QUADRATIC FORMS OVER COMMUTATIVE GROUP RINGS AND THE K-THEORY

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This work originated from the solution of a problem stated by S. P. Novikov [1]. Namely, he noticed that in classification of smooth manifolds and also in other problems of smooth topology of not simply connected manifolds, an algebraic problem about classification of quadratic forms with coefficients in group rings arises. This problem is considered in the second section of this work for the commutative case. It is very interesting that the invariants obtained can be easily described in terms of the K-functor. It is better to consider the given problem as a particular case of a more general problem, namely the equivalence problem of quadratic forms, the coefficients of which depend on a point of the manifold.

We consider, for example, the simplest setting up of the problem. We consider a commutative discrete group G, for example, a free abelian group with a finite number of generators. Let $\|a_{ik}(g)\| = A$ be a matrix the elements of which are complex valued (for example, finite) functions over the group (elements of the group ring). We assume that A is an Hermitian matrix, i.e., that $A^* = A$, where $A^* = \|\bar{a}_{ki}(g^{-1})\|$. We say that A and B are equivalent if an invertible matrix $X = \|x_{ik}(g)\|$ exists such that $B = X^*AX$. The problem is to determine conditions of equivalence of matrices, among other conditions for a matrix to be equivalent to

$$A \sim \begin{pmatrix} E_p & 0 \\ 0 & -E_q \end{pmatrix},$$

where E_p and E_q are identity matrices of orders p and q, respectively.

We discuss in brief the initial considerations in solution of this problem. Passing from the group G to the character group G*, we obtain that a quadratic form $a_{ik}(\chi)$, depending on the point $\chi \in G^*$ is given; here $a_{ki}(\chi) = \overline{a_{ik}(\chi)}$. There arises then the equivalence problem of quadratic forms $A(\chi) = \|a_{ik}(\chi)\|$, where χ runs through the character group. The idea of description of the invariants of the quadratic form $A(\chi)$ is to consider a stratification, the layer of which at every point χ consists of all methods of reduction of the quadratic form at the same point χ to a canonical form.

It turns out that the answer is formulated in terms of a K-functor of the character group of the free abelian group G. The majority of these elementary considerations do not depend on the fact that G is a character group. Thus, the classification of quadratic forms over group rings is reduced to the problem of classification of quadratic forms depending continuously on the point of a complex.

Analogous results hold for real symmetric and antisymmetric forms. They lead to a KR-functor and to a new functor, which we denote by SK.

We bring an example of a theorem. Assume that we have a matrix $\|a_{ij}(t)\|$, where $t = (t_1, \ldots, t_n)$ runs through a n-dimensional torus. Two forms $A(t) = \|a_{ij}(t)\|$ and $B(t) = \|b_{ij}(t)\|$ are called equivalent if a matrix $X(t) = \|x_{ik}(t)\|$ (Det $X(t) \neq 0$ for any t) exists such that $B(t) = X^*(t)A(t)X(t)$. Assume that Det $A(t) \neq 0$. The invariants of A(t) will then be, first of all, the signature p−q of the matrix A(t) at each point t. This signature is constant as a result of continuity. It turns out that in addition to the signature we have $2^{n-1}-1$ integral invariants of the form $\|a_{ik}(t)\|$ (if p and q are not simultaneously smaller than n). For example, on the circle (n = 1) the signature is the unique invariant.

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This work allows us also to compute simply the rank of the group of quadratic forms over group rings, obtained earlier by Shaneson from considerations of smooth topology.

1. A Useful Construction of a Classifying Space of the Group U(n)

Let X be a finite cell complex, and let A(x) be a function on the complex X assuming values in the spaces S_n of Hermitian non-degenerate matrices of order n. We put in correspondence to each such function A(x) an element of the group K(X).

1.1. Equivalence of Hermitian forms Depending on a Point of the Space. Let S(p, q), p + q = n be a subspace of the S_n, consisting of all matrices with signature equal p−q. Every matrix A $\in$ S(p, q), hence, it can be brought by transformations of the form A → C*AC to the canonical form

