INTEGRAL GEODESICS OF A FLOW ON LIE GROUPS

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The problem of determining the integrals of a flow on an n-dimensional solid body was posed in 1964 by V. I. Arnol'd (see also [1]). In the present work we calculate a series of such integrals.

§1. Notation

Let G be a Lie group; let X be the vector field of the group G, and f a smooth function. Then $X(f)$ denotes the derivative of the function f along the vector field X. If φ is a diffeomorphism of the group G, then φ induces an automorphism of the ring of functions and the space of vector fields:

$$\varphi(f)(x)=f(\varphi(x)), \qquad \varphi(X)(f)(x)=X(\varphi(f))(\varphi^{-1}(x)).$$

As usual L_g, R_g denote diffeomorphisms $L_g(x) = gx$, $R_g(x) = xg$, i.e., left and right translations. The space of left-invariant vector fields X, i.e., fields for which $L_g(x) = X$, is a Lie algebra $\mathfrak{g}$ of the group G. If we are given a metric on the manifold G, then for any vector of the fields X, Y it is possible to construct a function with values on the manifold G, equal to the scalar product $\langle X, Y\rangle$. The metric is said to be left-invariant if $\langle L_g(X), L_g(Y)\rangle = L_g(\langle X, Y\rangle)$. For any metric on the manifold G we can assign a covariant differentiation of vectors of the fields, by the following expression (see [2], p. 60):

$$2\langle X, \nabla_Z Y\rangle = Z(\langle X, Y\rangle) + \langle Z, [X, Y]\rangle + Y(\langle X, Z\rangle) + \langle Y, [X, Z]\rangle - X(\langle Y, Z\rangle) - \langle X, [Y, Z]\rangle. \tag{1}$$

Let TG be the tangent bundle for the group G. Then the left translation L_g induces a left translation of the tangent bundle $L_g : TG \to TG$ and a projection of the tangent bundle of the tangent space T_e to the unit of the group $L : TG \to T_e$.

§2. The Flow Equation

Assume we are given a left-invariant metric on the group G. It then is defined in a single-valued manner by the metric on the tangent space T_e and the mapping L.

The field X is said to be geodesic if $\nabla_X X \equiv 0$. It is clear that an integral curve of the geodesic field X is a geodesic curve, while conversely, any geodesic curve can be represented as the integral curve of some geodesic field.

Proposition 1. If X is a geodesic field then $L_g(X)$ is also a geodesic field.

Proposition 2. A vector field on the tangent bundle TG, corresponding to a flow equation, is left-invariant; i.e., the projection $L : TG \to T_e$ is consistent with this vector field.

Proposition 3. The equation for the geodesics of a vector field has the form

$$X(\langle X, Y\rangle) = \frac{1}{2} Y(\langle X, X\rangle) - \langle X, [Y, X]\rangle,$$

where X is some desired geodesic field and Y is some vector field.

COROLLARY. If X is a geodesic field, then $X(\langle X, X\rangle) \equiv 0$.

Let $x(t) \in G$ be a geodesic curve, $X(t) \in TG$ the tangent vector at the point $x(t)$, and let Y_e be any vector of the space T_e.

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Proposition 4. For any geodesic curve and any vector Y_e we have the identity

$$\langle R_{x(t)}(Y_e), X(t)\rangle \equiv \text{const}.$$

The functions standing in the left-hand side of the identity are obtained by integrals of the geodesics of the vector field in the tangent bundle T(G) and are referred to as kinetic moments.

Let us set $C(t) = L(X(t)) = L_{x(t)^{-1}}(X(t))$. It is clear that for every t, C(t) is a vector of the tangent space T_e. It can be called the angular velocity fixed to the system of coordinates of the body. We will now describe the flow equation of the vector C(t).

Proposition 5. Let $Y_e(t) \in T_e$ be any vector function. Then

$$\left\langle \frac{d}{dt}C(t),\ Y_e(t)\right\rangle \equiv \langle C(t), [C(t), Y_e(t)]\rangle. \tag{2}$$

COROLLARY. There exists an integral of the "kinetic energy" $(d/dt)\langle C(t), C(t)\rangle \equiv 0$ or $\langle C(t), C(t)\rangle \equiv \text{const}$.

§3. The Integral of the "Kinetic Moment"

Integrals of the "kinetic moment" are given on the tangent bundle TG. However, some of them cannot be "placed" upon the tangent space at the unit, T_e.

We will say that two vectors $C_1, C_2 \in T_e$ are equivalent if there exists an element $g \in G$ such that for any vector $D \in T_e$ we have the equation $\langle L_{g^{-1}}R_gD, C_1\rangle = \langle D, C_2\rangle$.

Proposition 6. Any function defined on the space T_e and assuming equal values on equivalent vectors, is by means of an integral of Eq. (2), dependent on the integral of the "kinetic moment." Other left-invariants of integrals, dependent on the integrals of the "kinetic moment," do not exist.

