ON FREDHOLM REPRESENTATIONS OF DISCRETE GROUPS

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The so-called Fredholm representations of the group π were investigated in [1]. In particular, an invariant π was constructed for each Fredholm representation T of the group $\xi_T \in K(B\pi)$. The natural question arises: how large is the subgroup of those elements of the group $K(B\pi)$ which have the form ξ_T for some Fredholm representation T of the group π. The answer to this question was not given in [1], and an indirect method was proposed for the proof of the homotypic invariance of higher signatures for a definite class of groups.

The method of constructing Fredholm representations of the group π which will permit getting rid of continuous families of Fredholm representations is considered herein.

THEOREM 1. Let π be a discrete group and $B\pi$ homotypically equivalent to a compact Riemann manifold with metric of nonpositive curvature over all two-dimensional directions. Then for any vector stratification $\xi \in K(B\pi)$ there exists a Fredholm representation T such that $\xi = \lambda\xi_T$, $\lambda \neq 0$.

Solov'ev [2] first proved Theorem 1 in the case $\pi = \mathbf{Z}^2$.

1. Preliminary Information

Let us recall that a pair of unitary representations T_1, T_2 in the Hilbert spaces H_1, H_2, respectively, with the Fredholm operator $F: H_1 \to H_2$, such that for any element $g \in \pi$ the operator $FT(g) - T(g)F$ is a compact operator, is called a Fredholm representation T of the group π. The element $\xi_T \in K(B\pi)$ is constructed according to the Fredholm representation T. Hence, if the element ξ_T is understood as an equivariant family of Fredholm operators $\Phi_x: H_1 \to H_2$, $x \in \widetilde{B\pi}$, then this family Φ_x is uniquely determined, to homotopy accuracy, by the following condition: the operators $\Phi_x - F$, $x \in \widetilde{B\pi}$, are compact operators. The family Φ_x can here be chosen linear in each simplex of some equivariant simplicial partition of the space $\widetilde{B\pi}$.

Let $T = (T_1, F, T_2)$ be a Fredholm representation such that F is an isomorphism, and for any element $g \in \pi$ the inequality

$$\| 1 - F^{-1} T_2(g) F T_1(g^{-1}) \| < 1. \tag{1}$$

is satisfied. Then the equivariant family Φ_x can be selected such that all the operators Φ_x, $x \in \widetilde{B\pi}$, would be reversible. In this case $\xi_T = 0$.

Furthermore, let T_u, $u \in X$, be a continuous family of Fredholm representations parametrized by the parameter u running over the topological space X. Then the family T_u determines an element $\xi_{T_u} \in K(X \times B\pi)$ where, if the representation T_u satisfies condition (1) for $u \in Y \subset X$, then we obtain the element $\xi_{T_u} \in K(X \times B\pi, Y \times B\pi)$.

Fredholm complexes of representations, i.e., sequences of unitary representations T_i in the Hilbert spaces H_i, $1 \leq i \leq n$, and a system of operators F_i generating the Fredholm complex

$$H_1 \xrightarrow{F_1} H_2 \xrightarrow{F_2} \cdots \xrightarrow{F_{n-1}} H_n,$$

Moscow State University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 9, No, 2, pp. 36-41, April-June, 1975. Original article submitted March 13, 1974.

where all the operators $F_i T_i(g) - T_{i+1}(g) F_i$ are compact, can be considered in place of the Fredholm representations.

The main technical difficulty we must overcome is the construction of the tensor product of two Fredholm representations. Let F_1 and F_2 be two Fredholm representations of the operator

$$F_1 : H_1^1 \to H_2^1, \quad F_2 : H_1^2 \to H_2^2.$$

Let us consider the diagram

$$\begin{array}{ccc} H_1^1 \otimes H_1^2 & \xrightarrow{F_1 \otimes 1} & H_2^1 \otimes H_1^2 \\ {\scriptstyle 1\otimes F_2}\downarrow & & \downarrow{\scriptstyle 1\otimes F_2} \\ H_1^1 \otimes H_2^2 & \xrightarrow[F_1 \otimes 1]{} & H_2^1 \otimes H_2^2 . \end{array}$$

It is clear that the complex of this diagram

$$H_1^1 \otimes H_1^2 \to (H_2^1 \otimes H_1^2) \oplus (H_1^1 \otimes H_2^2) \to H_2^1 \otimes H_2^2 \tag{2}$$

is a Fredholm complex. If a group π acts in the Hilbert spaces H_j^k so that two Fredholm representations are obtained, then complex (2) is not a Fredholm complex of representations if all four spaces H_j^k are infinite.

