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# HOMOTOPY INVARIANTS OF NONSIMPLY CONNECTED MANIFOLDS. I RATIONAL INVARIANTS

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Abstract. Invariants of nonsimply connected manifolds are investigated of the same type as the signature of a manifold, or of "higher signature" for manifolds with free Abelian fundamental group, which correspond to the possibility of surgery on a manifold. It is proved that these invariants depend only on the homotopy type of the manifold and a bordism class of manifolds.

## Introduction

We begin with a description of a typical problem connected with the Morse modifications. Let $f: M_1^n \to M_2^n$ be a map of degree 1 between closed (or with boundary) re-dimensional manifolds, let $\eta$ be a fiber bundle over the manifold $M_2^n$ and $F$ a stable fiber bundle trivialization $\nu(M_1^n) \oplus f^*(\eta)$, where $\nu(M_1^n)$ is the normalbundle of the manifold $M_1^n$. In this situation the question arises as to whether there exist a manifold with boundary $W^{n+1}$, $\partial W^{n+1} = M_1^n \cup \overline{M}_1^n$, a map $g: W^{n+1} \to M_2^n$ extending the map $f$ and a stable fiber bundle trivialization $G: \nu(W^{n+1}) \oplus g^*(\eta)$ extending the trivialization $F$ such that the map $g|\overline{M}_1^n$ is a homotopy equivalence. In [1] Wall studies the obstruction to the existence of such manifolds with boundary, or, as they say, obstructions to surgery on a manifold (see also [2], [3] and [4]). For each fundamental group $n$ one constructs groups $L_n(\pi)$. The obstruction to surgery $\alpha(M_1^n, M_2^n, f, F)$ is an element of the group $L_n(\pi)$, and, moreover, a necessary and sufficient condition for the existence of surgery on the manifold $M_1^n$ is that the element $\alpha(M_1^n, M_2^n, f, F)$ be zero. Note further that the element $\alpha(M_1^n, M_2^n, f, F)$ depends only on the "bordism" class of the manifold $M_2^n$ defined by the triple $(M_1^n, f, F)$.

In fact all problems about Morse modifications can be put in the form indicated above.

In the case of a simply connected manifold the groups $L_n(1)$ were described a long time ago ([5], [6]); they $L_{4k}(1) = \mathbf{Z}$, $L_{4k+2}(1) = \mathbf{Z}_2$ d $L_{2k+1}(1) = 0.$ e we will be interested in the following situation. As is well known, the group $L_{4k}(\pi)$ is the Grothendieck group generated by the nondegenerate symmetric scalar products on free modules over the group ring $\mathbf{Z}[\pi]$. In the case $\pi = 1$ the unique invariant of such scalar products is the signature, i.e. the difference in the dimensions of the subspaces where the scalar product is positive definite and negative definite. Thus when studying the obstruction to surgery of the map

$$f: M_1^n \longrightarrow M_2^n$$

and the stable trivialization *F:* ^Ш$_1$") ®*f*(rj)* the obstruction is defined as the signature of the scalar product on the group $\mathrm{Ker}(H_*(M_1^n) \to H_*(M_2^n))$. Here the homology can be considered with rational coefficients. On the other hand the signature $\sigma(M)$ can be defined for each manifold separately when studying the scalar product on its homology groups with rational coefficients. Then

$$\sigma(M_1^n, M_2^n, f, F) = \sigma(M_1^n) - \sigma(M_2^n).$$

This result explains to what degree the o b s t r u $\alpha(M_1^n, M_2^n, f, F)$ e n d s on the different elements $M_1^n, M_2^n, f, F$.

The aim of the present work is to investigate the degree to which the obstruction $\alpha(M_1^n, M_2^n, f, F) \in L_n(\pi)$ depends on the manifolds themselves, the map $f$ and the trivialization $F$.

To each manifold $M^n$ we associate an element $\sigma(M^n) \in L_n(\pi)$, $\pi = \pi_1(M^n)$. It will be shown that

1. α(*M], M\, {, F)=a{M'|)-a{M^.*
2. $\sigma(M^n)$ is a homotopy invariant.
3. $\sigma(M^n)$ is a bordism invariant of the space $K(\pi, 1)$ defined by the manifold $M^n$.

In the first part of the paper we study the construction of an invariant in the group $L_n^Q(\pi)$, analogous to the Wall group $L_n(\pi)$, in which in place of the group ring $Z[\pi]$ of the group *n* we take the group ring $Q[\pi]$ with rational coefficients; more precisely, with dyadic rational coefficients. These groups correspond to surgery modulo elements of finite order in the homotopy groups of the manifolds. On the other hand, reflecting the general idea of the method, the consideration of rational coefficients technically simplifies the construction.

In the second part of the paper we give a complete description of the invariant $\sigma(M)$ and consider a number of special cases. For example, it will be shown that on smooth manifolds with fundamental group $Z_p$ and dimension $4k-1$, $k \geq 2$, there exist a finite number of pairwise nonequivalent smooth structures. It is clear also that by examining the method it is possible to answer the question on the number of distinct free actions of finite groups on manifolds.

