SPECTRAL SEQUENCES OF ADAMS TYPE

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1. General Remarks. We shall consider a category of stable spectra of topological spaces, henceforth called simply spectra (see [1], say). Let be given the sequence of morphisms

$$\ldots \leftarrow X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \ldots$$

and the spectrum Z. Then the homological spectral sequence (see [3], pp. 400-403), whose r-th member equals

$$E_r^p = \operatorname{Im}([S^*Z, X_p/X_{p+r}] \to [S^*Z, X_p/X_{p+1}])/\operatorname{Im}([S^*Z, X_{p-r+1}/X_p] \to [S^*Z, X_p/X_{p+1}])$$

is defined in a natural manner. The cohomological spectral sequence is also defined reciprocally.

An interesting case is when the sequence

$$[X_0, S^*W] \leftarrow [S^{-1}X_0/X_1, S^*W] \leftarrow [S^{-2}X_1/X_2, S^*W] \leftarrow \ldots$$

is a free resolvent of the $[W, S^*W]$-modulus $[X_0, S^*W]$. It is easy to see that the term E_2 is $\operatorname{Ext}_{[W, S^*W]}([X_0, S^*W], [Z, S^*W])$, and hence lends itself to calculation by starting just from knowledge of the algebra $[W, S^*W]$ and the moduli $(X_0, S^*W]$ and $[Z, S^*W]$.

It is natural to expect that the above-mentioned spectral sequence will converge to a group adjoint to the group $[S^*Z, X_0]$ (or to the group $[X, S^*Z]$ in the reciprocal situation). In the case $W = K[Z_p]$ Adams [2] proved the convergence of the cohomological spectral sequence to the mentioned groups. We consider other spectral sequences, constructed by means of the spectra S^0, MU, and some others.

2. Fundamental Constructions. Let us give the following definition. We call the sequence of spectra

$$X = X_0 \leftarrow X_1 \leftarrow \ldots \leftarrow X_n \leftarrow \ldots \quad (1)$$

a cohomological W-resolvent of the spectrum X if the sequence

$$[X_0/X_1, S^*W] \leftarrow [X_1/X_2, S^{*-1}W] \leftarrow \ldots \leftarrow [X_n/X_{n+1}, S^{*+n}W] \leftarrow \ldots \quad (2)$$

is a free resolvent of the $[W, S^*W]$-modulus $[X_0, S^*W]$, and the spectra X_n/X_{n+1} are sums of the spectra S^kW.

If the first condition is discarded, the sequence (1) is called a free sequence; if the sequence (2) is exact, the sequence (1) is called an acyclic sequence.

Let the spectrum W satisfy the self-representability condition, and the spectrum X the Noether condition relative to the spectrum W.

Proposition 1.* For the spectra W and X and any n there exist realizations of homological and cohomological W-resolvents of the spectrum X in dimensionalities $\leq n$.

Proposition 2. Let be given a W-free sequence X_i of the spectrum X, a W-acyclic sequence Y_i of the spectrum Y, and the morphism $Y \to X$ in the case of cohomological, and $X \to Y$ in the case of homo-

*S. P. Novikov proved this result.

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logical sequences. Then there exist morphisms $Y_i \to X_i$ ($X_i \to Y_i$) which commute with the original morphisms of the sequences.

Proof. We have the commutative diagram

$$\begin{array}{ccc} X/X_1 & \leftarrow & X \\ \bar{f}\nwarrow & & \nearrow f \\ & Y & \end{array}$$

where $X/X_1 = \bigvee S^{n_i}W$. Then, it is possible to write the equality $f = \Sigma f_i$, where $f_i : Y \to S^{n_i}W$ is the projection on the component. Since the mapping $[Y, S^* W] \leftarrow [Y/Y_1, S^* W]$ is an epimorphism, there exist mappings $g_i : Y/Y_1 \to S^{n_i}W$, where $\varphi g_i = f_i$. This means that the diagram

$$\begin{array}{ccccc} X/X_1 & \overset{\chi}{\leftarrow} & X & \overset{\kappa}{\leftarrow} & X_1 \\ \bar{g}\uparrow & & \uparrow f & & \\ Y/Y_1 & \overset{\varphi}{\leftarrow} & Y & \overset{\psi}{\leftarrow} & Y_1 \end{array}$$

is commutative. Hence, $g\varphi\psi = 0 = \chi f \psi$; therefore, there exist a mapping $f_1 : Y_1 \to X_1$, such that $\kappa f_1 = f\psi$. The proof is continued further by induction.