G. G. Kasparov proposed a corrected construction of the tensor product of Fredholm representations. However, it is not at all suitable for our needs and we limit ourselves to a narrower class of Fredholm representations by introducing the following constraint. Let H be a Hilbert space and U a positive self-adjoint compact operator, where the space $H^\infty = \bigcap_{s=1}^{\infty} \operatorname{Im} U^s$ is compact everywhere in the space H. Let T be a unitary representation of the group π in the space H and let the condition

$$\| U^s T(g) \xi \| \leqslant C_s \| U^s \xi \|, \quad \| (U^s T(g) - T(g) U^s) \xi \| \leqslant C_s \| U^{s+1} \xi \| \tag{3}$$

be satisfied for any integer s, $-\infty < s < \infty$, $\xi \in H^\infty$. Let F: H → H be an (unbounded) operator, let the space H^∞ lie in its domain of definition

$$\| U^s F \xi \| \leqslant C_s \| U^{s-1} \xi \|, \tag{4}$$

$$\| U^s (FT(g) - T(g) F) \xi \| \leqslant C_s \| U^s \xi \|, \tag{5}$$

and let there exist an operator G such that

$$\| U^s (FG - 1) \xi \| \leqslant C_s \| U^{s+1} \xi \|, \tag{6}$$

$$\| U^s (GF - 1) \xi \| \leqslant C_s \| U^{s+1} \xi \|, \quad \| U^s G \xi \| \leqslant C_s \| U^{s+1} \xi \|. \tag{7}$$

We easily determine the Fredholm representation of the group π in the set (U, F, T). To do this let us introduce a norm scale in the space H^∞ by assuming

$$\| \xi \|_s = \| U^{-s} \xi \|, \tag{8}$$

and let H^s denote the completion of the space H^∞ in the norm $\| \xi \|_s$. Then $H^0 = H$, and since

$$\| \xi \|_{s-1} \leqslant C \| \xi \|_s, \tag{9}$$

the identical mapping of the space H^∞ is continued to the imbedding

$$i : H^s \subset H^{s-1}. \tag{10}$$

Condition (4) means that the operator F is continued to a continuous operator

$$F : H^s \to H^{s-1}, \tag{11}$$

and it follows from conditions (6) and (7) that the operator (11) is a Fredholm operator. Furthermore, let us set

$$T_s = U^s T U^{-s}. \tag{12}$$

The operators T_s are bounded, and

$$\| T_s \xi \|_s = \| \xi \|_s, \tag{13}$$

and the commutator $FT_S - T_{S-1}F: H^S \to H^{S-1}$ is compact.

2. Finite Representations

Let us consider the case of the family T_X of finite representations of a group π, parametrized by points x of a compact manifold M. Let $\xi \in K(M \times B\pi)$ be the stratification corresponding to the family T_X. Let $\mathrm{ch}\,\xi = \sum a_i \otimes b_i$, where $\{a_i\}$ is the basis in the cohomologies $H^*(M; Q)$, $b_i \in H^*(B\pi; Q)$. Let D be an elliptic pseudodifferential operator acting on the manifold M in the stratification sections η_1 and η_2, i.e., $D: \Gamma(\eta_1) \to \Gamma(\eta_2)$.