I would like to note that in writing the present article my valuable discussions with S. P. Novikov had a most stimulating significance, and so also did my joint work with I. M. Gel'fand in studying quadratic forms over rings of continuous functions.

## §1. Algebraic Poincare'complexes

Let *л* be a finitely generated group, and Л the group ring over the group *n* with coefficients in the ring of dyadic rational numbers $\mathbf{Z}[1/2]$, i.e. $\Lambda = \mathbf{Z}[1/2][\pi]$. An involution $\Lambda \to \Lambda$ which is generated by the involution in the group $\pi$: $\bar{x} = x^{-1}$ is defined in $\Lambda$. Let $M$ be a right $\Lambda$-module. Denote by $M^*$ the module $\mathbf{Hom}_\Lambda(M, \Lambda)$. If $M$ is a free module, then there exists a natural isomorphism $M \to (M^*)^*$. For any homomorphism $\phi$: $N \to M$ we will denote by $\phi^*$ the dual homomorphism.

Let us consider a complex of free Л-modules $C = \{C_i, d_i\}$ of length $n$:

$$C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} C_2 \leftarrow \ldots \xleftarrow{d_n} C_n.$$

We will say that an algebraic Poincaré complex is given if we are given homomorphisms

$$\xi_i : C^*_{n-i} \to C_i,$$

such that the relations

$$d_i\xi_i = (-1)^i \xi_{i-1}d^*_{n-i+1}, \tag{1}$$

$$\xi_i = (-1)^{(n-i)i} \xi^*_{n-i} \tag{2}$$

are satisfied and moreover the homology homomorphism

$$\xi_* : H(C^*) \to H(C),$$

induced by the homomorphisms $\xi = \{\xi_i\}$, is an isomorphism.

Analogously, considering a complex of free $\Lambda$-modules $C = \{C_i, d_i\}$ of length $n + 1$ and a subcomplex of free $\Lambda$-modules $C^0 = \{C^0_i, d_i\}$ of length $n$ which is a direct summand, we will say that an algebraic Poincare'pair is given if we are given homomorphisms

$$\xi_i : C^*_{n+1-i} \to C_i$$

such that the conditions

$$d_i\xi_i = (-1)^i \xi_{i-1}d^*_{n-i+2} \bmod C^0,$$

$$\xi_i = (-1)^{(n+1-i)i} \xi^*_{n+1-i}$$

are fulfilled and moreover the induced homorphism in homology is an isomorphism between the homology of the complex $C^*$ and the homology of the factor complex $C/C^\circ$:

$$\xi^* : H(C^*) \to H(C/C^0).$$

Lemma 1.1. *The complex $(C^0, d)$ admits a natural Poincare complex structure.*

Proof. Consider the homomorphism

$$\varphi_{i-1} = d_i\xi_i - (-1)^i \xi_{i-1}d^*_{n-i+2}.$$

Let $\beta$: $C \to C/C^0$ be the projection. Then by definition we have $\beta_{i-1}\phi_{i-1} = 0$. Since

$$\begin{aligned}\varphi^*_{i-1} &= \xi^*_i d^*_i - (-1)^i d_{n-i+2}\xi^*_{i-1} \\ &= (-1)^{(n+1-i)i}\xi_{n+1-i}d^*_i - (-1)^{(n+2-i)(i-1)+i} d_{n-i+2}\xi_{n-i+2} \\ &= (-1)^{(n-(i-1))(i-1)}(d_{n-i+2}\xi_{n-i+2} - (-1)^{(n-i+3)}\xi_{n+1-i}d^*_i) \\ &= (-1)^{(n-(i-1))(i-1)}\varphi_{n-(i-1)},\end{aligned}$$

then $\phi_{n-(i-1)}\beta^*_{i-1} = 0$. Consequently the homomorphism $\phi_{i-1}$ defines uniquely the homomorphism

$$\xi^0_{i-1}: C^{0*}_{n-(i-1)} \to C^0_{i-1}.$$

An immediate check confirms that the conditions (1) and (2) are fulfilled. To complete the proof we establish the following lemma. Let $\alpha: C^0 \to C$ be the inclusion homomorphism.

Lemma 1.2. *The diagram*

$$\begin{array}{ccccccc} \to H_i(C^0) & \xrightarrow{\alpha_*} & H_i(C) & \xrightarrow{\beta_*} & H_i(C/C^0) & \xrightarrow{\partial} & H_{i-1}(C^0) \to \\ \uparrow \xi^0_{i*} & & \uparrow (\xi\beta^*)_* & & \uparrow (\beta\xi)_* & & \uparrow (\xi^0_{i-1})_* \\ \to H_{n-i}(C^{0*}) & \xrightarrow{\partial^*} & H_{n+1-i}(C/C^{0*}) & \to & H_{n+1-i}(C^*) & \to & H_{n+1-i}(C^{0*}) \to \end{array}$$

*is commutative. All vertical homomorphisms are isomorphisms.*

Proof. The commutativity of the diagram is obvious. By definition the homomorphism $(\beta\xi)_*$ is an isomorphism. According to condition (2*) the homomorphism $(\xi\beta^*)_*$ is, up to sign, the dual of the homomorphism $(\beta\xi)_*$ and consequently is also an isomorphism. From the Five Lemma the homomorphism $(\xi^0)_*$ is also an isomorphism.

