TORIC TOPOLOGY IN THE WORK OF TARAS PANOV

1. Introduction

Since the 1970s, the study of torus actions has become increasingly important in various areas of pure mathematics, and has stimulated the formation of interdisciplinary links between algebraic geometry, combinatorial and convex geometry, commutative and homological algebra, differential topology and homotopy theory. As their net has spread wider, and the literature grown, a field of activity has emerged which merits the title *toric topology*. Toric topology is the study of algebraic, combinatorial, differential, geometric, and homotopy theoretic aspects of a particular class of torus actions, whose quotients are highly structured. A characteristic feature is the calculation of invariants in terms of combinatorial data associated to the quotients; a primary goal is to classify toric spaces by means of these invariants.

The initial impetus for these developments was provided by the theory of *toric varieties* in algebraic geometry. This theory gives a bijection between, on one hand, complex algebraic varieties that are equipped with an action of a complex torus with a dense orbit, and, on the other hand, *fans*, which are combinatorial objects. The fan allows one to completely translate various algebraic-geometric notions into combinatorics. A valuable aspect of this theory is that it provides many explicit examples of algebraic varieties, with applications in deep subjects such as resolution of singularities and mirror symmetry. The quotient of a projective variety by the action of the compact torus T^n is a convex simple polytope P. The polar polytope of P is necessarily simplicial, and its boundary is a simplicial complex K.

In symplectic geometry, since the early 1980s there has been much activity in the field of Hamiltonian group actions, largely following the Atiyah–Guillemin–Sternberg [6] convexity theorem and the Duistermaat–Heckman exact stationary phase formula [14]. Atiyah–Bott–Berline–Vergne exhibited the latter as a special case of localisation in equivariant cohomology, thus putting many of the activities on Hamiltonian group actions into the context of equivariant topology rather than symplectic geometry. Delzant [13], in 1988, showed that if the torus is of half the dimension of the manifold the moment map image determines the manifold up to equivariant symplectomorphism. In symplectic geometry, as in algebraic geometry, one translates various geometric constructions into the language of convex polytopes and combinatorics.

There is a tight relationship between the algebraic and the symplectic pictures: a projective embedding of a toric manifold determines a symplectic form and a moment map. The image of the moment map is a convex polytope that is dual to the fan. In both the smooth algebraic-geometric and the symplectic situations, the compact torus action is locally isomorphic to the standard action of $(S^1)^n$ on $\mathbb{C}^n$ by rotation of the coordinates. Thus the quotient of the manifold by this action is naturally a manifold with corners, stratified according to the dimension of the stabilisers, and each stratum can be equipped with data that encodes the isotropy torus action along that stratum.

Not only does this structure of the quotient provide a powerful means of investigating the action, but some of its subtler combinatorial properties may also be illuminated by the topology of the manifold. Taking into account the topological nature of this feature, there is no surprise that since the beginning of the 1990s the ideas and methodology of toric varieties and Hamiltonian actions have started penetrating back into algebraic topology.

Specifically, during the last two decades, the examples of smooth toric varieties and of symplectic toric manifolds have been generalised into several other classes of manifolds with torus action, mostly of purely topological nature. These more general manifolds are not necessarily algebraic or symplectic; thus there is more flexibility within these classes of spaces for topological or combinatorial applications, but on the other hand they still possess most of the important topological properties of their algebraic or symplectic predecessors. We now describe in more detail some of these generalisations.

Davis and Januszkiewicz' influential study [12] of toric varieties from a topological viewpoint led to the appearance of *quasitoric manifolds*. These manifolds are determined by two conditions: the T^n-action locally looks like the standard T^n-representation in the complex space $\mathbb{C}^n$, and that the orbit space Q is combinatorially a simple convex polytope. (Both conditions are satisfied for the torus action on a non-singular projective toric variety).

Similarly, Hattori and Masuda's generalisation of toric varieties led to a wider class of *torus manifolds* [17]. Apart from the usual conditions on the T^n-action such as smoothness and effectiveness, the fixed point set of a torus manifold is required to be non-empty. Perhaps surprisingly, these more general torus manifolds also admit a combinatorial treatment similar to that of classic toric varieties in terms of fans and polytopes. Namely, torus manifolds may be described by *multi-fans* and *multi-polytopes*.