$$I_{p,q}=\begin{pmatrix}E_p & 0\\ 0 & -E_q\end{pmatrix},$$

where E_p, E_q are identity matrices of orders p, q respectively. It is therefore possible to construct a mapping π: GL(n, C) → S(p, q), which puts in correspondence to each non-degenerate matrix C $\in$ L(n, C) the matrix $\pi(C) = C^*I_{p,q}C \in S(p, q)$. Let U(p, q) be the subgroup of the group GL(n, C), which leaves fixed the matrix $I_{p,q}$, i.e., $C^*I_{p,q}C = I_{p,q}$ for all C $\in$ U(p, q). It is easy to see that the mapping is a principal stratification with the layer U(p, q). Insofar as the group U(p, q) contracts to the group U(p) × U(q), and the group L(n, C) to the group U(n), it follows that the space S(p, q) is homotopically equivalent to the space U(n)/U(p) × U(q). Introducing the stabilization of the matrices S(p, q)→S(p+1, q+1) according to the formula $A\to\begin{pmatrix}A & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}$, we obtain that the space $S = \varinjlim S_n$ is homotopically equivalent to the classifying space Z × BU of an infinite dimensional unitary group U (U is understood to be $\varinjlim$ U(n)). Thus, the space S of Hermitian non-degenerate matrices is a classifying space of a unitary K-theory. An element of the group K(X) determined by the function A(x) will be denoted by [A].

If two functions A(x) and B(x) are homotopic, then they define the same element of the group K(X). We shall, however, be interested in what follows in another equivalence relation of functions. We say that two functions A(x), B(x): X → S_n are equivalent if there exists a function C(x): X → L(n, C), such that $C^*(x)A(x)C(x) = B(x)$, x $\in$ X.

PROPOSITION 1.1. Two functions A(x), B(x): X → S(p, q) are equivalent if and only if they are homotopic.

Proof. To every function A(x): X → S(p, q) there corresponds a principal U(p) × U(q)-stratification ξ_A, induced by the stratification π: L(n, C) → S(p, q). If the functions A(x) and B(x) are equivalent then the principal U(p) × U(q)-stratifications corresponding to them are also equivalent. Indeed, let C(x): X → L(n, C) be a function such that $C^*(x)A(x)C(x) = B(x)$. Let a: ξ_A → L(n, C), b: ξ_B → L(n, C) be mappings of the principal stratifications induced by the functions A, B, respectively. We construct a mapping f: ξ_A → ξ_B. If y $\in$ ξ_A is a point covering the point x $\in$ X, then f(y) $\in$ ξ_B is determined as a unique point covering the point x for which b(f(y)) = a(y)C(x).

It follows from here that the principal U(p)-stratifications which are projections, say, on the first factor of the group U(p) × U(q) are also equivalent. Insofar as the space S is a classifying space for U(p)-stratifications, the mappings A(x) and B(x) are homotopic.

Conversely, if the mappings A(x) and B(x) are homotopic, then there exists a principal U(p) × U(q)-stratification η on the complex X × I, and $\eta|X\times\{0\} = \xi_A$, $\eta|X\times\{1\} = \xi_B$. Hence, the stratifications ξ_A and ξ_B are equivalent. Let, as before, ξ_A → ξ_B be a mapping of the principal stratifications. Let the point y $\in$ ξ_A cover the point x $\in$ X. We set $C(x) = a(y)^{-1}b(f(y))$. The value of C(x) does not depend on the choice of the point y covering the point x, and, as it is easy to see, the relation $B(x) = C^*(x)A(x)C(x)$ holds. The functions A(x) and B(x) are consequently equivalent.

Thus, if we form the Grothendieck group, generated by all classes of equivalent functions A(x), the addition in which is defined as a direct sum of matrices, then this group is isomorphic to the group K(X). We notice the following property of the functions A(x).

PROPOSITION 1.2. Let ξ_A^1, ξ_A^2 be projections of the principal $U(p) \times U(q)$-stratification ξ_A on the first and second factors of the group $U(p) \times U(q)$. Then $[\xi_A^1] = -[\xi_A^2]$, where $[\xi_A^1]$ is an element of the group $\widetilde{K}(X)$, corresponding to the principal $U(p) \times U(q)$-stratification.

Proof. We can assume, without loss of generality, that the signature of the quadratic form $A(x)$ equals zero. Let π: $U(2n) \to U(2n)/U(n) \times U(n)$ be the principal $U(n) \times U(n)$-stratification. We want to prove that it is trivial over the group $U(2n)$. We associate with the stratification π the stratification with the group $U(2n)$ over the basis $X = U(2n)/U(n) \times U(n)$. The space of this stratification E can be constructed as a factor-space of the space $U(2n) \times U(2n)$ with respect to the equivalence relation

$$(ag,\ b) = (a,\ gb),\quad g \in U(n) \times U(n) \subset U(2n).$$

We construct a section in this stratification: f: $X \to E$. Let $x \in X$, $\pi(a) = x$. Then $f(x) = (a, a^{-1})$. The definition of the function f is correct, since if $\pi(a') = x$, then $a' = ag$; hence $(a', a'^{-1}) = (a, a^{-1})$.

