas. Let $u \in \Gamma^\infty(M, \xi)$ be a section. Put

$$\|u\|_s^2 = \sum_\alpha \|\varphi_\alpha u\|_s^2. \tag{148}$$

Proposition 8 says that the definition (148) defines a Sobolev norm, well defined up to equivalent norms. Hence the completion of the space of sections $\Gamma^\infty(M, \xi)$ does not depend on the choice of partition of unity or the choice of local coordinate system in each chart U_α. We shall denote this completion by $H_s(M, \xi)$.

Theorem 33 *Let M be a compact manifold, ξ be a vector bundle over M and $s_1 < s_2$. Then the natural inclusion*

$$H_{s_2}(M, \xi) \longrightarrow H_{s_1}(M, \xi) \tag{149}$$

is a compact operator.

Theorem 33 is called *the Sobolev inclusion theorem.*

Proof.

For the proof it is sufficient to work in a local chart since any section $u \in \Gamma^\infty(M, \xi)$ can be split into a sum

$$u = \sum_\alpha \varphi_\alpha u,$$

where each summand $\varphi_\alpha u$ has support in the chart U_α. So let u be a function defined on $\mathbf{R}^n$ with support in the unit cube $\mathbf{I}^n$. These functions can be considered as functions on the torus $\mathbf{T}^n$. Then a function u can be expanded in a convergent Fourier series

$$u(x) = \sum_l a_l e^{i(l,x)}.$$

Partial differentiation transforms to multiplication of each coefficient a_l by the number l_k, where l is a multiindex $l = (l_1, \ldots, l_n)$. Therefore,

$$\|u\|_s^2 = \sum_l |a_l|^2 \left(1 + |l_1|^2 + \ldots + |l_n|^2\right)^s. \tag{150}$$

By formula (150), the space $H_s(\mathbf{T}^n)$ is isomorphic to the Hilbert space l_2 of the square summable sequences by the correspondence

$$u(x) \mapsto \left\{ b_l = \frac{a_l}{(1 + |l_1|^2 + \ldots + |l_n|^2)^{s/2}} \right\}$$

Then the inclusion (149) becomes an operator $l_2 \longrightarrow l_2$ defined by the correspondence

$$\{b_l\} \mapsto \left\{ \frac{b_l}{(1+|l_1|^2+\ldots+|l_n|^2)^{(s_2-s_1)/2}} \right\}$$

which is clearly compact.

Theorem 34 *Let*

$$a(D) : \Gamma^\infty(M,\xi_1) \longrightarrow \Gamma^\infty(M,\xi_2) \tag{151}$$

be a pseudodifferential operator of order m. Then there is a constant C such that

$$\|a(D)u\|_{s-m} \leq C\|u\|_s, \tag{152}$$

that is, the operator $a(D)$ can be extended to a bounded operator on Sobolev spaces:

$$a(D) : H_s(M,\xi_1) \longrightarrow H_{s-m}(M,\xi_2). \tag{153}$$

Proof.
The theorem is clear for differential operators. Indeed, it is sufficient to obtain estimates in a chart, as the symbol σ has compact support. If $a(D) = \frac{\partial}{\partial x^j}$ then

$$\begin{aligned} \|\frac{\partial}{\partial x^j}u(x)\|_s^2 &= \int\limits_{\mathbf{R}^n} \left(1+|\xi|^2\right)^s \left|\widehat{\frac{\partial}{\partial x^j}u}\right|^2 d\xi = \\ &= \int\limits_{\mathbf{R}^n} \left(1+|\xi|^2\right)^s |\xi_j \hat{u}(\xi)|^2 d\xi \leq \int\limits_{\mathbf{R}^n} \left(1+|\xi|^2\right)^{s+1} |\hat{u}(\xi)|^2 d\xi = \\ &= \|u(x)\|_{s+1}^2. \end{aligned} \tag{154}$$

Hence the inequality (152) follows from (154) by induction.

For pseudodifferential operators, the required inequality can also be obtained locally using a more complicated technique. ∎

Using theorems 33 and 34 it can be shown that an elliptic operator is Fredholm for appropriate choices of Sobolev spaces.

Theorem 35 *Let $a(D)$ be an elliptic pseudodifferential operator of order m as in (151). Then its extension (153) is Fredholm. The index of the operator (153) is independent of the choice of the number s.*

Proof.
As in section 8.1.1 we can construct a new symbol b of order $-m$ such that both $a(D)b(D)-1$ and $b(D)a(D)-1$ are pseudodifferential operators of order -1. Hence by Theorem 31, $a(D)$ gives a Fredholm operator (153).