Proposition 3. Let be given a spectrum Z such that $\bigcap_n \operatorname{Im}([S^*Z, X_n] \to [S^*Z, X_m]) = 0$. Then the homological spectrum sequence converges to a group adjoint to $[S^* Z, X_0]$. (See [2], Proposition 2.1 on p. 385, and following.)

This proposition may be generalized as follows. Let us consider two classes of groups C_0 and C_1 in the category of abelian groups. Let the following conditions be satisfied: if $G_0 \in C_0$, $G_1 \in C_1$, and $f : G_1 \to G_0$ is a homomorphism, then f is certainly a null homomorphism; for any group G there exists a greatest subgroup $(G)_0 \in G$, $(G)_0 \in C_0$; finally, the class G satisfies axiom 1 from reference [4] (p. 125).

Proposition 4. Under the conditions of Proposition 3 let be given that $[S^* Z, X_m/X_{m+1}] \in C_1$ and

$$\bigcap_n \operatorname{Im}([S^*Z, X_n] \to [S^*Z, X_m]) = ([S^*Z, X_m])_0 \in C_0.$$

Then the homological spectral sequence converges to a group adjoint to $[S^* Z, X_0]/(S^* Z, X_0)_0$.

Proof. By virtue of section 2 in Chapter XV of the book [3] (p. 384), it must be proved that $Z^p_\infty = \bigcap_r Z^p_r$, where

$$Z^p_\infty = \operatorname{Im}([S^*Z, X_p] \xrightarrow{\varphi} [S^*Z, X_p/X_{p+1}]),$$
$$Z^p_r = \operatorname{Im}([S^*Z, X_p/X_{p+r}] \xrightarrow{\varphi_r} [S^*Z, X_p/X_{p+1}]).$$

It is clear that $Z^p_\infty \subset Z^p_r$. Let $0 \neq \alpha \in Z^p_r$, such that $\beta \in [S^* Z, X_p/X_{p+1}]$, for $\varphi_r(\beta) = \alpha$. We have the commutative diagram

$$\begin{array}{ccc} [S^*Z, X_p] & \xrightarrow{\varphi} & [S^*Z, X_p/X_{p+1}] \\ \downarrow & \searrow h & \uparrow \varphi_r \\ [S^*Z, X_p/X_{p+r+s}] & \xrightarrow{l} & [S^*Z, X_p/X_{p+r}] \\ \downarrow & & \downarrow k \\ [S^*Z, SX_{p+r+s}] & \to & [S^*Z, SX_{p+r}] \end{array}$$

If $\beta \in \operatorname{Im} h$, then $k(\beta) \neq 0$. Moreover, since $[S^* Z, X_p/X_{p+r}] \in C_1$, then $k(\beta) \in ([S^* Z, SX_{p+r}])_0$. There then exists a number s such that $\beta \in \operatorname{Im} l$. Furthermore, it is already clear that $\alpha \in Z^p_{r+s}$. The second part of the proof is trivial.

Let us give the following definition. Let be given a W-free sequence of spectra $X \leftarrow \ldots \leftarrow X_n \leftarrow \ldots \leftarrow X_0$. If for any spectrum Z we have the equality $\bigcap_n \operatorname{Ker}([X, S^* Z] \to [X_n, S^*Z]) = 0$, then we call the spectrum X as W-complex, and we call the sequence $\{X_n\}$ a W-partition of the spectrum X. For example, every CW-complex is an S^0-complex. If W-free sequence of spectra $\ldots \leftarrow X_n \leftarrow \ldots \leftarrow X_1 \leftarrow X_0$ is given, where X is the topological direct limit of the sequence $\{X_n\}$, the spectrum X is then a W-complex.

