INVOLUTION OF MANIFOLDS WITH A SET OF FIXED POINTS DIFFEOMORPHIC TO REAL PROJECTIVE SPACE

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It is proved that if, on a manifold with an involution, the subset of fixed points is diffeomorphic to an even-dimensional real projective space, then the manifold is bordant to the complex projective space in the class of nonoriented bordisms.

In the classification of smooth periodic transformations of manifolds, it is important to investigate the relation between various invariants of the manifold and the structure of the set of fixed points of the transformation. An effective method of doing this is to apply bordism theory for the description of the set structure of the fixed points. The present note gives an illustration of the application of bordism theory to a problem concerning fixed points of an involution.

We start with a description of the situation we shall study. All manifolds are assumed to be unoriented. We say that a manifold M with an involution is given if we have a smooth diffeomorphism $T: M \to M$, $T^2 = \mathrm{id}$. Two closed manifolds M and M' with involution are bordant if there is a manifold with a border W with an involution such that the boundary ∂W is equivalently diffeomorphic to the unconnected sum $M \cup M'$. If M is a manifold with an involution T, then the set N of fixed points of T, i.e., those points x such that $Tx = x$, is a finite union of submanifolds $N = \cup N_i$. A tubular neighborhood U_i of a submanifold N_i is diffeomorphic to the space of the normal vector fibering η_i of the imbedding of N_i in M, and the involution T induces a Z_2-fibering structure in this fibering. The bordism class with involution M determined the formal sum of bordisms (also nonoriented) of real fiberings $\Sigma[\eta_i]$ and is completely determined by this formal sum.

We are interested in the following problem. Let the set N of fixed points of the involution T consist of a single connected component and be diffeomorphic to the real projective space RP^{2n}. What can be said concerning the manifold M itself? An example of such an involution is the complex-conjugate transformation of coordinates in a complex projective space CP^{2n}. Conner and Floyd ([1], p. 228) assume that this is a unique situation; i.e., if the set N of fixed points of a manifold M with involution T is diffeomorphic to RP^{2n}, then M is bordant to the manifold CP^{2n} with the above involution. It is sufficient to prove that the normal fibering η of the imbedding of N in M is bordant to the normal fibering of the imbedding of RP^{2n} in CP^{2n}. Conner and Floyd [1] prove that dim $M = 4n$, and the fibering η is stably isomorphic to the fibering $(2s+1)\xi$ $(s \geq n)$, where ξ is the one-dimensional Hopf fibering over RP^{2n}.

The group of (unoriented) bordisms of real fiberings $\oplus\mathfrak{N}_*(BO(n))$ has a ring structure if we take multiplication to be the cartesian product of bases and the direct sum of fiberings. This ring is isomorphic to the ring of polynomials $\mathfrak{N}_*[x_n]$, where the x_n are one-dimensional Hopf fiberings over RP^n. In particular x_0 is the one-dimensional (trivial) fibering over one-point space.

THEOREM. If M^{4n} is a manifold with involution whose set of fixed points is diffeomorphic to the real projective space RP^{2n}, then the normal fibering of the imbedding of RP^{2n} in M^{4n} is bordant to the tangent fibering of RP^{2n}, and a manifold with involution M^{4n} is bordant to the complex projective space CP^{2n}, on which the involution is the taking of the complex conjugate of homogeneous coordinates.

Proof. Each vector fibering η over RP^{2n} is equal to $k\xi + N$. On the other hand each fibering η can be expressed in the form $\eta' + N$, where dim $\eta' = 2n$. Hence a bordism y representable by η can be written

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$y = y_0(x_0)^N$. Let y_k be the bordisms represented by the fiberings $k\xi$, $y_k = y_{0k}(x_0)^{k-2n}$. Let k_0 be the smallest number such that the complete Stiefel class is equal to unity: $w(k_0\xi) = 1$. Then the bordisms $\{y_{0k}\}$ $(1 \le k < k_0)$ run over the whole set of bordisms $\{y_{0k}\}$. We must prove that $\alpha(y_{0k}) \ne 0$ for $k \ne 2n+1$, where $\alpha : \mathfrak{N}_k(BO(n)) \to \mathfrak{N}_{n+k-1}(RP^\infty)$. This is equivalent to the fact that $\alpha(y_k)$ is a polynomial of degree higher than $k-2n$ for $k \ne 2n+1$. The element $\alpha(y_k)$ is represented by the projection P_k of the fibering $k\xi$ over RP^{2n} with the canonical fibering η imposed on it. We have to prove that, for $k \ne 2n+1$, there is a number $s > k-2n$ and a Stiefel class w of the manifold P_k such that $\langle c^s w, P_k\rangle \ne 0$.