The proof of Lemmas 1.1 and 1.2 is completed.

The Poincaré complex $(C^0, d, \xi^0)$ will be called the boundary of the Poincaré pair $(C, C^0, d, \xi)$.

Let two algebraic Poincaré complexes $\alpha_1 = (C_1, d_1, \xi_1)$ and $\alpha_2 = (C_2, d_2, \xi_2)$ of the same dimension be given. The complex $(C_1 \oplus C_2, d_1 \oplus d_2, \xi_1 \oplus \xi_2)$ will be called the disjoint sum $\alpha_1 \cup \alpha_2$. It is not difficult to confirm that this is a Poincaré complex. By a change of orientation in the Poincaré complex $\alpha = (C, d, \xi)$ we will mean that we are considering the new complex $-\alpha = (C, d, -\xi)$. We will say that two algebraic Poincaré complexes $\alpha_1$ and $\alpha_2$ are bordant if there exists a Poincaré pair $\beta = (C, C^0, d, \xi)$ such that its boundary $\partial\beta = (C^0, d, \xi^0)$ is $\alpha_1 \cup -\alpha_2$.

Lemma 1.3. *Let algebraic Poincaré complexes $\alpha_1$, $\alpha_2$ and $\alpha_3$ be given. If $\alpha_1$ is bordant to $\alpha_2$ and $\alpha_2$ is bordant to $\alpha_3$ then $\alpha_1$ is bordant to $\alpha_3$.*

Proof. Let

$$\beta = (C, C^0, d, \xi), \quad \beta' = (C', C^{0\prime}, d', \xi')$$

be two Poincaré pairs where $C^0 = F_1 \oplus F_2$, $d = d_1 \oplus d_2$, $\xi^0 = \eta_1 \oplus \eta_2$, $C^{0\prime} = F_2 \oplus F_3$, $d' = d_2 \oplus d_3$ and $\xi'^0 = -\eta_2 \oplus \eta_3$. We will construct a new Poincaré pair $\gamma = (A, A^0, \partial, \zeta)$. Put $A = (C \oplus C')/F_2$ and $A^0 = (C^0 \oplus C^{0\prime})/F_2 \oplus F_2$, where $F_2$ is embedded diagonally in $C \oplus C'$. It is clear that the modules $A$ and $A^0$ are free and that $A^0$ is a direct summand in $A$. The differential $\partial$ is by definition induced by the differential $d \oplus d'$ in the module $C \oplus C'$. The homomorphism $\zeta$ is defined as the composition

$$A^* \xrightarrow{\pi^*} (C \oplus C')^* \xrightarrow{\xi \oplus \xi'} (C \oplus C') \xrightarrow{\pi} A,$$

where $\pi: C \oplus C' \to A$ is the natural projection. Let us verify whether the conditions imposed on a Poincare pair are all fulfilled. In the given case condition (1') has the form

$$\partial_i\zeta_i-(-1)^i\,\zeta_{i-1}\partial^*_{n+2-i}\in A^0.$$

In fact, from condition (1') for the Poincaré pairs $\beta$ and $\beta'$ it follows that

$$\zeta^0_{i-1}=\partial_i\zeta_i-(-1)^i\,\zeta_{i-1}\partial^*_{n+2-i}\in A^0\oplus\pi(F'_2\oplus F''_2).$$

However the second component of the homomorphism $\zeta^0_{i-1}$ is obviously zero, because on $F'_2$ it is equal to $\eta_2$ and on $F''_2$ it is equal to $(-\eta_2)$. Condition (2') is automatically fulfilled. It is trivial to check that the homomorphism $\zeta\colon A^*\to A/A^0$ induces an isomorphism in homology.

Lemma 1.4. *An algebraic Poincaré complex* $\alpha$ *is bordant to itself.*

Proof. Let $\alpha=(C,d,\xi)$ be a Poincaré complex. Let us construct a Poincaré pair $\beta=(B,B^0,\delta,\eta)$. Put

$$B_i=C_i\oplus C_{i-1}\oplus C_i,$$

$$B^0_i=C_i\oplus C_i,$$

$$\delta_i=\begin{pmatrix} d_i & (-1)^i & 0\\ 0 & d_{i-1} & 0\\ 0 & (-1)^{i+1} & d_i\end{pmatrix},$$

$$\eta_i=\frac{1}{2}\begin{pmatrix} 0 & (-1)^{n+1}\xi_i & 0\\ (-1)^i\xi_{i-1} & 0 & (-1)^i\xi_{i-1}\\ 0 & (-1)^{n+1}\xi_i & 0\end{pmatrix}.$$

It is not difficult to verify that $\beta$ is a Poincaré pair and

$$\beta^0=(B^0,d,\eta^0)=\alpha\cup(-\alpha).$$

## §2. Modifications of algebraic Poincaré complexes

In this section we give one simple construction on a Poincaré pair analogous to "attaching" handles to a smooth manifold with boundary.