3. Algebra of Stable Homotopic Groups. Let us consider the spectrum $S_p^0 = S^0/_{f_p} S^0$, where f_p is "multiplication" by the prime p. Let Γ denote the algebra $[S^*S^0, S^0]$, and Γ_p the algebra $[S^*S_p^0, S_p^0]$. It is easy to see that the group Γ_p is p-primary, meaning that any group $[S^*S_p^0, X]$ is also p-primary.

The remark at the end of Section 2 shows the following theorem to be true.

THEOREM 1. For the CW-complex X and the spectrum Y a spectral sequence with the term

$$E_2 = \mathrm{Ext}_\Gamma([S^*S^0, X], [S^*S^0, X])$$

converges to a group adjoint to $[X, S^*Y]$.

Proposition 5. Let integer cohomologies of the CW-complex S be p-primary. Then the complex X is an S_p^0-complex.

Proof. Let $H^n(X) \neq 0$, and let n be minimal; then $H_{n-1}(X) \approx H^n(X)$. Thus, the group $\pi_{n-1}(X)$ is p-primary. Let us consider the subgroup $G \subset \pi_{n-1}(X)$ consisting of elements of order p. There exists a basis in this group, and therefore, a mapping $\bigvee S_p^{n-1} \to X$ giving an epimorphism of dimension n in the cohomologies. The order of the group $H^n(X/\bigvee S^{n-1})$ is strictly less than the order of the group $H^n(X)$. Repeating this process, in a finite number of steps we "subside" to the group H^n and we transfer to the dimension n+1. The sequence $X \leftarrow \ldots \leftarrow X_n \leftarrow \ldots \leftarrow X_0$ has thus been constructed for which $\lim X_n \subset X$, and there is an isomorphism in the cohomologies $H^*(X) \approx H^*(\lim X_n)$. This means that these spaces are weakly homotopically equivalent.

Proposition 6. Let X be some CW-complex, Z a spectrum, $\varphi : X \to Z$ a mapping whose order equals p^S, $1 \leq S \leq +\infty$. Then there exists an S_p^0-complex S' and a morphism $\psi : X' \to X$, such that $\varphi\psi \neq 0$.

Proof. Let f be a number such that the equation $p^f x = \varphi$ is not solvable. Let us consider the diagram

$$\begin{array}{ccccc} & & W & & \\ & & \uparrow \varphi & & \\ X & \xleftarrow{p^f} & X & \xleftarrow{\psi} & \bar{S}'(X/_{p^f}X) \end{array}$$

It is clear that $\varphi\psi \neq 0$. This means that the spectrum $X' = S(X/_{p^f}X)$ is the one desired. The fact that the spectrum X' is an S_p^0-complex results from proposition 5. We have thus proved:

THOEREM 2. For a CW-complex X and a spectrum Y the spectral sequence with the term

$$E_2 = \mathrm{Ext}_{\Gamma_p}([S^*S_p^0, X], [S^*S_p^0, Y])$$

reduces to a group adjoint to the group $[X, S^*Y]$, factored into subgroups of all elements whose order is mutually prime to the number p.

Let us utilize the knowledge of the algebra Γ in low dimensions to evaluate the second member of the spectral sequence converging to the algebra of integer cohomologies and cohomology mod 2. The algebra Γ has the following generator and relationships: in the dimension 1 one generator h_1, $2h_1 = 0$; the next generator h_2 in the dimension 3, $4h_2 = h_1^3$, $8h_2 = 0$. Furthermore, in the dimension 7 the generator h_3, $16h_3 = 0$. A new generator m appears in dimension 8, $2m = 0$. In the dimension 9 the elements are h_1m, $h_1^2h_3 = h_2^3$, δ, 2δ. Furthermore, $h_1^2m = 0$, $h_1\delta \neq 0$. And, finally, in the dimension 11 there is the generator x, $h_1^2\delta = 4x \neq 0$.