§4. Motion of a Rigid n-Dimensional Body with Stationary Points

Let us examine the particular case when the group G = SO(n) is that of orthogonal matrices. The space T_e can be identified with that of all skew-symmetric matrices, and the tangent space TG of the bundle, with matrices of the form AB or BA, where A is skew-symmetric, and B is orthogonal. Examining the motion of a solid n-dimensional body, whose configuration space is the group SO(n), it is easily shown that the metric on the group is left-invariant. For $n > 3$ not every left-invariant metric corresponds to the flow of a solid body, and only those metrics do, which are obtained in the following manner. Let I be some diagonal matrix with positive eigen-values λ_i. We then define a metric on the group by the formula

$$\langle X, Y\rangle = \operatorname{tr}(XIY' + YIX'),$$

where $X, Y \in TG$, and X' denotes the transposed matrix. It is easily seen that this metric is left-invariant.

Proposition 7. The flow equation on the tangent space T_e has the form

$$\frac{d}{dt}X\cdot I + I\frac{d}{dt}X = IXX - XXI. \tag{3}$$

Proposition 8. Equation (3) has the following integrals:

$$L_s(X) = \operatorname{tr}\left(\sum_{k=0}^{s} IXI^kXI^{(s-k)}\right), \quad s \geqslant 0.$$

By virtue of §3 part of the collection of integrals of the "kinetic moment" can be examined on the tangent space T_e. Since in the case of the group SO(n), equivalence of vectors in the space T_e is usually conjugacy of the matrices $\varphi(X)$ relative to interior automorphisms of the matrices, where $\varphi(X) = XI + IX$, it is possible to choose an independent system of invariants, and namely, coefficients of the polynomial $\det(\varphi(X) - \mu E)$ for odd order. We denote these coefficients by M_{n-2s}, $1 \leq s \leq n/2$.

THEOREM. The system of integrals M_{n-2s}, $1 \leq s \leq (n/2)$, L_s, $0 \leq s \leq n-2$, $s \neq 1$, is independent.

Thus the equation of flow of an n-dimensional solid body has $(n(n-1)/2) + n - 2$ independent integrals in the tangent bundle TG and $(n-2) + [n/2]$ independent integrals in the tangent space T_e.

§5. The Case $G = SO(4)$

In this case $\dim T_e = 6$, and the number of integrals is even, i.e., the invariant manifold of this flow is a closed surface. Since the set of vectors X, where dX/dt reduces to zero, has dimensionality equal to three, almost every surface is a torus. Let us now examine the invariant manifold in the tangent bundle. The integral of the "kinetic moment" has the form

$$F(C, A) = A(CI + IC)A' = P \equiv \text{const},$$

where A is orthogonal, and C is a skew-symmetric matrix. Let us fix the matrix P. Then the level manifold M_P of the function F is the image (left translation) of a projection in the tangent space T_e,

$$L : M_P \to L(M_P).$$

Let $\varphi: C \to CI + IC$ be an isomorphism of the space T_e onto itself. The mapping $A \to A\varphi^{-1}(A'PA)$ is a diffeomorphism between the group G and the manifold Mp. The continuous mapping $G \approx M_P \xrightarrow{\varphi} L(M_P)$, defined by the formula $A \to A'PA$, is a smooth stratification. The layer at the point P is a subgroup of the group G. In our case this subgroup is a two-dimensional torus. Thus the invariant manifold of geodesic flow for the group SO(4) is a 4-dimensional manifold with layers and a basis (the torus) almost everywhere in the tangent bundle of the group.

§6. Proof of the Theorem

For the proof we choose, in the tangent space T_e, a basis from the skew-symmetric matrix P_{ij}, for which, in the place (i, j) we have +1, and in (j, i) the value −1, while in the remaining points we have zero. It is easily seen that $[P_{ij}, P_{jk}] = P_{ik}$, $P_{ij} = -P_{ji}$.

If $X = \sum_{i<j} x_{ij}P_{ij}$ is any skew-symmetric matrix, then the integrals L_s have the form

$$L_s(X) = L_s(x_{ij}) = \sum_{i<j} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j}(\lambda_i^{s+1} - \lambda_j^{s+1})\, x_{ij}^2,$$

where λ_i, as indicated above, are the eigenvalues of the diagonal matrix I. We will show that the functions L_s, $0 \le s \le n-2$, are independent. For this it is necessary to compute the rank of the matrix consisting of the partial derivatives of the functions L_s at the points $x_{ij} = 1$, $i < j$.

The integrals of M_{n-2s} equal to the coefficients of the polynomial $\det(\varphi(X) - \mu(E))$ will be calculated at the points $x_{1i} = -x_{i1} = 1$, $x_{2k-1,\,2k} = -x_{2k,\,2k-1} = 1$, $x_{ij} = 0$ for the remaining indices (i, j).

Let the initial n be an even number.