COROLLARY 1.3. If $C(x) = A(x) \otimes B(x)$, then we have in the group $\widetilde{K}(X)$ the equality $[C] = 2[A][B]$.

1.2. Real Quadratic Forms. An analog of a real K-theory for complexes with involutions has been considered in [3]. We consider for every complex X with involution a complex vector stratification ξ, with an anticomplex involution τ, i.e., if $x \in \xi$, then $\tau(\lambda x) = \bar{\lambda}\tau(x)$. Grothendieck's group generated by the above stratifications is denoted by KR(X). For the theory of cohomologies constructed over the functor KR(X) cohomologies of a point are computed, which allows us to compute the groups KR(X) for various spaces.

The construction done in point 1.1 can be extended also to the case of the functor KR(X). We consider the functions $A(x)$: $X \to S(p, q)$, satisfying the condition $\overline{A(x)} = A(\tau(x))$. Two functions $A(x)$ and $B(x)$ are called equivalent if a function $C(x)$: $X \to L(n, C)$ exists such that $\overline{C}(x) = C(\tau(x))$, and $C^*(x)A(x)C(x) = B(x)$. It turns out that the classes of equivalent functions obtained in this way generate a group isomorphic to the group KR(X). Indeed, we have

PROPOSITION 1.4. A representing object for the functor KR(X) is the space $U(n)/U(p) \times U(q)$, $p + q = n$, on which the involution τ acts according to the formula

$$\tau(C) = \overline{C},\quad C \in U(n).$$

Proof. According to the results of [4], if π: $E \to X$ is a principal G-stratification in the category of spaces with involution, where G is a group with involution and the functor of the homotopic groups $\omega_q(E)$ is trivial, $0 \leq q \leq N$, N sufficiently large, then the space X is a representing object of the category of principal G-stratifications over an arbitrary complex with involution Y. We verify first that the mapping π: $U(n) \to U(n)/U(p) \times U(q)$ is a principal stratification in the category of spaces with involution. Let O(n) be the space of fixed points of the space U(n); let X_0 be the space of fixed points of the space X. Then $\pi(O(n)) = X_0$. Indeed, a point $x \in X$ can be represented as a p-dimensional complex subspace L of the space C^n. If $\tau(x) = x$, then the subspace L is invariant with respect to the involution of complex conjugacy, i.e., it is determined by equations with real coefficients. Hence, there exists a $y \in O(n)$, such that $\pi(y) = x$. Further, if $y_1, y_2 \in O(n)$ and $\pi(y_1) = \pi(y_2) \in X_0$, then $y_1 y_2^{-1} \in O(p) \times O(q)$. Thus, the fixed points form a principal $O(p) \times O(q)$-stratification:

$$\pi': O(n) \to O(n)/O(p) \times O(q).$$

Similarly, the fixed points form, in the space $U(n)/U(p)$, a set isomorphic to $O(n)/O(p)$. The mapping $U(n)/U(p) \to U(n)/U(p) \times U(q)$ is, therefore, a principal U(q)-stratification in the category of spaces with involution, and the functor of homotopic groups $\omega_q(U(n)/U(p))$ is trivial up to dimension p, which was what we were required to prove.

PROPOSITION 1.5. Two functions A(x) and B(x) are equivalent if and only if they are homotopic in the category of spaces with involution.

The proof is analogous to the proof of Proposition 1.1.

1.3. Real Antisymmetric Forms. A second analog of K-theory for spaces with involution is the group, generated by all functions $A(x)$: $X \to S_n$, for which equality $\overline{A(x)} = -A(\tau x)$, $x \in X$ holds. Two functions A(x) and B(x) are said to be equivalent if a function $C(x)$: $X \to L(n, C)$ exists such that $\overline{C(x)} = C(\tau(x))$ and $C^*(x)A(x)C(x) = B(x)$. We denote Grothendieck's group generated by the above functions by SK(X).

PROPOSITION 1.6. There exist natural homomorphisms α and β

$$\alpha: K(X) \to SK(X), \quad \beta: SK(X) \to K(X)$$

such that $\beta \circ \alpha(x) = x - \bar{x}$ $\alpha \circ \beta(x) = 2x$. $\bar{x}$ denotes here a complex-conjugate stratification.