Let an algebraic Poincaré pair $(C,C^0,d,\xi)$ of dimension $n+1$ be given. Let $\pi=\{\pi_j\}$, $\pi_j\colon C^0_j\to C_j$, be the inclusion of the submodule $C^0$ in the module $C$. We fix a certain free module $A$, the number $i$ and a homomorphism $\beta\colon A\to C^*_{n-i}$ such that $d^*_{n-i+1}\beta=0$. We construct a new Poincaré pair $(\overline{C},\overline{C}^0,\delta,\eta)$. Put

$$\overline{C}_j=C_j \text{ when } j\neq i+1,\ n-i-1,\ n-i,$$

$$\overline{C}_j=C_j\oplus A \text{ when } j=i+1,\ n-i-1,\ n-i,$$

$$\overline{C}^0_j=C^0_j \text{ when } j\neq i+1,\ n-i-1,$$

$$\overline{C}^0_j=C^0_j\oplus A \text{ when } j=i+1,\ n-i-1.$$

The inclusion $\pi\colon\overline{C}^0\to\overline{C}$ is defined by

$$\bar{\pi}_{i+1} = \begin{pmatrix} \pi_{i+1} & 0 \\ 0 & 1 \end{pmatrix},$$

$$\bar{\pi}_{n-i-1} = \begin{pmatrix} \pi_{n-i+1} & 0 \\ 0 & 1 \end{pmatrix},$$

$$\bar{\pi}_{n-i} = \begin{pmatrix} \pi_{n-1} \\ 0 \end{pmatrix},$$

$$\bar{\pi}_j = \pi_j \text{ for the remaining numbers } j.$$

We define a homomorphism $\alpha\colon A \to C_i^0$ by

$$\alpha = \xi_i^0 \pi_{n-i}^* \beta.$$

In this case we have the relation

$$\pi_i \alpha = (d_{i+1}\xi_{i+1} - (-1)^{i+1} \xi_i d_{n+1-i}^*)\beta.$$

We define the boundary homomorphism in the following way:

$$\delta_{i+1} = (d_{i+1}, \pi_i\alpha),$$

$$\delta_{i+2} = \begin{pmatrix} d_{i+2} \\ 0 \end{pmatrix},$$

$$\delta_{n-i-1} = (d_{n-i-1}, 0),$$

$$\delta_{n-i} = \begin{pmatrix} d_{n-i} & 0 \\ \beta^* & 1 \end{pmatrix},$$

$$\delta_{n-i+1} = \begin{pmatrix} d_{n-i+1} \\ 0 \end{pmatrix},$$

$$\delta_j = d_j \text{ for all remaining numbers } j.$$

Further, put

$$\eta_{i+1} = \begin{pmatrix} \xi_{i+1} & -\xi_{i+1}\beta \\ 0 & 1 \end{pmatrix},$$

$$\eta_{i+2} = (\xi_{i+2}, 0),$$

$$\eta_{n-i-1} = \begin{pmatrix} \xi_{n-i-1} \\ 0 \end{pmatrix},$$

$$\eta_{n-i} = \begin{pmatrix} \xi_{n-i} & 0 \\ -\beta^*\xi_{n-i} & (-1)^{(n-i)(i+1)} \end{pmatrix},$$

$$\eta_j = \xi_j \text{ for all remaining numbers } j.$$

Special cases occur when $n = 2k + 1$, $i = k - 1$ and when $n = 2k$, $i = k - 1$. In the first case we put

$$\delta_{i+2} = \delta_{n-i-1} = \begin{pmatrix} d_{i+2} & 0 \\ 0 & 0 \end{pmatrix}$$

$$\eta_{i+2} = \eta_{n-i-1} \begin{pmatrix} \xi_{i+2} & 0 \\ 0 & 0 \end{pmatrix}.$$

In the second case we put

$$\bar C_k = C_k \oplus A \oplus A,$$

$$\bar C_{k+1} = C_{k+1} \oplus A,$$

$$\bar C_k^0 = C_k^0 \oplus A \oplus A,$$

$$\delta_k = (d_k, \pi_{k-1}\alpha, 0),$$

$$\delta_{k+1} = \begin{pmatrix} d_{k+1} & 0 \\ 0 & 0 \\ \beta^* & 1 \end{pmatrix},$$

$$\eta_k = \begin{pmatrix} \xi_k & -\xi_k\beta \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$\eta_{k+1} = \begin{pmatrix} \xi_{k+1} & 0 & 0 \\ -\beta^*\xi_{k+1} & 1 & 0 \end{pmatrix}.$$

An immediate check of the conditions (1') and (2') shows that the Poincare pair $(\bar C, \bar C^0, \delta, \eta)$ is well defined. The boundary of this pair $(\bar C^0, \delta, \eta^0)$ is, according to Lemma 1.1, a Poincaré complex. We will say that the complex $(\bar C^0, \delta, \eta^0)$ is obtained as a result of a modification on the complex $(C^0, d, \xi^0)$ with respect to the homomorphism $\pi^*_{n-i}\beta$.