The cohomologies of P_k form the ring

$$H^*(P_k, Z_2) = Z_2[u, c] / \{(c+u)^k, u^{2n+1}\}.$$

The complete Stiefel class of tangent fibering is equal to $w = (1+u)^{2n+1}(1+c+u)^k$ (cf. [1], Sec. 24).

Let r be the smallest number of which $2^r > 2n$. It is easily seen that, if $2n+1 \le k < 2n+1+2^r$, then y_{0k} runs through the whole set of bordisms $\{y_{0k}\}$. We shall prove that there are i, j, $i+j = 2n+k-1$, $k-2n+1 \le j \le k-2$, $2n+1 \le j$, such that $\langle w_{(i)}c^j, P_k\rangle \ne 0$, where $w_{(i)}$ is the Newton generator of the characteristic classes of tangent fibering. Recalling that k is odd, we have $w_{(i)} = u^i + (u+c)^i$. Since $i = 2n + k - 1 - j \ge 2n+1$, then $w_{(i)} = (u+c)^i$. Hence $w_{(i)}c^j = (u+c)^i c^j = (u+c)^k u^{2n} C_j^{2n}$. We take a number j such that $j \ge k-2n+1$, $j \ge 2n+1$, $j \le k-2$ for odd $k(2n+1+2^r > k > 2n+1)$ and $C_j^{2n} = 1$. Such a number exists. Let $2n$ and k have the binary representations

$$2n = 1\alpha_{s-1}\ldots\alpha_1 0,\quad \alpha_i = 0,1,\quad k = \beta_{s+1}\beta_s\ldots\beta_s 1.$$

Here $\beta_{s+1} = 1$ or $\beta_{s+1} = 0$, $\beta_s = 1$. Consider the following two cases. 1) $\beta_{s+1} = 1$. Then $m = \beta_s \ldots \beta_1 1 < 2n$. Let $\beta_s = \alpha_s, \ldots, \beta_t = \alpha_t$, $\beta_{t-1} = 0$, $\alpha_{t-1} = 1$ for some number t. Then we must take $j = \beta_s \ldots \beta_t \alpha_{t-1} \ldots 1$. Plainly $j \ge 2n+1$, $j \ge k-2n+1$, $j \le k-2$ and $C_j^{2n} = 1$. 2) $\beta_{s+1} = 0$, $\beta_s = 1$. Let $\beta_s = \alpha_s$, $\beta_{s-1} = \alpha_{s-1}, \ldots, \beta_t = \alpha_t$, $\alpha_{t-1} = 0$, $\beta_{t-1} = 1$. Set $j = \alpha_s \ldots \alpha_t 1 \alpha_{t-2} \ldots \alpha_1 0$. Then $j \ge 2n+1$, $j \ge k-2n+1$, $j \le k-2$, and $C_j^{2n} = 1$, and the theorem is proved.

Remark. For each manifold M^n, there is a manifold with involution N^{2n} such that M^n is its set of fixed points. For N^{2n} we must take $M^n \times M^n$, and the involution T is defined by $T(x, y) = (y, x)$. Then the normal fibering of the imbedding of M^n in N^{2n} is isomorphic to the tangent fibering of M^n. The question arises of whether there are other manifolds with involution N' having a set of fixed points diffeomorphic to the fixed manifold M^n. There is a theorem which asserts that, if $M^n = RP^{2k}$, then N' is bordant to N. However when $M^n \ne RP^{2k}$ this is not in general true. For example if $M^5 = (RP^4 + RP^2 \times RP^2)RP^1$, besides then the manifold with involution $M^5 \times M^5$, there is another manifold with involution of dimension 8, bordant to the manifold $(RP^2)^4 + (RP^4)^2$ (and not bordant to $M^5 \times M^5$), whose fixed points are bordant to M^5. Calculations with examples of this type reduce to the methods used in [2] and [3].

LITERATURE CITED

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