For either a quasitoric or torus manifold M, the faces of the quotient Q form a simplicial poset S with respect to reverse inclusion. In the case of quasitoric manifolds, this poset is the poset of faces of a genuine simplicial complex K which is dual to the polytope Q.

The concept of a *GKM-manifold* is closely related to Hamiltonian torus actions. According to [16], a compact $2n$-dimensional manifold M with an effective torus action $T^k \times M \to M$ $(k \leq n)$ is called a GKM-manifold if the fixed point set is finite, M possesses an invariant almost complex structure, and the weights of the tangential T^k-representations at the fixed points are pairwise linearly independent. These manifolds are named after Goresky, Kottwitz and MacPherson, who studied them in [15]. They showed that the "one-skeleton" of such a manifold M, that is, the set of points fixed by at least a codimension-one subgroup of T^k, has the structure of a labelled graph (Γ, α), and that the most important topological information about M, such as its Betti numbers or equivariant cohomology ring, can be read directly from this graph. These graphs have since become known as *GKM-graphs* (or *moment graphs*); and their study has been of independent combinatorial interest since the appearance of Guillemin and Zara's paper [16]. The idea of associating a labelled graph to a manifold with a circle action also featured in the work of Musin.

Stanley was one of the first to realise the full potential of torus actions for combinatorial applications, using it to prove McMullen's conjectured *g-theorem* for face vectors of simplicial polytopes, and the *Upper Bound Conjecture* for triangulated spheres. His work ensured that commutative algebra and homological techniques have been intertwined with combinatorial geometry ever since, and forms a core part of his influential book [25].

Many of Stanley's ideas extend to the topological context, and the *face ring* or *Stanley–Reisner algebra* $\mathbb{Z}[K]$ of K is a crucial ingredient in the computation of the integral cohomology ring of a quasitoric manifold M. In performing this calculation, Davis and Januszkiewicz associate an auxiliary T^m-space $\mathcal{Z}_K$ to any complex K on m vertices, and considered its homotopy quotient (or Borel construction) $DJ(K)$. Their definition of $\mathcal{Z}_K$ is inspired by Vinberg's universal space for reflection groups (and so is analogous to that of the *Coxeter complex*). Davis and Januszkiewicz show that the cohomology ring $H^*(DJ(K))$ (or the *equivariant cohomology* $H_T^*(M)$) is isomorphic to $\mathbb{Z}[K]$ for any K. They also deduce that the ordinary cohomology $H^*(M)$ is obtained from $\mathcal{Z}[K]$ by factoring out certain linear forms, exactly as in the situation with toric varieties.

Since the appearance of Stanley's book, an increasing body of work has confirmed that $\mathbb{Z}[K]$ encodes many subtle combinatorial properties of K. The study of such algebras has now acquired its own momentum, and has added a geometrical flavour to the well-developed study of *Cohen–Macaulay rings*. In particular, a notion of a *Cohen–Macaulay complex* K, whose face ring $\mathbb{Z}[K]$ is Cohen–Macaulay, has now become an object of an independent topological study. Many of these developments are surveyed in [7], where the importance of the homological viewpoint is made clear. For example, the dimensions of the bigraded components of the vector spaces $\mathrm{Tor}_{k[v_1,\ldots,v_m]}(k[K],k)$ are known as the algebraic Betti numbers of $k[K]$, for any field k. These numbers are quite subtle invariants: they depend on a combinatorics of K rather than on the topology of its realisation $|K|$, and fully determine the "ordinary" topological Betti numbers of $|K|$. Hochster's theorem gives an expression for the algebraic Betti numbers in terms of the homology of full subcomplexes of K.

2. Toric Topology in the work of Taras Panov

A detailed account of quasitoric manifolds, moment-angle complexes, their role in toric topology, and applications to combinatorial geometry and homological algebra is contained in monograph [9] of Buchstaber and Panov, published with the AMS in University Lecture Series. In 2004, its extended Russian edition [4] appeared. Some results have since been further developed in works of Panov, partly joint with other colleagues and collaborators.