Then the symbol $\sigma(D)$ of the operator D is the homomorphism $\sigma(D): q^*(\eta_1) \to q^*(\eta_2)$, where $q: T^*M \to M$ is the projection of the cotangential stratification to the manifold M in the manifold M. Here, if $\xi \in T^*M$, $\xi \neq 0$, then $\sigma(D)_\xi$ is an isomorphism. Let $D \otimes \xi$ denote the family of pseudodifferential operators parametrized by points of the space $B\pi$, whose symbol equals

$$\sigma(D) \otimes 1 : q^*(\eta_1 \otimes \xi) \to q^*(\eta_2 \otimes \xi). \tag{14}$$

The family $D \otimes \xi$ is restored uniquely by its symbol (14) to the accuracy of lesser order operators. It follows directly from the construction of the operator by means of its symbol that we obtain an equivalent family of elliptic operators over the universal covering $\widetilde{B\pi}$, which differ from each other at diverse points by a lesser order operator.

Therefore, family (14) defines an element of the group $K(B\pi)$, which is interpreted as a generalized analytical index $\mathrm{ind}_a(D \otimes \xi)$ of the family from the elliptic operator viewpoint. It follows from [3] that

$$\mathrm{ind}_a(D \otimes \xi) = p_!(\sigma(D) \otimes 1) \in K(B\pi), \tag{15}$$

where $p: T^*M \times B\pi \to B\pi$ is a natural projection.

On the other hand, by selecting the operator D such that the element $\mathrm{ch}(\sigma(D))$ would be a dual element $a_i \in H^*(M)$, i.e., $\mathrm{ch}(\sigma(D))\, a_j = \lambda\delta_{ij}u$, $\lambda \neq 0$, where $u \in H^{2n}(T^*M; Q)$ is a fundamental cocycle, we obtain

$$\mathrm{ch}\,\mathrm{ind}_a(D \otimes \xi) = \mathrm{ch}\,p_!(\sigma(D) \otimes 1) = \mathrm{ch}\,p_!(\sigma(D)\,\xi) =$$
$$= p_*(\mathrm{ch}(\sigma(D)\,\xi)\,T(M)) = p_*(\mathrm{ch}(\sigma(D))\cdot\mathrm{ch}\,\xi\cdot T(M)) = \lambda p_*(u) \otimes b_i = \lambda b_i.$$

We can thus formulate the following assertions.

THEOREM 2. Let T_X be a family of finite representations of the group π, parametrized by the space X. Let $\mathrm{ch}\,\xi_{T_x} = \sum a_i \otimes b_i$ $\{a_i\}$ be a basis in the cohomologies $H^*(X; Q)$. Then there are Fredholm representations $T^i = (T_1^i, F^i, T_2^i)$ of the group π such that $\mathrm{ch}\,\xi_{T^i} = \lambda_i b_i$, $\lambda_i \neq 0$.

THEOREM 3. Under the conditions of Theorem 2 let the family T_X satisfy condition (1) for $x \in Y \subset X$, $\{a_i\}$ be a basis in $H^*(X, Y; Q)$, $\mathrm{ch}\,\xi_{T_x} = \sum a_i \otimes b_i$. Then there exist Fredholm representations T^i of the group π such that $\mathrm{ch}\,\xi_{T^i} = \lambda_i b_i$, $\lambda_i \neq 0$.

To prove Theorem 2 it is sufficient to realize a class of cohomologies a_i by a singular smooth closed manifold M, i.e., by a mapping $f: M \to X$ such that $f^*(a_i) = \lambda u \in H^n(M)$, where u is the fundamental cocycle of the manifold M. The mapping f induces a family $P_y = T_{f(y)}$ of finite representations of the group π parametrized by points of the smooth manifold M. It is clear that the mapping $g = f \times 1: M \times B\pi \to X \times B\pi$ transfers the stratification ξ_{T_X} into the stratification ξ_{P_y}. Therefore,

$$\mathrm{ch}\,\xi_{P_y} = \lambda u \otimes b_i + \sum_j v_j \otimes c_j, \quad \dim v_j < n.$$

Hence, the assertion of Theorem 2 is reduced to the case when the domain of variation of the parameter is a smooth manifold.

