Exponential actions, holomorphic foliations and irrational toric geometry

Taras Panov

Steklov Mathematical Institute
Moscow State University
Higher School of Economics, Moscow

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Vector configurations and exponential actions

$V \cong \mathbb{R}^k$ a k-dimensional real vector space
$\Gamma = \{\gamma_1, \dots, \gamma_m\}$ a configuration of m vectors in the dual space V^*.
Allow repetitions, but assume that $\gamma_1, \dots, \gamma_m$ span V^*.

The exponential action of V on $\mathbb{R}^m$:

$$V \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$$
$$(\mathbf{v}, \mathbf{x}) \mapsto \mathbf{v} \cdot \mathbf{x} = \left(e^{\langle \gamma_1, \mathbf{v} \rangle} x_1, \dots, e^{\langle \gamma_m, \mathbf{v} \rangle} x_m\right).$$

A classical dynamical system taking its origin in the works of Poincaré.
The linear properties of Γ determine the topology of the foliation of $\mathbb{R}^m$ by the orbits of the action.

Applications

Exponential actions $V \times \mathbb{R}^m \to \mathbb{R}^m$ and their holomorphic versions arise in several important constructions of algebraic geometry and topology:

- Leaf spaces of holomophic foliations & topology intersections of real and Hermitian quadrics
  (topology & holomorphic dynamics)
- The quotient construction of toric varieties (the Batyrev–Cox construction)
  (toric geometry)
- Smooth and complex-analytic structures on moment-angle manifolds and their partial quotients
  (complex geometry and toric topology)

The moment-angle complex

$\mathcal{K}$ an abstract simplicial complex on the set $[m] = \{1, 2, \ldots, m\}$
$I = \{i_1, \ldots, i_k\} \in \mathcal{K}$ a simplex; always assume $\varnothing \in \mathcal{K}$.

Consider the m-dimensional unit polydisc:

$$\mathbb{D}^m = \{(z_1, ..., z_m) \in \mathbb{C}^m : |z_i|^2 \leqslant 1 \text{ for } i = 1, ..., m\}.$$

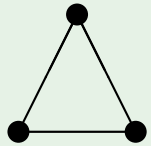
The moment-angle complex is

$$\mathcal{Z}_{\mathcal{K}} := \bigcup_{I \in \mathcal{K}} \left(\prod_{i \in I} \mathbb{D} \times \prod_{i \notin I} \mathbb{S} \right) \subset \mathbb{D}^m,$$

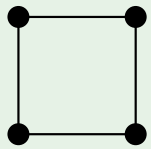
where $\mathbb{S}$ is the boundary of the unit disk $\mathbb{D}$.

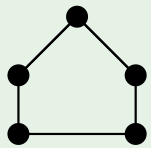
$\mathcal{Z}_{\mathcal{K}}$ has a natural action of the torus $T^m = S^1 \times \cdots \times S^1$.
When $\mathcal{K}$ is a simplicial subdivision of sphere (e.g., the boundary of a simplicial polytope), $\mathcal{Z}_{\mathcal{K}}$ is a topological manifold, called the moment-angle manifold.

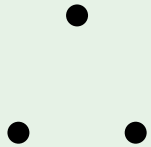
Example

1. Let $\mathcal{K} =$  (the boundary of a triangle). Then

$$\mathcal{Z}_{\mathcal{K}} = (\mathbb{D} \times \mathbb{D} \times \mathbb{S}) \cup (\mathbb{D} \times \mathbb{S} \times \mathbb{D}) \cup (\mathbb{S} \times \mathbb{D} \times \mathbb{D}) = \partial(\mathbb{D}^3) \cong S^5.$$