Hirzebruch genera of toric and quasitoric manifolds. The work of Buchstaber and Ray reveals that toric manifolds are influential players in the complex cobordism theory, a classical subject in algebraic topology. Unlike toric varieties, quasitoric manifolds may fail to be complex; however, they always admit a stably (or weakly almost) complex structure, and their cobordism classes generate the complex cobordism ring [11]. A stably complex structure on a quasitoric manifold is defined in purely combinatorial terms, namely, by an orientation of the polytope and a *characteristic function* from the set of codimension-one faces of the P^n to primitive vectors in $\mathbb{Z}^n$. The characteristic function of a quasitoric manifold may be thought of as the replacement of the fan corresponding to a toric variety in toric geometry. In his work [5] Panov obtains effective combinatorial formulae calculating several important Hirzebruch genera of quasitoric manifolds in terms of the characteristic function. These formulae extend certain known Riemann–Roch–Hirzebruch type results of toric geometry to quasitoric manifolds. In the case of the top Chern number c_n and the Todd genus Panov's formulae lead to obstructions to the existence of an equivariant almost complex structure on a quasitoric manifold, which sheds some light on a problem of Davis and Januszkiewicz [12].

Moment-angle complexes. The study of *moment-angle complexes*, of which T. Panov is one of the important contributors, is one of the key ingredients in most modern applications of toric topology. It is also closely related to the study of quasitoric manifolds. In their work [12], Davis and Januszkiewicz assigned an auxiliary T^m-space $\mathcal{Z}_K$ to an arbitrary simplicial complex K on m vertices. It has become evident that the spaces $\mathcal{Z}_K$ are of great independent interest in toric topology, and they become known as moment-angle complexes [9]. They arise in homotopy theory as *homotopy colimits* [23], in symplectic topology as level surfaces for the *moment maps* of Hamiltonian torus actions [22], and in the theory of arrangements as *complements of coordinate subspace arrangements* [9, Ch. 8]. The construction of moment-angle complexes gives rise to a functor from the category of simplicial complexes and simplicial maps to the category of spaces with torus actions and equivariant maps. This functor provides an effective way to study invariants of triangulations by the methods of equivariant topology. If K is a triangulation of an $(n-1)$-dimensional sphere, then $\mathcal{Z}_K$ is an $(m+n)$-dimensional manifold. If $K = \partial P$ is a dual triangulation of the boundary of a simple polytope, then for arbitrary quasitoric manifold M^{2n} with orbit space P there is a principal T^{m-n}-bundle $\mathcal{Z}_K \to M^{2n}$.

Cohomology of moment-angle complexes and face rings. The cohomology ring of a moment-angle complex was calculated by Buchstaber and Panov in [1] in terms of the combinatorics of the simplicial complex K. The cohomology algebra $H^*(\mathcal{Z}_K)$ is shown to be isomorphic to the Tor-*algebra* $\mathrm{Tor}_{\mathbb{Z}[v_1,\dots,v_m]}(\mathbb{Z}[K],\mathbb{Z})$, where $\mathbb{Z}[K]$ is the face ring of K. Through this isomorphism the canonical "algebraic" bigrading in Tor acquires a geometrical realisation; it corresponds to a "topological" bigrading of cells in a certain cellular decomposition of $\mathcal{Z}_K$. A further analysis leads to an effective description of the Tor-algebra in terms of the Koszul complex, which allows us to use computer packages like `Macaulay2` or `BISTELLAR` for calculations in combinatorial homological algebra and combinatorial geometry.

Subspace arrangement complements. Due to the interdisciplinary nature of the study of moment-angle complexes, Buchstaber and Panov's results on the cohomology of $\mathcal{Z}_K$ got instant applications in several other fields. The realisations of $\mathcal{Z}_K$ as a non-compact toric variety and as the level surface for a toric moment map lead to cohomology calculations in the algebraic geometry of toric varieties. Moment-angle complexes are also shown to be equivariant deformation retracts of the complements of arrangements of coordinate subspaces in a complex space, and therefore, the cohomology of these complements may also be effectively described. It should be mentioned that all previous results on the cohomology of coordinate subspace arrangement complements either did not describe the multiplicative structure (like the general Goresky–MacPherson theorem), or just provided a description of the product of two given cocycles in combinatorial terms (like the results of de Longueville). The Buchstaber–Panov theorem on moment-angle complexes provides a complete global description of the cohomology ring of a coordinate subspace arrangement complement.

