Symmetries in complex cobordism theory related to stable equivalence classes of SU-bundles

A.V. Ershov

Let $F(u, v) \in \mathrm{MU}^*(\mathrm{pt})[[u, v]]$ be the formal group of complex cobordisms. Suppose X is a finite CW-complex, then for any line bundles ζ_1, ζ_2 over X we have the following relation

$$c_1(\zeta_1 \otimes \zeta_2) = F(c_1(\zeta_1), c_1(\zeta_2)),$$

where c_1 denotes the first Chern class in complex cobordism theory.

In the first part of my talk I will discuss the following generalization of this relation to the case of virtual U-bundles of virtual dimension 1. Fix a virtual SU-bundle ξ over X of virtual dimension 1 and consider the set of virtual bundles $\mathcal{L}(\xi) := \{\zeta \otimes \xi \mid \zeta \text{ is a line bundle over } X\}$. The set $\mathcal{L}(\xi)$ can be equipped with the group structure in the following way: for $\widehat{\xi}_i := \zeta_i \otimes \xi \in \mathcal{L}(\xi)$, $i = 1, 2$ the product $\widehat{\xi}_1 \circ \widehat{\xi}_2$ is the virtual bundle $\widehat{\xi}_1 \otimes \widehat{\xi}_2 \otimes \xi^{-1} = \zeta_1 \otimes \zeta_2 \otimes \xi \in \mathcal{L}(\xi)$. We claim that there is a formal group $\widetilde{F}(x, y) \in \mathrm{MU}^*(\mathrm{BSU})[[x, y]]$ such that $c_1(\widehat{\xi}_1 \circ \widehat{\xi}_2) = (\varphi_\xi^* \widetilde{F})(c_1(\widehat{\xi}_1), c_1(\widehat{\xi}_2))$, where $\varphi_\xi \colon X \to \mathrm{BSU}$ is a classifying map for ξ. Moreover, the formal group $\widetilde{F}(x, y)$ is a re-parameterization of the usual formal group $F(u, v)$ by means of the formal series $h(u) \in \mathrm{MU}^*(\mathrm{BSU})[[u]]$ whose explicit form will be given.

In the second part of the talk for any virtual SU-bundle ξ over X of virtual dimension 1 we construct a ring automorphism T_ξ of the component $\mathrm{MU}^{ev}(X) := \oplus_{q \in \mathbb{Z}} \mathrm{MU}^{2q}(X)$ of the cobordism ring $\mathrm{MU}^*(X)$. The automorphism T_ξ has the following properties:

- it depends naturally on the pair (X, ξ);
- T_ξ maps the coefficients subring $\mathrm{R} := \mathrm{MU}^*(\mathrm{pt}) \subset \mathrm{MU}^{ev}(X)$ isomorphically to $\varphi_\xi^*(\mathrm{K}) \subset \mathrm{MU}^{ev}(X)$, where $\mathrm{K} \subset \mathrm{MU}^*(\mathrm{BSU})$ is the subring generated by the coefficients of $\widetilde{F}(x, y) \in \mathrm{MU}^*(\mathrm{BSU})[[x, y]]$;
- $\mathrm{T}_\xi(c_1(\zeta)) = c_1(\xi \otimes \zeta)$ for any line bundle ζ over X.

Furthermore, we prove that the assignment $\xi \mapsto T_\xi$ is the group homomorphism

$$(1 + \widetilde{\mathrm{KSU}}(X))^\times \to \mathrm{Aut}_{ring}(\mathrm{MU}^{ev}(X)),$$

i.e. $\mathrm{T}_\eta \circ \mathrm{T}_\xi = \mathrm{T}_{\eta \otimes \xi}$ for all virtual SU-bundles ξ, η over X of virtual dimension 1.