2. Let $\mathcal{K} =$  (the boundary of a square). Then $\mathcal{Z}_{\mathcal{K}} \cong S^3 \times S^3$.

3. Let $\mathcal{K} =$  Then $\mathcal{Z}_{\mathcal{K}} \cong (S^3 \times S^4) \# \cdots \# (S^3 \times S^4)$ (5 times).

4. Let $\mathcal{K} =$  (three disjoint points). Then

$$\mathcal{Z}_{\mathcal{K}} = (\mathbb{D} \times \mathbb{S} \times \mathbb{S}) \cup (\mathbb{S} \times \mathbb{D} \times \mathbb{S}) \cup (\mathbb{S} \times \mathbb{S} \times \mathbb{D}) \simeq S^3 \vee S^3 \vee S^3 \vee S^4 \vee S^4$$

(not a manifold).

We define an open subset $U(\mathcal{K}) \subset \mathbb{C}^m$ in a similar way:

$$U(\mathcal{K}) := \bigcup_{I \in \mathcal{K}} \Big(\prod_{i \in I} \mathbb{C} \times \prod_{i \notin I} \mathbb{C}^\times \Big), \qquad \mathbb{C}^\times = \mathbb{C} \setminus \{0\}.$$

$U(\mathcal{K})$ is a toric variety with the corresponding fan given by

$$\Sigma_{\mathcal{K}} = \{\mathbb{R}_{\geqslant} \langle e_i : i \in I \rangle : I \in \mathcal{K}\},$$

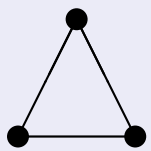
where e_i denotes the i-th standard basis vector of $\mathbb{R}^m$.

Theorem

(a) $U(\mathcal{K}) = \mathbb{C}^m \setminus \bigcup_{\{i_1,\dots,i_k\} \notin \mathcal{K}} \{z_{i_1} = \cdots = z_{i_k} = 0\}$

(the complement of an arrangement of coordinate subspaces);

(b) *There is a T^m-equivariant deformation retraction $U(\mathcal{K}) \xrightarrow{\simeq} \mathcal{Z}_{\mathcal{K}}$.*

E.g., $\mathcal{K} =$  Then $U(\mathcal{K}) = \mathbb{C}^3 \setminus \{z_1 = z_2 = z_3 = 0\} \xrightarrow{\simeq} S^5 = \mathcal{Z}_{\mathcal{K}}$.

Simplicial fans and exponential action

Suppose $\mathcal{K}$ is the underlying complex of a complete simplicial (not necessarily rational) fan Σ in $W^* \cong \mathbb{R}^n$.

Then the deformation retraction $U(\mathcal{K}) \to \mathcal{Z}_{\mathcal{K}}$ can be realised as the projection onto the orbit space of a smooth free and proper action of a non-compact subgroup $R \subset (\mathbb{C}^\times)^m$ isomorphic to $\mathbb{R}^{m-n}$, as described next.

A simplicial Σ fan is defined by generators $a_1, \dots, a_m \in W^*$ of its one-dimensional cones and a simplicial complex $\mathcal{K}$ on $[m] = \{1, \dots, m\}$.

For $I = \{i_1, \dots, i_k\} \in [m]$ we let $\mathrm{A}_I = \{a_i : i \in I\}$. Then $\Sigma = \{\operatorname{cone} \mathrm{A}_I : I \in \mathcal{K}\}$ provided that the fan condition is satisfied: $\operatorname{relint} \operatorname{cone} \mathrm{A}_I \cap \operatorname{relint} \operatorname{cone} \mathrm{A}_J = \varnothing$ for $I, J \in \mathcal{K}$, $I \neq J$.

$$0 \longrightarrow V \xrightarrow{\Gamma^*} \mathbb{R}^m \xrightarrow{A} W^* \longrightarrow 0, \qquad A(e_i) = a_i, \quad V \cong \operatorname{Ker} A$$

Gale duality $\updownarrow$

$$0 \longrightarrow W \longrightarrow \mathbb{R}^m \xrightarrow{\Gamma} V^* \longrightarrow 0, \qquad \Gamma(e_i) = \gamma_i$$

Gale dual vector configurations
$\mathrm{A} = \{a_1, \dots, a_m\}$ in $W^* \cong \mathbb{R}^n$ and $\Gamma = \{\gamma_1, \dots, \gamma_m\}$ in $V^* \cong \mathbb{R}^{m-n}$.