In the case of Theorem 3, let us realize the class of cohomologies a_i by a singular smooth manifold M with boundaries ∂M, and let us construct the elliptic operator D analogously to [4] by a symbol which is constant in the neighborhood of the boundary ∂M. If $\sigma: \pi^*(\eta_1) \to \pi^*(\eta_2)$ is a homogeneous symbol of degree 0, then the pseudodifferential operator D of order 0 is restored by the symbol ∂M and satisfies the following additional condition: there exists a neighborhood $U \supset \partial M$ such that if $f \in \Gamma(\eta_1)$, $\mathrm{supp}\,f \subset U$, then $Df = f$, but if $\mathrm{supp}\,f \cap \partial M = \phi$, then $\mathrm{supp}\,Df \cap \partial M = \phi$. If D' is another such operator, then the difference $D - D'$ is a pseudodifferential operator of order (−1), where $\mathrm{supp}\,(D - D')f \cap \partial M = \phi$ for any section f. The formula for the Atiyah−Zinger index

$$\mathrm{ind}_a D = p_!(\sigma)$$

is valid for such a class of operators, where $p: T^*M \to \{pt\}$ is the mapping at a point and σ simultaneously denotes an element from the group $K(T^*M, \partial T^*M)$. To prove the formula presented it is sufficient to continue the operator D in the second copy of the manifold M glued to M along the boundary, by using an identity operator, and to note that the index of the operator D is not changed in this procedure.

Using a group of characters of the free Abelian group $\pi = Z^n$ (see [5]), we obtain from Theorem 2 the following corollary.

COROLLARY. Every class of even cohomologies $a \in H^{2*}(T^n; Q)$ of the n-dimensional torus T^n can be represented as $a = \lambda \mathrm{ch}\xi_T$, $\lambda \neq 0$, for some Fredholm representation T.

Solov'ev [2] first established this result for the case n = 2.

3. Infinite Representations

Now, let us consider the family of Fredholm representations defined at the end of Sec. 1., i.e., the system (U, F, T) satisfying conditions (3)-(7) and parametrized by points of the manifold M.

Exactly as in Sec. 2, let us consider the elliptic operator D (of order 1) in the manifold M. Let us assume that the families $F_x T_x$ are smooth, i.e., their derivatives also satisfy conditions (3)-(7).

Therefore, if $D: \Gamma(\eta_1) \to \Gamma(\eta_2)$, then we obtain the diagram

$$\begin{array}{ccc} \Gamma(\eta_1 \otimes H) & \xrightarrow{D\otimes 1} & \Gamma(\eta_2 \otimes H) \\ {\scriptstyle 1\otimes F_x}\uparrow & & \uparrow{\scriptstyle 1\otimes F_x} \\ \Gamma(\eta_1 \otimes H) & \xrightarrow[D\otimes 1]{} & \Gamma(\eta_2 \otimes H), \end{array}$$

where $D \otimes 1$ is understood as a pseudodifferential operator with the symbol $\sigma(D) \otimes 1$ (see [6]).

Let Δ be the Laplace operator in the stratification sections η_1 and η_2. Let us introduce a norm in the space $\Gamma(\eta_k \otimes H)$ by assuming

$$\|u\|_s^2 = \int \|(\Delta^{1/2} + U^{-1})^s u\|^2 d\mu, \tag{16}$$

where μ is a smooth measure on the manifold M, induced by a Riemann lattice, and the norm under the integral sign is understood in the sense of a norm in the stratification layer $\eta_k \otimes H$.

Let H_k^s be the completion of the space $\Gamma(\eta_k \otimes H)$ in norm (16). Then the diagram

$$\begin{array}{ccc} H_1^s & \xrightarrow{D\otimes 1} & H_2^{s-1} \\ {\scriptstyle 1\otimes F_x}\uparrow & & \uparrow{\scriptstyle 1\otimes F_x} \\ H_1^{s+1} & \xrightarrow[D\otimes 1]{} & H_2^s \end{array} \tag{17}$$

consists of bounded operators and is commutative to the accuracy of compact operators. In fact, the second assertion follows from the circumstance that the commutator of diagram (17) can be interpreted as a zero order pseudodifferential operator with a compact symbol.