The calculations $E_2 = \mathrm{Ext}_\Gamma(Z, Z)$ yield the following results (we write only nonzero terms):

$$\begin{array}{ll} E_2^{2,1} = \{Sq^3\}, & E_2^{2,3} = \{{}^1/_2\, Sq^5\}, \\ E_2^{3,4} = \{Sq^7,\ Sq^5Sq^2\}, & E_2^{4,2} = \{(Sq^3)^2\}, \\ E_2^{4,4} = \{{}^1/_2\, Sq^5Sq^3\}, & E_2^{6,3} = \{(Sq^3)^3\}, \\ E_2^{6,5} = \{{}^1/_2\, Sq^5(Sq^3)^2\}, & E_2^{8,4} = \{(Sq^3)^4\}. \end{array}$$

It is easy to note that the differential d_2 transforms the element $\frac{1}{2}Sq^5$ into the element Sq^3Sq^3 into zero, etc. This affords a foundation for assuming (see [5]) that there is a secondary cohomological operation Φ such that $2\Phi = Sq^5$. There also exist elements which vanish only for the third differential, for example an element from the term $E_3^{2,7}$.

Let us now consider $\mathrm{Ext}_\Gamma(Z_2, Z_2)$. Calculations yield the following results. There are elements reciprocal to the non-decomposable elements of the algebra Γ in the group Ext', i.e., h_0, h_1, h_2, h_3, m, δ, x; the first four are the operations Sq^1, Sq^2, Sq^4, Sq^8. Their groupings and Massey products are in the groups Ext^i, $i > 1$. In particular, there is an element X, reciprocal to the element $h_1 \otimes h_1^2 + h_1^2 \otimes h_1$, in the group $\mathrm{Ext}^{2,3}$. Since the operation $Sq^1Sq^4 + Sq^4Sq^1$ is decomposable into three cofactors, we then have the relation $Sq^1Sq^4 = Sq^4Sq^1$ in the group $\mathrm{Ext}^{2,3}$. The differential d_2 transforms the element X into the element $Sq^1Sq^2Sq^1Sq^2 + Sq^2Sq^1Sq^2Sq^1$, the element $Sq^1X = XSq^1$ into the element $Sq^1Sq^2Sq^1Sq^2Sq^1$, etc. The question arises: What filtration do the operations Sq^2 have? It can be shown that the operation Sq^{16} has the filtration 2 (see [6], say), and is reciprocal to the element $h_3 \otimes 2h_3$. It can also be shown that the operation Sq^{32} has filtration ≥ 3.

4. Convergence of a Spectral Sequence for an Algebra of Operations of U-Cobordisms.

Let us consider the spectrum $MU = \{MU(k)\}$, where MU(k) are Thom complexes of bundles of the space BU(k) (see [7]). We shall study the homological spectral sequence, i.e., the spectral sequence associated with the free MU-resolvent of the spectrum X.

Proposition 7. Let the integer cohomologies of the spectrum X have no torsion, and let $[S^nS_0, X]$ be a nonzero group of minimal dimension n. Let $\varphi : S^nS_0 \to X$, $\varphi \neq 0$. Then there exists a mapping f: $X \to S^nMU$ such that $f\varphi \neq 0$.

For the proof we must consider the Postnikov system for the spectrum S^nMU. All the obstructions have finite order. This means such a mapping exists.

Proposition 8. If integer cohomologies of an n-connected spectrum X have no torsion, the spectrum X is then a MU-complex.

Proof. Let n_0 be the least number such that $[S^{n_0}S^0, X] \neq 0$. This is a free abelian group. Let us select its basis $x_1, \ldots, x_s$. Let us map the spectrum X into $\bigvee_{i=1}^{s} (S^{n_0}MU)_i$ so that the generator x_i would go over into the generator of the i-th component of the group $[S^{n_0}S^0, \bigvee S^{n_0}MU]$. We have the exact sequence

$$\ldots \to [S^{n_0}S^0, X_1] \to [S^{n_0}S^0, X] \to [S^{n_0}S^0, \bigvee S^{n_0}MU] \to 0,$$

where φ is an isomorphism. This means that $[S^{n_0}S^0, X_1] = 0$. It may turn out that the spectrum X_1 will already have torsion in the cohomologies. Let us write the exact sequence in cohomologies:

$$0 \leftarrow H^{n_0+2}(X_1) \leftarrow H^{n_0+2}(X) \leftarrow H^{n_0+2}(\bigvee S^{n_0}MU)$$
$$\leftarrow H^{n_0+1}(X_1) \leftarrow H^{n_0+1}(X) \leftarrow H^{n_0+1}(\bigvee S^{n_0}MU) = 0.$$