Set

$$R = \exp(\operatorname{Ker} A) = \left\{(e^{\langle \gamma_1, v\rangle}, \dots, e^{\langle \gamma_m, v\rangle}) \colon v \in V\right\} \subset (\mathbb{R}^\times)^m \subset (\mathbb{C}^\times)^m$$

Theorem

(1) *The exponential action of R on $U(\mathcal{K}) \subset \mathbb{C}^m$ is free and proper iff $\{\mathcal{K}; a_1, \dots, a_m\}$ defines a simplicial fan $\Sigma = A(\Sigma_\mathcal{K})$ in W^*.*

(2) *If Σ is a complete fan ($|\Sigma| = W^*$), then the quotient $U(\mathcal{K})/R$ is T^m-equivariantly homeomorphic to the moment-angle manifold $\mathcal{Z}_\mathcal{K}$.*

Holomorphic exponential actions

$\widetilde{V} = (V, \mathcal{J}) \cong \mathbb{C}^\ell$, where $\mathcal{J}\colon V \to V$ is a complex structure on $V \cong \mathbb{R}^{m-n}$ (assuming $m - n = 2\ell$ is even).

$\Gamma = \{\gamma_1, \dots, \gamma_m\}$ a vector configuration in $\widetilde{V}^* = \mathrm{Hom}_{\mathbb{C}}(\widetilde{V}, \mathbb{C})$.
Complex bilinear pairing $\widetilde{V}^* \times \widetilde{V} \to \mathbb{C}$

$$\langle \gamma, v \rangle_{\mathbb{C}} = \langle \gamma, v \rangle_{\mathbb{R}} - i \langle \gamma, \mathcal{J} v \rangle_{\mathbb{R}}.$$

The complex-analytic subgroup

$$H = \left\{ (e^{\langle \gamma_1, v \rangle_{\mathbb{C}}}, \dots, e^{\langle \gamma_m, v \rangle_{\mathbb{C}}}) \colon v \in \widetilde{V} \right\} \subset (\mathbb{C}^\times)^m$$

acts holomorphically on

$$U(\mathcal{K}) = (\mathbb{C}, \mathbb{C}^\times)^{\mathcal{K}} = \mathbb{C}^m \setminus \bigcup_{\{i_1, \dots, i_p\} \notin \mathcal{K}} \{\mathbf{z} \colon z_{i_1} = \dots = z_{i_p} = 0\}$$

Theorem

(1) *The holomorphic exponential action of* $H \cong \mathbb{C}^\ell$ *on* $U(\mathcal{K}) \subset \mathbb{C}^m$ *is free and proper iff* $\{\mathcal{K}; a_1, \dots, a_m\}$ *defines a simplicial fan* $\Sigma = A(\Sigma_\mathcal{K})$ *in* W^*.

(2) *If* Σ *is a complete fan* ($|\Sigma| = W^*$), *then the quotient* $U(\mathcal{K})/H$ *is* T^m*-equivariantly homeomorphic to the moment-angle manifold* $\mathcal{Z}_\mathcal{K}$.