Face vectors and Dehn–Sommerville equations. Buchstaber and Panov's calculation of $H^*(\mathcal{Z}_K)$ has strong ties with the class of combinatorial problems related to *face vectors* of polytopes and triangulations. Many important properties of face vectors may be described by expressing them in terms of Betti numbers of moment-angle complexes. The well-known *Dehn–Sommerville relations* for the numbers f_i of faces of dimensions $0 \le i \le n-1$ in an $(n-1)$-dimensional sphere triangulation K arise as a consequence of the bigraded Poincaré duality, discovered in the theory of moment-angle complexes. These relations may be written in the form $h_i = h_{n-i}$, where $h(K) = (h_0, h_1, \ldots, h_n)$ is the so-called *h-vector* of the triangulation, whose components are certain linear combinations of the numbers f_i. In the case of triangulations of arbitrary manifolds a more subtle analysis of the bigraded Poincaré duality for moment-angle complexes led to the *generalised Dehn-Sommerville equations* of the form $h_{n-i} - h_i = (-1)^i(\chi(K) - \chi(S^{n-1}))\binom{n}{i}$ valid for arbitrary triangulated manifolds.

Toric topology and geometric invariant theory. Recently the theory of moment-angle complexes has been bridged to the study of algebraic group actions. In work of Panov [22] *Kempf–Ness-type sets* are constructed for certain algebraic torus actions on quasiaffine varieties. In the classical situation of algebraic group actions on affine varieties, the concept of a Kempf–Ness set is used to replace the categorical quotient by the quotient with respect to a maximal compact subgroup. It turns out that moment-angle complexes play the role of Kempf–Ness sets for a class of algebraic torus actions on quasiaffine varieties (coordinate subspace arrangement complements) arising in Batyrev and Cox' "geometric invariant theory" approach to toric varieties. In the case of smooth projective toric varieties the corresponding "toric" Kempf–Ness sets can be described as complete intersection of real quadrics in a complex space. The calculations with moment-angle complexes apply to obtain an explicit description of the cohomology of these Kempf–Ness sets.

Analogous polytopes and cobordisms of quasitoric manifolds. In work of Buchstaber–Panov–Ray [10] the theory of analogous polytopes is applied to the study of quasitoric manifolds, in the context of stably complex manifolds with compatible torus action. The theory of analogous polytopes was initiated by Alexandrov in the 1930s, and extended more recently by Khovanskii and Pukhlikov. By way of application, we give an explicit construction of a quasitoric representative for every complex cobordism class as the quotient of a free torus action on a real quadratic complete intersection. The latter complete intersection is nothing but a yet another disguise of the moment-angle complex. We suggest a systematic description for omnioriented quasitoric manifolds in terms of combinatorial data, and explain the relationship with non-singular projective toric varieties (otherwise known as toric manifolds). By expressing the Buchstaber and Ray's approach [11] to the representability of cobordism classes in these terms, we simplify and correct two of their original proofs concerning quotient polytopes; the first relates to framed embeddings in the positive cone, and the second involves modifying the operation of connected sum to take account of orientations. Analogous polytopes provide an informative setting for several of the details.

Homotopy-theoretical aspects of toric topology. Different constructions of *homotopy direct limits* are gaining considerable importance in applications of homotopy theory. In work [23] of Panov, Ray and Vogt it is proven that the classifying space and loop functor commute with the homotopy direct limit functor (note that this is not true for the classical direct limit functor). As the corollary, the authors obtain models for the loop spaces on moment-angle complexes and their Borel constructions, now known as the *Davis–Januszkiewicz spaces*. The models have the form of homotopy direct limits of diagrams of tori in the category of topological groups. Applications of these models include a calculation of the Ext-cohomology $\mathrm{Ext}_{\mathbb{Z}[K]}(\mathbb{Z}, \mathbb{Z})$ of Stanley–Reisner rings. This calculation relies upon an isomorphism between the Ext-cohomology of a Stanley–Reisner ring and the Pontrjagin homology ring of the loops on the corresponding Davis–Januszkiewicz space.

