# MANIFOLDS WITH THE ACTION OF THE GROUP $Z_p$ AND FIXED POINTS

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UDC 513.83

We study the admissible sets of fixed submanifolds under the action of the group $Z_p$ in the quasi-complex manifold $M^{2n}$, $n < p$.

§1. We use the following notation: $\Omega_*$ is the ring of unitary cobordisms, $L_p(i_1, \ldots, i_n)$ is a $(2n-1)$-dimensional lens with the weights $i_k \in Z_p^*$, and $L_p^\infty$ is an infinite dimensional lens. Let $A = \Omega[x_1, \ldots, x_{p-1}]$ be a graded ring, where $\deg x_i = 2$. There exists a homomorphism

$$\alpha\colon A \to U_*(L_p^\infty),$$

associating with every monomial $\gamma x_1^{n_1} \ldots x_p^{n_{p-1}}$ the element

$$\gamma L_p(\underbrace{1,\ldots,1}_{n_1},\ldots,\underbrace{p-1,\ldots,p-1}_{n_{p-1}}).$$

Put $B = \ker \alpha$. With every homogenous element $z \in B$ we associate a quasi-complex manifold M in the following way: put $z = \Sigma \gamma_n x^n$, where $n = (n_1, \ldots n_{p-1})$ is a multi-index.

Let $\Sigma_n$ be the sphere $S^{2|n|-1}$, where the action of the group $Z_p$ on $\Sigma_n$ is given by the weights $x^n$, and let $D_n$ be the disc with the boundary $\Sigma_n$ with linearly extended action of $Z_p$. Since $z \in B$, the manifold $\bigcup_n \gamma n \times \Sigma_n$ has the fiber with the action of $Z_p$ without fixed points. Then

$$M = \Gamma \cup \bigcup_n \gamma_n \times D_n.$$

M is acted upon by $Z_p$, where the $\gamma_n$ are the fixed manifolds with the weights of the representation in the normal fiber $x^n$. Obviously, B is a subring, and the above confrontation defines a ring homomorphism $\beta$: $B \to \Omega_* \otimes Z_p$. In the sequel we shall always argue modulo the elements of degree $\geq 2p$.

We consider the formal series

$$f(t) = \sum_{n=0}^{\infty} \frac{1}{n+1} [CP^n]\, t^{n+1}.$$

Let g(t) be the series which is defined by the equation $g(f(t)) = t$. Put

$$H_k(t) = tg(kf(t))^{-1} = \sum_{n=0}^{\infty} h_n^k t^n,$$

and

$$G_n(x_1) = \sum_{n=0}^{\infty} h_n^k x_1^{1-n}.$$

If $\varphi(t)$ is a formal Laurent series, then we denote by $[\varphi(t)]$ the regular part of the series without the free term. Furthermore, put

$$F_n(x_1) = [x_1^{n_1} G_2^{n_2} \cdot \ldots \cdot G_{p-1}^{n_{p-1}}] \;.$$

and denote by $B_p$ the image of B in $A \otimes Z_p$.

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M. V. Lomonosov Moscow State University. Translated from Matematicheskie Zametki, Vol. 4, No. 4, pp. 381-386, October, 1968. Original article submitted May 22, 1968.

THEOREM 1. The elements

$$v_n = x^n - F_n(x_1), \sum_{i=2}^{p-1} n_i \geqslant 1, \quad n_i \geqslant 0,$$

and the elements $px^n$ generate the group B additively. The elements

$$v^{k,s} = x_1^s x_k - F_{(s,0,\ldots,1,\ldots,0)}(x_1), \quad s \geqslant 0, \quad k \in Z_p^*,$$

generate the ring $B_p$ multiplicatively.

THEOREM 2. The equation

$$\beta(px^n) = 0$$

holds, and the homomorphism $\beta$ may be represented in the form of the composition

$$B \to B_p \xrightarrow{\beta'} \Omega_* \otimes Z_p.$$

The element $\beta(v_n)$ is equal to the free term of the Laurent series $G_1^{n_1} G_2^{n_2} \ldots G_{p-1}^{n_{p-1}}$.