Corollary

Let $\mathcal{K}$ *be the underlying complex of a complete simplicial fan (a star-shaped sphere triangulation) and* $m - n = 2\ell$ *is even.*
Then the moment-angle complex $\mathcal{Z}_\mathcal{K} = (D^2, S^1)^\mathcal{K}$ *has a structure of a compact complex manifold.*

Example (holomorphic torus)

Exponential action of $H \cong \mathbb{C}$ on $\mathbb{C}^2$ defined by a configuration $\{\gamma_1, \gamma_2\}$:

$$H = \{(e^{\gamma_1 v} z_1, e^{\gamma_2 v} z_2) \colon v \in \mathbb{C}\}, \qquad (z_1, z_2) \mapsto (e^{\gamma_1 v} z_1, e^{\gamma_2 v} z_2).$$

Let $\mathcal{K} = \{\varnothing\}$ on $[2] = \{1, 2\}$, so that $U(\mathcal{K}) = (\mathbb{C}^\times)^2$. If $\mathbb{R}\langle\gamma_1, \gamma_2\rangle = \mathbb{R}^2$, then the restriction of the action to $(\mathbb{C}^\times)^2$ is free and proper. The quotient

$$(\mathbb{C}^\times)^2/\widetilde{V} \cong \mathbb{C}/(\mathbb{Z}\langle\gamma_1, \gamma_2\rangle) \cong T^2$$

is a one-dimensional complex torus.

More generally, let $\Gamma = \{\gamma_1, \dots, \gamma_m\}$ be a real basis in $\widetilde{V} \cong \mathbb{C}^\ell$. Then A is a configuration of m zero vectors in $W^* = \{0\}$, $\mathcal{K} = \{\varnothing\}$ on $[m] = [2\ell]$ and $U(\mathcal{K}) = (\mathbb{C}^\times)^m$. The quotient $(\mathbb{C}^\times)^m/\widetilde{V} \cong \overline{\widetilde{V}}/\Lambda \cong T^{2\ell}$ is a holomorphic torus. Any holomorphic torus can be obtained in this way.

Proposition

A complex moment-angle manifold $\mathcal{Z}_K = U(\mathcal{K})/\widetilde{V}$ is non-Kähler, unless it is a holomorphic torus.

Irrational toric varieties

If the space V is rational with respect to $\Gamma = \{\gamma_1, \dots, \gamma_m\}$, then

$$L = \{v \in V : \langle \gamma_k, v \rangle \in \mathbb{Z} \text{ for } k = 1, \dots, m\}.$$

is a full rank lattice in V, and $L^* = \mathbb{Z}\langle \gamma_1, \dots, \gamma_m \rangle \subset V^*$.

The Gale dual rational vector configuration $\mathrm{A} = \{a_1, \dots, a_m\}$ spans a full rank lattice $N = M^*$ in W^*.

The algebraic torus $\mathbb{C}^\times_L = L \otimes_{\mathbb{Z}} \mathbb{C}^\times = V_{\mathbb{C}}/(2\pi i L)$ embeds in $(\mathbb{C}^\times)^m$ as the closed algebraic subgroup

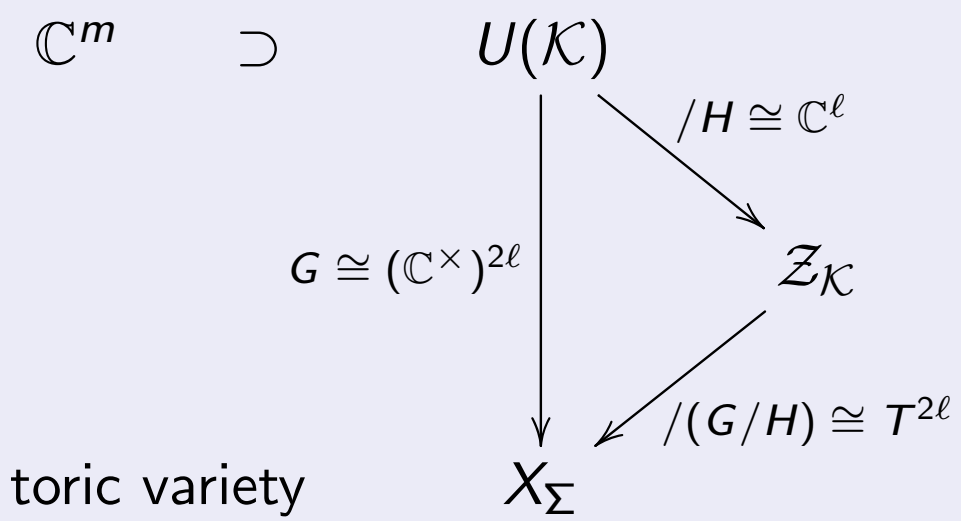
$$G = \exp \Gamma^*_{\mathbb{C}}(V_{\mathbb{C}}) = \exp(\operatorname{Ker} A_{\mathbb{C}} : \mathbb{C}^m \to W^*_{\mathbb{C}})$$