Let us note that modifications can be carried out on Poincaré complexes which are not the boundaries of a Poincaré pair. In addition, modifications can be carried out on Poincaré pairs themselves. Let $(C, C^0, d, \xi)$ be a Poincaré pair of dimension $n$.

Put

$$\bar C_j = C_j \oplus A, \quad j = i+1,\ n-i-1,$$

$$\bar C_j = C_j \oplus A, \quad j = i+1,\ n-i-1,$$

$$\delta_{i+1} = (d_{i+1}, \alpha),$$

$$\delta_{i+2} = \begin{pmatrix} d_{i+2} \\ 0 \end{pmatrix},$$

$$\delta_{n-i-1} = (d_{n-i-1}, \text{[illegible]}$$

where $\beta\colon A \to C^*_{n-i}$ is a homomorphism such that $d^*_{n-i+1}\beta = 0$,

$$\eta_{i+1} = \begin{pmatrix} \xi_{i+1} & 0 \\ 0 & 1 \end{pmatrix},$$

$$\eta_{n-i-1} = \begin{pmatrix} \xi_{n-i-1} & 0 \\ 0 & (-1)^{(n-i+1)(i+1)} \end{pmatrix}.$$

Proposition 2.1. *Let $\alpha$ be an algebraic Poincaré complex and $\bar\alpha$ the result of a modification of it. Then the complex $\alpha$ is bordant to the complex $\bar\alpha$.*

The proposition follows easily from Lemma 1.4.

Proposition 2.2. *Let $\alpha$ be an algebraic Poincare' complex. Then the complex $\alpha$ is* *bordant to another Poincaré complex $\beta$ for which all the homology, with the exception of the homology in the middle dimension {for even n) or in the two middle dimensions {for odd n), is zero. Analogously every Poincare' pair can be modified to a pair for which the homology is zero except in the middle {or the two middle) dimensions.*

Proof. Let $H_i(C)$ be the first nontrivial homology group of the complex $C$, $i < (n-1)/2$. Then the module $\operatorname{Ker} d^*_{n-i+1}$ is a projective module. Consequently there exists an epimorphism $\beta$ from a certain free module $A$ onto the module $\operatorname{Ker} d^*_{n-i+1} \subset C^*_{n-i}$. Let us carry out a modification of the complex $C$ with respect to the homomorphism $\beta$. We obtain a new complex $\bar{C}$ for which $H_j(C) = 0$ for $j \leq i$. The rest of the proof is carried out by induction.

## §3. Homotopically equivalent algebraic Poincare'complexes

Two Poincaré complexes $\alpha = (C, d, \xi)$ and $\alpha' = (C', d', \xi')$ will be called homotopically equivalent if there exists a homomorphism $f: C \to C'$ such that $fd = d'f$ and $\xi' = f \xi f^*$ and moreover $f$ induces an isomorphism in homology.

Lemma 3.1. *Homotopically equivalent algebraic Poincare complexes are cobordant.*

Proof. As in Lemma 1.3, let $\beta = (C, C^0, d, \xi)$ and $\beta' = (C', C^{0\prime}, d', \xi')$ be algebraic Poincare pairs where $C^0 = F_1 \oplus F_2$, $d = d_1 \oplus d_2$, $\xi^0 = \eta_1 \oplus \eta_2$, $C^{0\prime} = F_2' \oplus F_3$, $d' = d_2' \oplus d_3$, $\xi'^0 = -\eta_2' \oplus \eta_3$ and the complexes $(F_2, d_2, \eta_2)$ and $(F_2', d_2', \eta_2')$ are homotopically equivalent, i.e. there exists a homomorphism $f$: $F_2 \to F_2'$ such that $fd_2 = d_2' f$, $\eta_2' = f\eta_2 f^*$. Define the inclusion $i$: $F_2 \to C^0 \oplus C^{0\prime}$ by the formula $i(x) = (x, f(x))$. Put $A = (C \oplus C')/i(F_2)$ and $A^0 = C^0 \oplus C^{0\prime}/(F_2 \oplus F_2')$, and define the homomorphism $\zeta$ as the composition

$$A^* \to (C \oplus C')^* \xrightarrow{\xi \oplus \xi'} (C \oplus C') \to A.$$

The remainder of the proof is analogous to the proof of Lemma 1.3. The existence of Poincaré pairs $\beta$ and $\beta'$ is guaranteed by Lemma 1.4.