§2. Proof of Theorem 1. It is known [2] that all elements of degree > 0 of $U_*(L_p^\infty)$ have the order p, and the module $U_*(L_p^\infty)$ is generated over $\Omega_*$ by the elements $\alpha(x_1^s)$, $s \geq 0$. Consequently, the elements $px^n$ belong to B, and the $\alpha(x^n)$ are represented as a linear combination over $\Omega_*$ of the elements $\alpha(x_1^s)$, i.e., there exist polynomials $F_n(x_1)$ such that $v_n = x^n$, and the $F_n(x_1)$ lie in B. The fact that the elements $v_n$ and $px^n$ generate B additively is evident. It remains to prove that the $F_n(x_1)$ have the form required in §1.

Clearly, all coefficients of the $F_n(x_1)$ are determined uniquely modulo p. Put

$$v^{k,s} = x_1^s x_k - F^{k,s}(x_1), F^{k,s}(x_1) = f_0^{k,s} x_1^{s+1} + f_1^{k,s} x_1^s + \cdots + f_s^{k,s} x_1.$$

LEMMA 1. The equation $f_i^{k,s} = f_i^{k,s'}$ holds for arbitrary $s, s' \geq i$.

Proof. For $i = 0$ Lemma 1 follows from the spectral sequence of Atiyah-Hirzebruch. Assume the lemma is true for a certain i. Put

$$y = v^{k,i} = x_1^i x_k - (f_0^{k,i} x_1^{i+1} + \cdots + f_i^{k,i} x_1).$$

Then we have for $s \geq i$

$$v^{k,s} = x_1^{s-i} y - (f_{i+1}^{k,s} x_1^{s-i} + \cdots + f_s^{k,s} x_1).$$

We consider the expression

$$v^{k,s} v^{k,i+1} - v^{k,s+1} v^{k,i} = -(f_{i+1}^{k,s} x_1^{s+i-1} y + \cdots + f_s^{k,s} x_1^2 y) - f_{i+1}^{k,i+1} x_1^{s+1-i} y$$
$$+ f_{i+1}^{k,i+1}(f_{i+1}^{k,s} x_1^{s+1-i} + \cdots + f_s^{k,s} x_1^2) + (f_{i+1}^{k,s+1} x_1^{s+1-i} y + \cdots + f_{s+1}^{k,s+1} x_1 y).$$

Replacing $x_1^j y$ in the above expression by $v^{k,j}$ and $x_1$ and applying the homomorphism $\alpha$ we obtain in the highest dimension the equation

$$0 = (f_{i+1}^{k,s+1} - f_{i+1}^{k,i+1})(f_{i+1}^{k,s+1} - f_{i+1}^{k,s}).$$

From here we obtain the equation $f_{i+1}^{k,s} = f_{i+1}^{k,i+1}$ by induction. Lemma 1 is proved.

Put $f_i^k = f_i^{k,s}$ for $s \geq i$. We assume

$$G_k(x_1) = \sum_{i=0}^{\infty} f_i^k x_1^{1-i}.$$

LEMMA 2. The polynomials $F_n(x_1)$ have the form

$$F_n(x_1) = [x_1^{n_1} G_2^{n_2} \cdots G_{p-1}^{n_{p-1}}].$$

Proof. Lemma 2 holds in the case $|n| - n_1 = 1$. The proof is carried out by induction on $|n| - n_1$. Put

$$n = (n_1, n_2, \ldots, n_{p-1}),$$
$$n' = (n_1, n_2, \ldots, n_{p-2}, n_{p-1} - 1),$$
$$n'' = (n_1 + 1, n_2, \ldots, n_{p-2}, n_{p-1} - 1),$$
$$G^n = x_1^{n_1} G_2^{n_2} \cdots G_{p-1}^{n_{p-1}}.$$

We consider the expression

$$v_{n'}v^{p-1,0} = (x^{n'} - F_{n'})(x_{p-1} - f_0^{p-1}x_1) = x^n - x_{p-1}\tilde{F}_{n'} - f_0^{p-1}x^{n''} + f_0^{p-1}x_1F_{n'}$$

$$= v_n + F_n - x_{p-1}F_{n'} - f_0^{p-1}(v_{n''} + F_{n''}) + f_0^{p-1}x_1F_{n'}.$$

Put

$$G^{n'} = g_0x_1^N + g_1x_1^{N-1} + \cdots + g_N + g_{N+1}x_1^{-1} + \cdots$$

Then

$$x_{p-1}F_{n'} = \sum_{i=0}^{N-1} v^{p-1,\,N-i}g_i + \sum_{i=0}^{N-1} g_i\Big(\sum_{j=0}^{N-i} f_j^{p-1}x_1^{N+1-i-j}\Big).$$

