A generalization of the topological Brauer group

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Let X be a finite CW-complex, and k a positive integer greater than 1. By A_{k^m}, B_{k^r} we denote locally trivial bundles over X with fibers the matrix algebras $M_{k^m}(\mathbb{C})$, $M_{k^r}(\mathbb{C})$, respectively. On the set of such bundles consider the following equivalence relation:

$$A_{k^m} \sim B_{k^r} \Leftrightarrow \text{ there are vector bundles } \xi_{k^t},\ \eta_{k^v} \text{ with fibers } \mathbb{C}^{k^t},\ \mathbb{C}^{k^v}, \text{ respectively,}$$

$$\text{such that } A_{k^m} \otimes \mathrm{End}(\xi_{k^t}) \cong B_{k^r} \otimes \mathrm{End}(\eta_{k^v}).$$

Then the k-primary component $\mathbf{Br}_k(X)$ of the topological Brauer group $\mathbf{Br}(X)$ of X is the group of equivalence classes of this relation with respect to the operation induced by the tensor product of bundles.

In my talk some generalization of this group will be introduced. More precisely, let k, l be coprime positive integers greater than 1. By $\mathbb{M}_{k,\,l}$ denote the union of all subalgebras in the matrix algebra $M_{kl}(\mathbb{C})$ that are isomorphic to the matrix algebra $M_k(\mathbb{C})$. Consider $\mathbb{M}_{k,\,l}$ as a topological space with respect to the topology induced by the canonical embedding $\mathbb{M}_{k,\,l} \subset M_{kl}(\mathbb{C})$. Let $\mathrm{Aut}(\mathbb{M}_{k,\,l})$ be the group of homeomorphisms of $\mathbb{M}_{k,\,l}$ such that their restriction to any subalgebra $A \subset \mathbb{M}_{k,\,l},\ A \cong M_k(\mathbb{C})$, is a $\mathbb{C}$-algebras isomorphism with a subalgebra $B \subset \mathbb{M}_{k,\,l},\ B \cong M_k(\mathbb{C})$. We equip $\mathrm{Aut}(\mathbb{M}_{k,\,l})$ with the compact-open topology related to the natural action $\mathrm{Aut}(\mathbb{M}_{k,\,l}) \times \mathbb{M}_{k,\,l} \to \mathbb{M}_{k,\,l}$.

We construct our generalized Brauer group $\mathbf{GBr}$ by means of locally trivial bundles with fibers $\mathbb{M}_{k,\,l}$ for different k, l as above. The natural embedding $M_{k^m}(\mathbb{C}) \cong M_{k^m}(\mathbb{C}) \underset{\mathbb{C}}{\otimes} \mathbb{C}E_{l^n} \hookrightarrow \mathbb{M}_{k^m,\,l^n}$ can be extended to the functor $\mathbf{F}_{k^m,\,l^n}$ which to any $M_{k^m}(\mathbb{C})$-bundle A_{k^m} over X associates the $\mathbb{M}_{k^m,\,l^n}$-bundle $\mathbf{F}_{k^m,\,l^n}(A_{k^m})$ over X.

On the set of $\mathbb{M}_{k^m,\,l^n}$-bundles (for arbitrary $m,\ n \geq 1$) over X we consider the following equivalence relation

$$\mathcal{A}_{k^m,\,l^n} \sim \mathcal{B}_{k^r,\,l^s} \Leftrightarrow \text{ there are vector bundles } \xi_{k^t},\ \eta_{k^v}$$

$$\text{with fibers } \mathbb{C}^{k^t},\ \mathbb{C}^{k^v}, \text{ respectively, and integers } u,\ w \geq 1 \text{ such that}$$

$$\mathcal{A}_{k^m,\,l^n} \otimes \mathbf{F}_{k^t,\,l^u}(\mathrm{End}(\xi_{k^t})) \cong \mathcal{B}_{k^r,\,l^s} \otimes \mathbf{F}_{k^v,\,l^w}(\mathrm{End}(\eta_{k^v})). \tag{1}$$

Then the k-primary component $\mathbf{GBr}_k(X)$ of $\mathbf{GBr}(X)$ is defined to be the group of equivalence classes of this relation with respect to the product induced by the tensor product of $\mathbb{M}_{k,\,l}$-bundles.

Note that the k-primary component $\mathbf{GBr}_k(X)$ of the generalized Brauer group, similarly to $\mathbf{Br}_k(X)$, can be treated as the group of obstructions to the lifting of locally trivial bundles $\mathcal{A}_{k^m,\,l^n}$ with fibers $\mathbb{M}_{k^m,\,l^n}$ to bundles of the form $\mathbf{F}_{k^m,\,l^n}(\mathrm{End}(\xi_{k^m}))$ for some vector $\mathbb{C}^{k^m}$-bundle ξ_{k^m}. Compared to the classical case, we see that a whole new step is added to the procedure of lifting in (1): first of all, we have to lift locally trivial bundles $\mathcal{A}_{k^m,\,l^n}$ to bundles of the form $\mathbf{F}_{k^m,\,l^n}(A_{k^m})$ for some $M_{k^m}(\mathbb{C})$-bundle A_{k^m} (the second step essentially coincides with the classical case).

Suppose a finite CW-complex X is obtained by applying the forgetful functor to some algebraic variety $\widetilde{X}$ over $\mathbb{C}$. Then the group $\mathbf{Br}(X)$ is not only a homotopy invariant of X but also a birational invariant of $\widetilde{X}$. I believe that *the group* $\mathbf{GBr}(X)$ *should be a birational invariant of* $\widetilde{X}$, *too.*