Corollary 3.2. *Let the Poincare complex $\alpha = (C, d, \xi)$ have the form $C_i = \bar{C}_i \oplus A$, $C_{i+1} = \bar{C}_{i+1} \oplus A$ and $C_j = \bar{C}_j$ for the remaining numbers $j$. Let*

$$d_{i+1} = \begin{pmatrix} \bar{d}_{i+1} & 0 \\ 0 & 1 \end{pmatrix}.$$

*Then the complex $\bar{\alpha} = (\bar{C}, \bar{d}, \bar{\xi})$ is a Poincare complex bordant to the complex $\alpha$.*

We are now in a position to strengthen Proposition 2.2.

Proposition 3.3- *Let $\alpha$ be an algebraic Poincare complex. Then the complex $\alpha$ is bordant to a Poincare complex $\beta = (C, d, \xi)$ for which the modules $C_i$ are zero except in the middle dimension {for even n) or the two middle dimensions {for odd n).*

The proof springs immediately from Proposition 2.2 and Corollary 3.2.

## §4. The construction of an element of the Wall groups $L_n^Q(\pi)$. Even-dimensional case

Theorem 4.1. *To each algebraic Poincare complex there can be associated an element of the group $L_n^Q(\pi)$ which depends only on the bordism class of the complex.*

Proof. By modifying the complex if necessary, according to Proposition 3.3 we may suppose that there is only one module $C_k$, $n = 2k$, which is not zero. Thus the homomorphism $\xi_k\colon C_k^* \to C_k$ is an isomorphism, where $\xi_k^* = (-1)^k \xi_k$ according to condition (2) of §1. This means that there is a nondegenerate scalar product given on the free module $C_k$ with values in the ring $\Lambda$ which is symmetric or cosymmetric depending on the sign of $(-1)^k$. Consequently we can associate with the complex $\alpha$ an element of the group $L_n^Q(\pi)$. Let us show that this element is correctly defined and does not depend on the representative of the given bordism class. For this it is sufficient to show that if $\alpha = \partial\beta$ then the complex a corresponds to the zero element of the group $L_n^Q(\pi)$.

Thus, let $\alpha = \partial\beta$. Inasmuch as only the module $C_k^0$ in the complex $\alpha$ is nonzero, it is possible to modify the pair $\beta$ so that only the modules $C_k$ and $C_{k+1}$ are nonzero. Let us consider an arbitrary homomorphism of the free module $A$

$$\beta\colon A \to C_{k+1}^*.$$

$$\beta\colon A \to C_{k+1}^*.$$

Put

$$\alpha = \xi_k\beta,$$

$$\overline{C}_k = C_k \oplus A \oplus A,$$

$$\overline{C}_{k+1} = C_{k+1} \oplus A,$$

$$\overline{C}_k^0 = C_k^0 \oplus A \oplus A,$$

$$\overline{C}_{k+1}^0 = 0,$$

Further put

$$\delta_{k+1} = \begin{pmatrix} d_{k+1} & \alpha \\ 0 & 1 \\ \beta^* & 0 \end{pmatrix},$$

$$\eta_k = \begin{pmatrix} \xi_k & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\eta_{k+1} = \begin{pmatrix} \xi_{k+1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is not difficult to verify that we obtain a Poincare pair whose boundary $(\overline{C}_k^0, \eta_k^0)$ defines the same element in the group $L_n^Q(\pi)$ as the complex $(C_k, \xi_k)$. By choosing the homomorphism $\beta$ in a suitable way we can arrange that the homology $H_k(C, C^0)$ is trivial. Consequently we can suppose that $C_k^0 = C_k$. Thus we have the diagram

$$\begin{array}{ccc} C_k^0 & \xleftarrow{d_{k+1}} & C_{k+1} \\ \xi_k \uparrow & & \uparrow \xi_k^* \\ C_{k+1}^* & \xleftarrow[d_{k+1}^*]{} & C_k^{0*} \end{array}$$

Inasmuch as the homomorphism $\xi$ induces an isomorphism between the homology of the complex $C^*$ and the homology of the complex $C/C^0$, it follows that homomorphisms $d_{k+1}^*$, $\xi_k^*$ are monomorphisms and moreover the product homomorphism

$$d_{k+1}^* \oplus \xi_k^* : C_k^{0*} \to C_{k+1}^* \oplus C_{k+1}$$

is an isomorphism. Making a change of coordinates with the help of the isomorphism $(d_{k+1}^* \oplus \xi_k^*)$, we quickly confirm that the scalar product on the module $C_k^0$ is trivial.

## §5. The construction of an element of the Wall group $L_n^Q(\pi)$. Odd-dimensional case

Theorem 5.1. *To each algebraic Poincaré complex there can be associated an element of the group $L_n^Q(\pi)$ depending only on a bordism class of complexes.*

Proof. Recall that the Wall group $L_n^Q(\pi)$ in the odd-dimensional case $n = 2k + 1$ is a group of invertible matrices of order $2m$ preserving the scalar product given by the matrix

$$\begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix},$$

factorized by the subgroup generated by the so-called elementary transformations. The matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and matrices of the type

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix},$$

where $A$ is an invertible matrix, are related to elementary transformations. It is easy to see that if $C = (-1)^{k+1}C^*$ then the matrix

$$\begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix}$$

is elementary.