Cohomology of torus manifolds. In [20] Masuda and Panov obtaine a series of results relating the cohomological properties of torus manifolds to the combinatorics of their orbit quotients; we give a summary of this results here. The cohomology ring of a torus manifold M is generated by the degree-two classes if and only if the orbit space Q is a homology polytope (that is, all faces, including Q itself, are acyclic and every non-empty intersection of faces is connected). In this case the cohomology ring itself has the structure familiar from toric geometry; in particular the cohomology vanishes in odd degrees. However, torus manifolds M with vanishing odd-degree cohomology constitute a more general class. This class is characterised by the property that the equivariant cohomology of M is a *Cohen–Macaulay ring*, i.e. a finitely generated free module over the equivariant cohomology of point. The orbit space of a torus manifold with $H^{odd}(M) = 0$ is not necessarily a homology polytope, as a simple example of a torus action on even-dimensional sphere shows. This led to another concept of a *face-acyclic* manifold with corners Q, in which all faces are still acyclic, but their intersection may fail to be connected. It is proven that $H^{odd}(M) = 0$ if and only if the orbit quotient Q is face-acyclic. In this case the equivariant cohomology ring is shown to be isomorphic to the face ring of the *simplicial poset* [24] of faces of Q. Thereby, a new class of Cohen–Macaulay rings arises in the study of torus actions. These rings are more general than Stanley–Reisner rings of polytopes and are not generated by the degree-two elements.

Simplicial posets and applications to combinatorial commutative algebra. Combinatorial structures arising in the orbit spaces of manifolds with torus actions include not only polytopes and simplicial complexes, but also more sophisticated ones, like labelled graphs or *simplicial posets*. The topological study of the torus action may in some cases lead to new interesting results of purely combinatorial or algebraic nature. In the context of simplicial posets, this approach has been explored in recent works of Buchstaber–Panov [3], Masuda [19] and Maeda–Masuda–Panov [18]. For several important classes of manifolds acted on by the torus, the information about the action can be encoded combinatorially by a regular n-valent graph with vector labels on its edges, which we refer to as the *torus graph*. An important family of torus graphs arises from the orbit spaces of torus manifolds, which explains the terminology. By analogy with the theory of *GKM-graphs* originating from the work of Goresky–Kottwitz–MacPherson [15] on symplectic torus actions, we introduce in [18] the notion of *equivariant cohomology* of a torus graph, and show that it is isomorphic to the face ring of the associated simplicial poset. This extends a series of previous results on the equivariant cohomology of torus manifolds. As a primary combinatorial application, we show that a simplicial poset is *Cohen–Macaulay* if its face ring is Cohen–Macaulay. This completes the algebraic characterisation of Cohen–Macaulay posets initiated by Stanley [24]. We also study *blow-ups* of torus graphs and manifolds from both the algebraic and the topological points of view.

Semifree circle actions and Bott towers. A *Bott tower* is the total space of a tower of fibre bundles with base $\mathbb{C}P^1$ and fibres $\mathbb{C}P^1$. Every Bott tower of height n is a smooth projective toric variety whose moment polytope is combinatorially equivalent to an n-cube. A circle action is *semifree* if it is free on the complement to fixed points. In work [21] we show that a (quasi)toric manifold over an n-cube with a semifree circle action and isolated fixed points is a Bott tower. Then we show that every Bott tower obtained in this way is topologically trivial, that is, homeomorphic to a product of 2-spheres. This extends a recent result of Ilinskii, who showed that a smooth compact toric variety with a semifree circle action and isolated fixed points is homeomorphic to a product of 2-spheres, and makes a further step towards our understanding of a problem motivated by Hattori's work on semifree circle actions. Finally, we show that if the cohomology ring of a quasitoric manifold (or Bott tower) is isomorphic to that of a product of 2-spheres, then the manifold is homeomorphic to the product.

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