Proof. We consider the matrix $J = \frac{1}{\sqrt{2}}\begin{pmatrix} -i & -1 \\ 1 & i \end{pmatrix}$. The following relations:

$$J^* = J^{-1},$$

$$C(x) = J^* \begin{pmatrix} A(x) & 0 \\ 0 & -\bar{A}(\tau x) \end{pmatrix} J = \frac{1}{2}\begin{pmatrix} A(x) - \bar{A}(\tau x) & i(A(x) + \bar{A}(\tau x)) \\ i(A(x) + \bar{A}(\tau x)) & A(x) - \bar{A}(\tau x) \end{pmatrix}$$

hold for this matrix. It is clear that $C(\tau x) = -\overline{C(x)}$. The mapping α puts in correspondence to the function A(x) the function C(x). To equivalent matrices correspond equivalent ones. The mapping β reduces, in fact, to "forgetting" of the involution on the complex X.

COROLLARY 1.7. $SK(S^{4k+2}) \otimes Q = Q$, $SK(S^{4k}) \otimes Q = 0$. Q is here the field of rational numbers.

2. Quadratic Forms Over Commutative Group Rings

The results of the previous section allow us to determine the invariants of quadratic forms, the coefficients of which belong to a commutative group ring, in homotopic terms. Let G be an abelian finitely generated group, and Z(G) its group ring. We consider a nondegenerate symmetric or antisymmetric scalar product in a finitely generated free Z(G)-module M. The scalar product will be assumed trivial if it can be represented in some basis by the matrix $\begin{pmatrix} 0 & E \\ \pm E & 0 \end{pmatrix}$ where E is the identity matrix. All such Z(G)-modules with a scalar product generate a Grothendieck group L(G). The problem consists of describing the group L(G) for various groups G. We fix in the module M an arbitrary Z(G)-basis and obtain by the scalar product a matrix $A = \|a_{ik}\|$, the elements of which belong to the ring Z(G) and satisfy the condition $a^*_{ik} = \pm a_{ki}$, where the involution * is generated by the correspondence $g \to g^{-1}$, $g \in G$.

In order to pass to homotopic terms of description of invariants of the group L(G) we replace the algebraic problem by an analytic one. First we replace the ring of integers Z by the field of real or complex numbers. Secondly, we replace the group ring of the abelian group G by a ring consisting of continuous functions on the character group G* of the group G. Distinct completions of the group ring lead to distinct subrings of the ring of continuous functions on the character group G*; however, this probably influences only slightly the homotopic invariants determined by those subrings.

We shall concentrate in the case of a free abelian group G, for example, on the ring of series $\sum_{g \in G} a(g)g$, the coefficients a(g) of which are such that $P(g) \cdot a(g)$ tends to zero for any polynomial P(g) (the elements g are considered as elements of an integral lattice of an Euclidean space). The obtained ring $\hat{C}(G)$ is isomorphic to a ring of functions of the class C^∞ on the character group G* (G* is a torus in this case). We denote the corresponding group of scalar products by $L_C(G)$. We consider the subring $\hat{R}(G)$ of the ring $\hat{C}(G)$, consisting from series with real coefficients. We denote the group of scalar products corresponding to the ring $\hat{R}(G)$ by $L_{R,0}(G)$ in the symmetric case and by $L_{R,1}(G)$ in the antisymmetric case.

We are now in the position to formulate the basic theorem on homotopic description of invariants of a group of scalar products over a commutative group ring. Let an involution $\tau(\chi) = \chi^{-1}$ be defined on the character group G*.

THEOREM. There exist the isomorphisms:

$$L_C(G) \approx K(G^*), \quad L_{R,0}(G) \approx KR(G^*), \quad L_{R,1}(G) \approx SK(G^*).$$

This theorem is a particular case of the assertions of §1, in points 1.1, 1.2, and 1.3, respectively.

Remark. If one is interested only in the ranks of the groups $L_C(G)$, $L_{R,0}(G)$, $L_{R,1}(G)$, then the group $L_C(G)$ decomposes into a direct sum of groups $L_{R,0}(G)$ and $L_{R,1}(G)$: $L_C(G) \otimes Q \approx L_{R,0}(G) \otimes Q + L_{R,1}(G) \otimes Q$. Groups of scalar products over the integer ring Z(G) are computed up to elements of finite order in [2]. Not complicated computations show that the groups $L_{R,0}(G) \otimes Q$ and $L_{R,1}(G) \otimes Q$ are isomorphic to the corresponding groups of scalar products over the ring Z(G). This shows that the method described in the present article does not lead to an essential loss of information on groups of scalar products over group rings.

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