Now let a Poincare complex $\alpha$ be given. According to Proposition 3-3 the complex a is bordant to a complex of the type

$$\begin{array}{ccc} C_k & \xleftarrow{d_{k+1}} & C_{k+1} \\ \xi_k \uparrow & & \uparrow \xi_{k+1} \\ C_{k+1}^* & \xleftarrow[d_{k+1}^*]{} & C_k^* \end{array}.$$

In a module $C_{k+1} \oplus C^*_{k+1}$ we introduce a scalar product by means of the matrix

$$\begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix}.$$

It is clear that the imbedding $\phi\colon C^*_k \to C_{k+1} \oplus C^*_{k+1}$, $\phi = \xi_{k+1} \oplus d^*_{k+1}$, is an embedding onto a direct summand, and that the scalar product is trivial on the image $\operatorname{Im}\phi$. Consequently there exists a transformation $A$ of the module $C_{k+1} \oplus C^*_{k+1}$ which preserves the scalar product and $A = (\phi, \psi)$, i.e. an element of the group $L^Q_n(\pi)$ has been defined.

Let us show that if the complex $\alpha$ is the boundary of a Poincaré pair then the element defined above of the group $L^Q_n(\pi)$ is zero. Let $\alpha = \partial\beta$ and $\beta = (C, C^0, d, \xi)$. Let us modify the pair $\beta$ so that only $C^0_k = C_k$ and $C_{k+1}$ are nonzero. Adding, if necessary, a module in dimension $k+2$, we can suppose that the horaomorphism $\xi_k$: $C^*_{k+2} \to C^0_k$ is an epimorphism. By carrying out a modification of the pair $\beta$ with respect to a certain map $A \to C^*_{k+2}$ we may suppose that the homomorphism $d_{k+1}$: $C_{k+1} \to C^0_k$ maps a certain submodule $A \subset C_{k+1}$, $A \cap C^0_{k+1} = 0$, epimorphically onto the module $C^0_k$. After this we can again "split off" the module $C_{k+2}$ and make it zero. Thus we may suppose that $C^0_k = A$, $C_{k+1} = A \oplus B \oplus C$, $C^0_{k+1} = B$, $d_k = (1, d, 0)$ and $\xi_{k+1} = \|\xi_{ij}\|$, where $\xi^*_{k+1} = (-1)^{k+1}\xi_{k+1}$.

Consider the new Poincaré pair

$$\bar C^0_k = \bar C_k = A \oplus C,$$

$$\bar C_{k+1} = A \oplus C \oplus B \oplus C,$$

$$\bar C^0_{k+1} = B \oplus C,$$

$$\delta_{k+1} = \begin{pmatrix} 1 & 0 & d & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix},$$

$$\eta_{k+1} = \begin{pmatrix} \xi_{11} & \xi_{13} & \xi_{12} & \xi_{13} \\ \xi_{31} & \xi_{33} & \xi_{32} & \xi_{33} \\ \xi_{21} & \xi_{23} & \xi_{22} & \xi_{23} \\ \xi_{31} & \xi_{33} & \xi_{32} & \xi_{33} \end{pmatrix}.$$

It is easy to see that the embedding

$$\bar\varphi : \bar C^{0*}_k \to \bar C^0_{k+1} \oplus \bar C^{0*}_{k+1}$$

is defined by the matrix

$$\bar\varphi = \begin{pmatrix} d^* & 0 \\ 0 & -1 \\ \xi^0_{k+1} & 0 \\ 0 & 0 \end{pmatrix}$$

and consequently defines the same element of the group $L^Q_n(\pi)$. Thus we arrive at the pair $(C, C^0, d, \xi)$ which can be represented in the form

$$A = C_k^0, \quad C_{k+1}^0 = B,$$
$$C_{k+1} = A \oplus B,$$
$$d_{k+1} = (1, d),$$
$$\eta_{k+1} = \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix}.$$

It is easy to see that $\xi_{12} = (-1)^{k+1}\xi_{21}^*$ is an isomorphism. By carrying out a suitable change of coordinates in the module $A$ we may put $B = A^*$ and $\xi_{12} = 1$. Then $d = (-1)^{k+1}d^*$. The embedding $\phi$: $C_k^{0*} \to C_{k+1}^0 \oplus C_{k+1}^{0*}$ is given by the matrix

$$\begin{pmatrix} 1 + \xi_{22}d^* \\ d^* \end{pmatrix}.$$

The change of coordinates in the module $C_{k+1}^0 \oplus C_{k+1}^{0*}$ given by the matrix

$$\begin{pmatrix} 1 & \xi_{22} \\ 0 & 1 \end{pmatrix},$$

changes the matrix of the embedding $\phi$ into

$$\begin{pmatrix} 1 \\ d^* \end{pmatrix}.$$

The theorem is proved.