To prove that the index of $a(D)$ does not depend of the number s, consider the special operator $(1+\Delta)^k$ with symbol $\left(1+|\xi|^2\right)^k$. Since the norm $\|(1+\Delta)^k u\|_s$ is equivalent to the norm $\|u\|_{s+2k}$, the operator

$$(1+\Delta)^k : H_{s+2k}(\xi) \longrightarrow H_s(\xi)$$

is an isomorphism. Then the operator

$$A = (1+\Delta)^{-k}\sigma(D)(1+\Delta)^k : H_{s+2k}(\xi_1) \longrightarrow H_s(\xi_1) \longrightarrow$$
$$\longrightarrow H_{s-m}(\xi_2) \longrightarrow H_{s+2k-m}(\xi_2)$$

differs from $\sigma(D)$ by a compact operator and therefore has the same index. ∎

Corollary 2 *The kernel of an elliptic operator $\sigma(D)$ consists of infinitely smooth sections.*

Proof.
Indeed, by increasing the number s we have a commutative diagram

$$\begin{array}{ccc} H_s(\xi_1) & \xrightarrow{\sigma(D)} & H_{s-m}(\xi_2) \\ \uparrow \cup & & \uparrow \cup \\ H_{s+1}(\xi_1) & \xrightarrow{\sigma(D)} & H_{s+1-m}(\xi_2) \end{array} \tag{155}$$

An increase in the number s can only decrease the dimension of the kernel, **Ker** $\sigma(D)$. Similarly, the dimension of the cokernel may only decrease since the cokernel is isomorphic to the kernel of the adjoint operator $\sigma(D)^*$. Since the index does not change, the dimension of kernel cannot change. Hence the kernel does not change in the diagram (155). Thus the kernel belongs to the $\bigcap_s H_s(\xi_1) = \Gamma^\infty(\xi_1)$.

8.2 The Atiyah–Singer formula for the index of an elliptic operator

In the sense explained in previous sections, an elliptic operator $\sigma(D)$ is defined by a symbol

$$\sigma : \pi^*(\xi_1) \longrightarrow \pi^*(\xi_2) \tag{156}$$

which is an isomorphism away from a neighborhood of the zero section of the cotangent bundle T^*M. Since M is a compact manifold, the symbol (156) defines a triple $(\pi^*(\xi_1), \sigma, \pi^*(\xi_2))$ which in turn defines an element

$$[\sigma] \in K_c(T^*M).$$

Theorem 36 *The index* **index** $\sigma(D)$ *of the Fredholm operator $\sigma(D)$ depends only on the element $[\sigma] \in K_c(T^*M)$.*

The mapping

$$\textbf{index} \ : K_c(T^*M) \longrightarrow \mathbf{Z}$$

is an additive homomorphism.

Proof.
A homotopy of an elliptic symbol gives a homotopy of Fredholm operators and, under homotopy, the index does not change. Assume that the symbol σ is an isomorphism not only away from zero section but everywhere. The operator $\sigma(D)$ can be decomposed into a composition of an invertible operator $(1+\Delta)^{m/2}$ and a operator $\sigma_1(D)$ of the order 0. The symbol σ_1 is again invertible everywhere and is therefore homotopic to a symbol σ_2 which is independent of the cotangent vector and also invertible everywhere. Then the operator $\sigma_2(D)$ is multiplication by the invertible function σ_2. Therefore, $\sigma_2(D)$ is invertible. Thus

$$\mathbf{index}\ \sigma(D) = 0.$$

Finally, if $\sigma = \sigma_1 \oplus \sigma_2$ then $\sigma(D) = \sigma_1(D) \oplus \sigma_2(D)$ and hence

$$\mathbf{index}\ \sigma(D) = \mathbf{index}\ \sigma_1(D) + \mathbf{index}\ \sigma_2(D).$$