$$= \left\{(t_1, \dots, t_m) \in (\mathbb{C}^\times)^m : \prod_{i=1}^m t_i^{\langle a_i, w \rangle} = 1, \quad w \in M\right\}.$$

The fan $\Sigma = A(\Sigma_{\mathcal{K}})$ is rational and defines a toric variety

$$X_\Sigma = U(\mathcal{K})/G = \mathcal{Z}_{\mathcal{K}}/F.$$

Rational case:



moment-angle manifold

A perturbation of the vector configuration A destroys the rationality of the fan $\Sigma = \{\mathcal{K}, \mathrm{A}\}$, the subgroup $G \subset (\mathbb{C}^\times)^m$ ceases to be closed, the closed holomorphic tori in the fibres of the bundle $\mathcal{Z}_\mathcal{K} \to X_\Sigma$ "open up", and the fibre bundle turns into a holomorphic foliation $\mathcal{F}$ of $\mathcal{Z}_\mathcal{K}$ with noncompact leaves $F = G/H$.

The holomorphic foliated manifolds $(\mathcal{Z}_\mathcal{K}, \mathcal{F})$ model irrational deformations of toric varieties.

Basic de Rham and Dolbeault cohomology of the foliated manifold $(\mathcal{Z}_\mathcal{K}, \mathcal{F})$ is described in the same way as the cohomology of the toric quotient $X_\Sigma = \mathcal{Z}_\mathcal{K}/F$ in the rational case:

Theorem (Ishida–Krutowski–P)

There is an isomorphism of algebras:

$$H^{*,*}_{\mathrm{bas},\bar\partial}(\mathcal{Z}_\mathcal{K}, \mathcal{F}) \cong \mathbb{C}[v_1, \ldots, v_m]/(I_\mathcal{K} + J_\Sigma),$$

where $v_i \in H^{1,1}_{\mathrm{bas},\bar\partial}(\mathcal{Z}_\mathcal{K}, \mathcal{F})$, $i = 1, \ldots, m$,
$I_\mathcal{K}$ *is the Stanley–Reisner ideal, generated by*

$$v_{i_1} \cdots v_{i_k} \quad \text{with } \{i_1, \ldots, i_k\} \notin \mathcal{K},$$

and J_Σ *is the ideal generated by the linear forms*

$$\sum_{i=1}^{m} \langle \mathbf{a}_i, \mathbf{w} \rangle v_i, \quad \mathbf{w} \in W.$$

In the rational case, the E_3 term of the Borel spectral sequence of the holomorphic principal bundle $\mathcal{Z}_\mathcal{K} \to X_\Sigma$ with fibre $F \cong T^{2\ell}$ gives a dga model for the Dolbeault cohomology $H^{*,*}_{\bar\partial}(\mathcal{Z}_\mathcal{K})$:

$$\big(H^{*,*}_{\bar\partial}(F) \otimes H^{*,*}_{\bar\partial}(X_\Sigma), d\big),$$

where $H^{*,*}_{\bar\partial}(X_\Sigma) \cong \mathbb{C}[v_1, \dots, v_m]/(I_\mathcal{K} + J_\Sigma)$ (Danilov),

$$H^{*,*}_{\bar\partial}(F) = \Lambda\big(\widetilde{V} \oplus \overline{\widetilde{V}}\big) = \Lambda[\xi_1, \dots, \xi_\ell, \eta_1, \dots, \eta_\ell]$$