The symbol of diagram (17) is a Fredholm symbol for each value of the tangential vector to the manifold M since the vertical operators are Fredholm operators. If $\xi \in T^*M$, $\xi \neq 0$, then since Ker F_x and Coker F_x lie in $H^\infty \subset H^s$, the horizontal operators realize the isomorphisms

$$\sigma(D) \otimes 1 : q^*\eta_1 \otimes \mathrm{Ker}\, F_x \to q^*\eta_2 \otimes \mathrm{Ker}\, F_x,$$
$$\sigma(D) \otimes 1 : q^*\eta_1 \otimes \mathrm{Coker}\, F_x \to q^*\eta_2 \otimes \mathrm{Coker}\, F_x.$$

Therefore, diagram (17) forms an elliptic complex.

Finally, the commutators of the operators of diagram (17) with the representation T of the group π are compact operators.

Just as in Sec. 2, we obtain the following assertion.

THEOREM 4. Let $T = (U, F_X, T_X)$ be a family of Fredholm representations of the group π parametrized by points of the space X and satisfying condition (1) in the subspace $Y \subset X$. Let $\{a_i\}$ be a basis in the cohomologies $H^*(X, Y; Q)$, $\mathrm{ch}\, \xi_T = \sum a_i \otimes b_i$, $b_i \in H^*(B\pi; 0)$.

Then Fredholm representations T^i of the group π exist such that $\mathrm{ch}\, \xi_{T^i} = \lambda_i b_i$, $\lambda_i \neq 0$.

4. Proof of Theorem 1

Let us take a family of Fredholm complexes constructed in Sec. 6 of [1] as the family of Fredholm representations T_x, $x \in T^*B\pi$. It has been proved in [1] that the family T_X satisfies condition (1) for sufficiently large, in the norm, cotangent vectors $\xi \in T^*B\pi$, and $\mathrm{ch}\, T_x = \sum a_i \otimes b_i$, where $\{b_i\}$ is the dual basis to the basis in the space $T^*B\pi$.

It is sufficient for us to verify compliance with conditions (3)-(7). Let us recall the construction of the family of Fredholm complexes of the representations of the group π. The space $\widetilde{T^*B\pi}$ is a Riemann manifold, diffeomorphic to the Euclidean space in which we introduce the coordinates (x, ξ), $x \in \widetilde{B\pi}$, ξ is the cotangent vector. Let η be the complexification of the cotangent stratification to the manifold $\widetilde{B\pi}$, raised to the space $\widetilde{T^*B\pi}$, $\Lambda^i\eta$ are the external degrees of the stratification η. Let us consider the complex

$$\Lambda : \Lambda^0\eta \xrightarrow{a_0} \Lambda^1\eta \xrightarrow{a_1} \cdots \xrightarrow{a_{n-1}} \Lambda^n\eta$$

($\dim B\pi = n$), for which the homomorphisms a_k are determined at the point (x, ξ) as external multiplication by the vector $(i\xi + \omega(x))$, and $\omega(x) = \sum x^i dx^i / \sqrt{1 + |x|^2}$.

Finally, the family of Fredholm complexes H is defined as a stratification over $T^*B\pi$, obtained from a complex by using the direct image for the projection $p : \widetilde{T^*B\pi} \to T^*B\pi$,

$$H : H_0 \xrightarrow{A_0} H_1 \xrightarrow{A_1} \cdots \xrightarrow{A_{n-1}} H_n, \quad H_k = p_!(\Lambda^k\eta), \qquad A_k = p_!(a_k).$$

As the operator U we consider an operator obtained as the direct image of the operator of multiplication by the function $1/[1 + \rho(x, 0)]$, where $\rho(x, 0)$ is the range from the point 0 to the point x along the geodesic. Condition (3) follows from the fact that $\rho(gx, 0) \leqslant \rho(x, 0) + \rho(0, g0)$.

The conditions

$$\|U^s A_k \xi\| \leqslant C \|U^s \xi\|, \ \|U_\xi^s (A_k T(g) - T(g) A_k) \xi\| \leqslant C \|U^{s+1} \xi\|$$

are valid in place of conditions (4) and (5). Therefore, for (4) and (5) to be satisfied, the substitution $A_k \to U^{-1}A_k$ is necessary. Theorem 1 is proved completely.

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