## §6. Modifications of smooth manifolds

For each smooth manifold $M^n$ with fundamental group there is defined an algebraic Poincaré complex in the sense defined in §1. One utilizes for this the $\cap$-product between chains and cochains (see for example t'] ). Conditions (2) and (2') can be fulfilled by using the ring $\Lambda$ with coefficients modulo 2. If a homotopy equivalence between two manifolds is given then the algebraic Poincaré complexes corresponding to them are also homotopically equivalent. Thus the results of §§3, 4 and 5 give the following theorem.

Theorem 6.1. *There exists a function associating with each manifold $M^n$ an element $\sigma(M^n) \in L_n^Q(\pi)$, where $\pi_1(M^n) = \pi$, and $\pi_1(M^n)$ preserves die orientation of $M^n$. The element $\sigma(M^n)$ is a homotopy invariant and an invariant of the bordism $\Omega_*(K(\pi, 1))$.*

Theorem 6.2. *Let $f$: $M_1^n \to M_2^n$ be a map of degree 1 and $F$ a stable fiber bundle trivialization $\tau(M_1^n) \oplus f^*(\xi)$, where $\xi$ is a certain fibration over $m_2^n$ and $f_*$: $\pi_1(M_1^n) \to \pi_1(M_2^n)$ is an isomorp $\alpha(M_1, f, F) \in L_n(\pi)$ be an obstruction to surgery on the manifold $M_1$, and let $q$: $L_n(\pi) \to L_n^Q(\pi)$ be the natural inclusion. Then*

$$q(\alpha(M_1, f, F)) = \sigma(M_1^n) - \sigma(M_2^n).$$

Proof. The corresponding map between algebraic Poincaré complexes is an epimorphism. Let us consider the complex $C_3 = \mathrm{Ker}\, f$. It is clear that $C_3^* \approx \mathrm{Coker}\, f^*$, and therefore $\xi^1$ induces the structure of a Poincaré complex o ı $C_3$. ı u s we have the diagram

$$\begin{array}{ccccc} C_3 & \rightarrow & C_1 & \rightarrow & C_2 \\ \uparrow & & \uparrow & & \uparrow \\ C_3^* & \leftarrow & C_1^* & \leftarrow & C_2^* \end{array}.$$

Further, to each modification of the triple $(M_1, f, F)$ there corresponds a certain modification of the complexes $C_3$ and $C_1$ which commutes with the inclusion $g$. Thus for even $n = 2k$ we modify the complex $C_1$ until $\hat{C}_3$ consists only of a group in the middle dimension. Inasmuch as the homomorphism $g$ induces a monomorphism in homology, we can represent the complex $C_1$ in the form of a direct s ı $C_3 \oplus C_2$, e r e the differential $d'$ has the form

$$d' = \begin{pmatrix} 0 & d \\ 0 & d^2 \end{pmatrix}$$

and $d \neq 0$ only in dimensions $k + 1$. The homomorphism $\xi^1$ in view of the self-duality can be represented in the form

$$\xi^1 = \begin{pmatrix} \xi^3 & 0 \\ 0 & \xi^2 \end{pmatrix}.$$

Further, in every dimension $i < k$ we carry out simultaneous modifications in the complexes $C_2$ and $C_3$, identifying them by means of the isomorphism $f$. Then the unique nondiagonal differential $d'_{k+1}$ maps the module $C_{2,k+1} \approx C_{3,k+1}$ onto a direct summand of the m o d ι $C_{2,k}$ h i c h does not intersect the module $C_{3,k}$. Thus by a suitable choice of the direct decomposition it is possible to modify the complexes $C_2$ and $C_3$ up to the middle dimension.

For odd $n = 2k + 1$ we carry out a modification up to the two middle dimensions in an analogous way, and decompose the complex $C_1$ into the direct sum $C_1 = C_2 \oplus C_3$. Then the element of the group $L_n^Q(\pi)$ is defined by an embedding matrix of the form

$$\begin{pmatrix} \varphi_3 & \varphi \\ 0 & \varphi_2 \end{pmatrix},$$

where $\phi_3$ and $\phi_2$ are embeddings onto a direct summand in the module with trivial scalar product. Thus the homomorphism $\phi$ may be represented in the form $\phi = \alpha\phi_2$. Consequently

$$\begin{pmatrix} \varphi_3 & \varphi \\ 0 & \varphi_2 \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varphi_3 & 0 \\ 0 & \varphi_2 \end{pmatrix},$$

i.e. the element of the group $L_n^Q(\pi)$ defined by the matrix

$$\begin{pmatrix} \varphi_3 & \varphi \\ 0 & \varphi_2 \end{pmatrix}$$

is equal to the direct sum $(\phi_3) + (\phi_2)$. Theorem 6.2 is proved.

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