Thus, if the mapping $H^{n_0+2k}(\bigvee S^{n_0}MU) \to H^{n_0+2k}(X)$ were an epimorphism, then the cohomologies of the spectrum X would be without torsion. Since all the differentials are trivial for the spectral sequence whose member E_2 equals $H^*(X, U^*(X))$, then there exists a mapping $X \to \bigvee S^{n_0}MU \bigvee S^{n_i}MU$, where $n_i > n_0 + 1$, such that the above-mentioned homomorphism will be an epimorphism. We henceforth construct the sequence by induction.

Proposition 9. If the periodic part of the groups $H^*(X)$ for the CW-complex X is found only in a finite number of dimensions, there then exists an MU-free sequence $X \leftarrow X_1 \leftarrow \ldots \leftarrow X_n \ldots$, such that

$$\bigcap_n \mathrm{Im}\,([S^*S^0, X_n] \to [S^*S^0, X]) = 0.$$

Proof. It is sufficient to construct a sequence such that its cohomologies have no torsion for a certain member X_n. Let n_0 be the greatest number such that $H^{n_0}(X)$ has torsion. Let X^{n_0-1} be an $(n_0 - 1)$ skeleton of the space X. Let us consider the mapping $X \leftarrow X/X^{n_0-1}$ where the space X/X^{n_0-1} already has no torsion, and the mapping $H^i(X) \leftarrow H^i(X/X^{n_0-1})$ is an epimorphism for $i \geq n_0$. Furthermore, there exists a mapping $X/X^{n_0-1} \to \bigvee MU$. which is epimorphic in cohomologies. Thus, we have the mapping, $X \to \bigvee MU$, where it is epimorphic in dimensions $i \geq n_0$ in cohomologies. Let X_1 be the "kernel" of this mapping. We have the exact sequence

$$\leftarrow H^{n_0+1}(\bigvee MU) \leftarrow H^{n_0}(X_1) \leftarrow H^{n_0}(X) \leftarrow H^{n_0}(\bigvee MU) \leftarrow .$$

This means the groups $H^i(X_1)$ are torsion-free for $i \geq n_0$. After n_0 steps we obtain the torsion-free complex Xn_0 in cohomologies.

Since for any spectrum X and number n there exists the mapping $f: X \to Y$, which is a homotopic equivalence in the dimensions $\leq n$, and the cohomologies Y are torsion-free in dimensions $\geq n'$, we have then proved the following:

THEOREM 3. For any CW-complex X there exists a spectral sequence whose member E_2 equals $\mathrm{Ext}_{[MU, S^*MU]}([X, S^*MU], [Y, S^*MU])$ and which converges to a group adjoint to the group $[S^*Y, X]$.

LITERATURE CITED

1. E. L. Lima, The Spanier-Whitehead duality in new homotopy categories, Summa Brasil. Math., Vol. 4, 91-148 (1959).
2. J. F. Adams, On the structure and application of the Steenrod algebra, Comment. Math. Helv., Vol. 32, No. 3, 180-214 (1958).
3. A. Cartan and S. Eilenberg, Homological Algebra, Moscow (1960).
4. J. P. Serre, Homotopic groups and classes of abelian groups, Collection: "Partitioned Spaces" [Russian translation], Moscow, pp. 124-159 (1958).
5. C. R. F. Maunder, Cohomology operations of the N-th kind, Proc. London Math. Soc. (3), Vol. 13, No. 49, 125-154 (1963).
6. H. Toda, Composition methods in homotopy groups of spheres, Princeton (1962).
7. S. P. Novikov, Homotopic properties of a Thom complex, Matem. Sb., Vol. 57, No. 4, 407-442 (1962).