Substituting this expression into the above one and applying the homomorphism $\alpha$, we get

$$F_n = \sum_{i=0}^{N-1} g_i\Big(\sum_{j=0}^{N-i} f_j^{p-1}x_1^{N+1-i-j}\Big) + f_0^{p-1}F_{n''} - f_0^{p-1}x_1F_{n'}.$$

On the other hand,

$$[G^n] = [G^{n'}G_{p-1}] = \Big[\Big(\sum_{i=0}^{\infty} g_ix_1^{N-i}\Big)\Big(\sum_{i=0}^{\infty} f_i^{p-1}x_1^{1-i}\Big)\Big] = f_0\Big(\sum_{i=0}^{N-i} g_ix_1^{N-i+1}\Big) + \sum_{i=1}^{N-1} f_i^{p-1}\Big(\sum_{j=0}^{N-i} g_ix_1^{N+1-i-j}\Big) = F_n.$$

Lemma 2 is proved.

We now fix a certain number N and consider the lens $L_p^{2N-1}$. Let $u \in U^2(L_p^{2N-1})$ be a "geometric" cobordism [2], and let

$$D:\ U_i(L_p^{2N-1}) \to U^{2N-1-i}(L_p^{2N-1})$$

be a Poincaré duality (see [1]). It is easy to see that

$$D(\alpha(x_k^n)) = u^{N+1-n}(H_k(u))^n$$

(we recall that $u^{N+1} = 0$). Let

$$\varphi_k:\ L_p^{2N-1} \to L_p^{2N-1}$$

be a mapping corresponding to the multiplication in $Z_p$ by $k^{-1} \in Z_p^*$. The following equation holds [2]

$$\varphi_{k,*}(\alpha(x_1^n)) = \alpha(x_k^n).$$

We consider the equation

$$\varphi_{k,*}(\varphi_k^*(u)\cdot 1) = u\varphi_{k,*}(1).$$

We have

$$\varphi_{k,*}(u^s) = u^sH_k^{N+1-s}(u),\ \varphi_k^*(u) = g(k^{-1}f(u)) = \sum_{i=1}^{\infty} \lambda_i^k u^i \text{ (see [2], Appendix).}$$

Then

$$\sum_{i=1}^{\infty} \lambda_i^k u^i H_k^{N+1-i}(u) = uH_k^{N+1}(u).$$

Since the element $H_k(u)$ is invertible in $U^*(L_p^{2N+1})$, we get

$$\sum_{i=1}^{N} \lambda_i^k\Big(\frac{u}{H_k(u)}\Big)^i = u \qquad (\mathrm{mod}\ u^{N+1}).$$

Consequently,

$$g\Big(k^{-1}f\Big(\frac{u}{H_k(u)}\Big)\Big) = u$$

or

$$H_k(u) = \frac{u}{g(kf(u))}.$$

Theorem 1 is proved completely.

§3. Proof of Theorem 2. Put $n = (n_1, \ldots, n_{p-1})$, $n' = (n_1 + 1, n_2, \ldots, n_{p-1})$, and let $\gamma$ be the free term of the series $G^n$. Then $F_{n'} = F_n x_1 + \gamma x_1$. We have the equation

$$v_{n'}v^{k,s} - v^{k,s+1}v_n = \alpha^k_{s+1}v_{n'} - \gamma v^{k,s+1}.$$

We apply the homomorphism $\beta$:

$$\beta(v_{n'})(\beta(v^{k,s}) - \alpha^k_{s+1}) = \beta(v^{k,s+1})(\beta(v_n) - \gamma).$$

Choosing s so that $\deg v^{k,s+1} > \deg v_{n'}$ and k so that $\beta(v^{k,s+1})$ is an indecomposable element of the product (see [1]), we obtain that in the above equation the left and the right sides are equal to zero. We show that $\beta(px_1^n) = 0$. It is clear that $\beta(px_k^n) = \beta(px_1^n)$. On the other hand,

$$(x_k^n - k^{-n}x_1^n + \ldots) \in B,$$

i.e.,

$$\beta(px_k^n - pk^{-n}x_1^n + \cdots) = 0,$$

or

$$\beta(px_k^n)(1 - k^{-n}) + \cdots = 0.$$

From here we obtain the assertion by induction on n. Theorem 2 is proved.

## LITERATURE CITED


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2. S. P. Novikov, "Methods of the algebraic topology under the point of view of the theory of cobordisms," Izv. Akad. Nauk SSSR, Ser. Matem., 31, No. 4 (1967), pp. 855-951.