Then

$$\det(\varphi(X) - \mu E) = \prod_{i=1}^{n/2} (\mu^2 + (\lambda_{2i-1} + \lambda_{2i})^2 x_{2i-1,\,2i}^2) \times \left(1 + \frac{\mu^2}{\mu^2 + (\lambda_1 + \lambda_2)^2 x_{12}^2} \sum_{i=2}^{n/2} \frac{(\lambda_1 + \lambda_{2i-1})^2 x_{1,\,2i-1}^2 + (\lambda_1 + \lambda_{2i})^2 x_{1,\,2i}^2}{\mu^2 + (\lambda_{2i-1} + \lambda_{2i})^2 x_{2i-1,\,2i}^2}\right).$$

In the case of odd n we obtain a formula of similar type. It is easily seen that the integral M_{n-2} coincides with the integral L_1.

Let us denote by σ_s the s-th order elementary symmetric polynomials in the variables $(\lambda_{2i-1} + \lambda_{2i})^2$, $2 \le i \le (n/2)$, $1 \le s \le (n/2)-1$. Analogously we denote by $\widehat{i}\sigma_s$ and $\widehat{ij}\sigma_s$ the s-th elementary symmetric polynomials in the variables $(\lambda_{2k-1} + \lambda_{2k})^2$, $2 \le k \le (n/2)$, $k \ne i$ or $k \ne i$, $k \ne j$. The coordinates of the gradient of the function L_s (respectively, the function M_s) at the points indicated above will be denoted by $L_{s,i,j}$ (respectively, by $M_{s,i,j}$), where (i, j) is the index of the corresponding coordinate.

The calculation of these gradients is done by using the following equation:

$$L_{s,\,i,\,j} = \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j}(\lambda_i^{s+1} - \lambda_j^{s+1})$$

for $i = 1$ or for $i = 2k + 1$, $j = 2k$;

$$L_{s,\,i,\,j} = 0$$

for the remaining values (i, j), which, however, are not of interest to us;

$$M_{n-2s,1,i} = (\lambda_1 + \lambda_2)^2 \sigma_{s-1};$$

$$M_{n-2s,1,2i} = (\lambda_1 + \lambda_i)^2_{\hat{i}} \sigma_{s-1}, \quad i \geqslant 3;$$

$$M_{n-2s,2i-1,2i} = (\lambda_{2i-1} + \lambda_{2i})^2 ({}_{\hat{i}}\sigma_{s-1} + (\lambda_1 + \lambda_2)^2_{\hat{i}} \sigma_{s-2}) + \sum_{\substack{k=2 \\ k \neq i}}^{n/2} (\lambda_{2i-1} + \lambda_{2i})^2 ((\lambda_1 + \lambda_{2k-1})^2 + (\lambda_1 + \lambda_{2k})^2)_{\hat{i}\hat{k}} \sigma_{s-2}, \quad i \geqslant 2.$$

We now form the matrices of numbers $L_{s,i,j}$ and $M_{n-2,s,i,j}$, of s columns and index (i, j) along the rows. By means of a series of elementary transformations the elements of the matrix are replaced by the following numbers:

$$L_{s,i,j} \to \lambda_i^{s+1} - \lambda_j^{s+1}, \qquad M_{n-2s,1,j} \to (\lambda_1^2 - \lambda_j^2)_{\hat{1}} \sigma_{s-1}, \quad j \geqslant 3,$$

$$M_{n-2s,1,2} \to (\lambda_1^2 - \lambda_2^2) \sigma_{s-1}, \qquad L_{s,2i-1,2i} \to 0,$$

$$M_{n-2s,2i-1,2i} \to (\lambda_{2i-1}^2 - \lambda_{2i}^2) ((\lambda_1 + \lambda_2)^2_{\hat{i}} \sigma_{s-2} + \sum_{\substack{k=2 \\ k \neq i}}^{n/2} ((\lambda_1 + \lambda_{2k-1})^2 + (\lambda_1 + \lambda_{2k})^2)_{\hat{i}\hat{k}} \sigma_{s-2}).$$

Now our matrix is of block tridiagonal type, whose diagonal consists of two blocks. The determinant of the first block is obviously not equal to zero. Now let $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_{2j-1}\lambda_{2j} = 1$. Then setting $a_{i,s} = M_{n-2s,2i-1,2i} (\lambda_{2i-1}^2 - \lambda_{2i}^2)^{-1}$, we obtain the elements of the second block in the following form:

$$a_{i,s} = {}_{\hat{i}}\sigma_{s-2} + \sum_{k=1, k \neq 2}^{n/2} ((\lambda_{2k-1} + \lambda_{2k})^2 - 2)_{\hat{i}\hat{k}} \sigma_{s-2}, \quad 2 \leqslant i \leqslant n/2, \quad 2 \leqslant s \leqslant n/2.$$

Moreover, we have

$$a_{i,s} = (1 - n + 2s)\, {}_{\hat{i}}\sigma_{s-2} + (s-1)\, {}_{\hat{i}}\sigma_{s-1}.$$

It is now easily seen that the matrix $a_{i,s}$ is nondegenerate. The proof for even n is complete. The case of odd n reduces to the previous one by letting $\lambda_n \to 0$.

LITERATURE CITED

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