∎

The total space of the cotangent bundle T^*M has a natural almost complex structure. Therefore, the trivial mapping $p : M \longrightarrow \mathrm{pt}$ induces the direct image homomorphism

$$p_* : K_c(T^*M) \longrightarrow K_c(\mathrm{pt}) = \mathbf{Z}.$$

Theorem 37 *Let $\sigma(D)$ be an elliptic pseudodifferential operator. Then*

$$\mathbf{index}\ \sigma(D) = p_*[\sigma]. \tag{157}$$

The formula (157) can be written as

$$\mathbf{index}\ \sigma(D) = \big(\mathbf{ch}\ [\sigma]T^{-1}(T^*M), [T^*M]\big)$$

where $[T^*M]$ is the fundamental (open) cycle of the manifold T^*M. For an oriented manifold M we have the Thom isomorphism

$$\varphi : H^*(M) \longrightarrow H^*_c(T^*M). \tag{158}$$

Therefore, the formula (158) has the form

$$\mathbf{index}\ \sigma(D) = \big(\varphi^{-1}\mathbf{ch}\ [\sigma]T^{-1}(T^*M), [M]\big) \tag{159}$$

The formula (159) was proved by M.F.Atiyah and I.M.Singer [**?**] and is known as the *Atiyah–Singer formula*.

Proof.
The proof of the Atiyah–Singer formula is technically complicated. Known proofs are based on studying the algebraic and topological properties of both the left and right hand sides of (157). In particular, Theorem 36 shows that both the left and right hand sides of the (157) are homomorphisms on the group $K_c(T^*M)$. Let $j : M_1 \longrightarrow M_2$ be a smooth inclusion of compact manifolds and let σ_1 be an elliptic symbol on the manifold M_1. Assume that σ_2 is a symbol on the manifold M_2 such that

$$[\sigma_2] = j_*[\sigma_1]. \tag{160}$$

Theorem 38 *If two symbols σ_1 and σ_2 satisfy the condition (160) then the elliptic operators $\sigma_1(D)$ and $\sigma_2(D)$ have the same index,*

$$\textbf{index}\ \sigma_1(D) = \textbf{index}\ \sigma_2(D).$$

Theorem 37 follows from Theorem 38. In fact, Theorem 37 holds for $M = \mathrm{pt}$. Then the inclusion

$$j : \mathrm{pt} \longrightarrow \mathbf{S}^n.$$

gives the direct image

$$j_* : K_c(\mathrm{pt}) \longrightarrow K_c\left(T^*\mathbf{S}^n, T^*_{s_0}\right)$$

which is an isomorphism.Therefore, for a symbol σ defining a class

$$[\sigma] \in K_c\left(T^*\mathbf{S}^n, T^*_{s_0}\right)$$

we have

$$\begin{aligned}&[\sigma] = j_*[\sigma'],\ [\sigma'] = p_*[\sigma],\\ &\textbf{index}\ \sigma(D) = \textbf{index}\ \sigma'(D) = [\sigma'] = p_*([\sigma]).\end{aligned}$$

Finally, let $j : M \longrightarrow \mathbf{S}^n$ an inclusion and $q : \mathbf{S}^n \longrightarrow \mathrm{pt}$ be the natural projection. Then

$$p = qj,\ j_*([\sigma]) = [\sigma'],$$

and by Theorem 38

$$\textbf{index}\ \sigma(D) = \textbf{index}\ \sigma'(D) = q_*[\sigma'] = q_* j_*[\sigma] p_*[\sigma].$$

Thus the Atiyah–Singer formula follows from Theorem 38.

Let us explain Theorem 38 for the example when the normal bundle of the inclusion $M_1 \longrightarrow M_2$ is trivial. The symbol $[\sigma_2] = j_*[\sigma_1]$ is invertible outside of a neighborhood of submanifold $M_1 \subset T^*M_2$. Suppose that the symbols are of order 0. Then the symbol σ_2 can be chosen so that it is independent of the cotangent vector outside of a neighborhood U of submanifold M_1. Then we can chose the operator $\sigma_2(D)$ such that if $\textbf{supp}\ \cap U = \emptyset$ then $\sigma_2(D)(u)$ is multiplication by a function. Hence we can substitute for the manifold M_2, the Cartesian product $M_1 \times T^k$, where T^k is torus, equipped with an elliptic operator of the same index. Now we can use induction with respect to the integer k and that means that it is sufficient consider the case when $k = 1$.

The problem is now reduced to the existence of an elliptic operator $\tau(D)$ on the circle such that

$$\textbf{index}\ \tau(D) = 1$$

and

$$[\tau] \in K_c\left(T^*\mathbf{S}^1,\ T^*_{s_0}\right)$$

is a generator.

8.3 Signature of manifolds

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