$\xi_j \in H^{1,0}_{\bar\partial}(F) = \widetilde{V}$, $\eta_j \in H^{0,1}_{\bar\partial}(F) = \overline{\widetilde{V}}$, $dv_j = d\eta_j = 0$, $d\xi_j = c(\xi_j)$,

$c\colon H^{1,0}_{\bar\partial}(F) \to H^{1,1}_{\bar\partial}(X_\Sigma)$ is the holomorphic first Chern class map (the transgression). Similarly, in the irrational case,

Theorem (P–Ustinovsky, Krutowski–P)

The Dolbeault cohomology algebra of $\mathcal{Z}_\mathcal{K}$ is given by

$$H^{*,*}_{\bar\partial}(\mathcal{Z}_\mathcal{K}) \cong H\big(\Lambda(\widetilde{V} \oplus \overline{\widetilde{V}}) \otimes H^{*,*}_{\mathrm{bas},\bar\partial}(\mathcal{Z}_\mathcal{K}, \mathcal{F}), d\big),$$

where $d(\overline{\widetilde{V}}) = 0$ and $d\colon \widetilde{V} \to H^{1,1}_{\mathrm{bas},\bar\partial}(\mathcal{Z}_\mathcal{K}, \mathcal{F})$ is the foliated Chern class.

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[3] Hiroaki Ishida, Roman Krutowski and Taras Panov. *Basic cohomology of canonical holomorphic foliations on complex moment-angle manifolds*. Internat. Math. Research Notices 2022 (2022), no. 7, 5541–5563.

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De Rham and Dolbeault cohomology (rational case)

The face ring (the Stanley–Reisner ring) of $\mathcal{K}$ is

$$\mathbb{C}[\mathcal{K}] := \mathbb{C}[v_1, ..., v_m]/I_{\mathcal{K}} = \mathbb{C}[v_1, ..., v_m]/(v_{i_1} \cdots v_{i_k} : \{i_1, \ldots, i_k\} \not\in \mathcal{K}),$$

where $\mathbb{C}[v_1, ..., v_m]$ is the polynomial algebra, $\deg v_i = 2$, and $I_{\mathcal{K}}$ is the Stanley–Reisner ideal.

Proposition

The T^m-equivariant cohomology is given by

$$H^*_{T^m}(\mathcal{Z}_{\mathcal{K}}) = H^*_{T^m}(U(\mathcal{K})) \cong \mathbb{C}[\mathcal{K}].$$

The toric variety X_Σ is Kähler (equivalently, projective) if and only if Σ is the normal fan of a nonsingular (Delzant) polytope P.

Theorem (Danilov)

The Dolbeault cohomology of complete nonsingular X_Σ is given by

$$H^{*,*}_{\bar\partial}(X_\Sigma) \cong \mathbb{C}[v_1, ..., v_m]/(I_\mathcal{K} + J_\Sigma),$$

where $v_i \in H^{1,1}_{\bar\partial}(X_\Sigma)$, $I_\mathcal{K}$ is the Stanley–Reisner ideal,
J_Σ is the ideal generated by the linear forms $\sum_{k=1}^m \langle a_k, w\rangle v_k$,
$a_k = A(e_k)$ are the generators of 1-dim cones of Σ, $w \in W$.

The nonzero Hodge numbers are given by $h^{p,p}(X_\Sigma) = h_p$,
where $h(\Sigma) = (h_0, h_1, \ldots, h_n)$ is the *h*-vector of Σ.

Theorem (Buchstaber-P.)

The de Rham cohomology ring of $\mathcal{Z}_{\mathcal{K}}$ is given by

$$
\begin{aligned}
H^*(\mathcal{Z}_{\mathcal{K}}) &\cong \operatorname{Tor}_{\mathbb{C}[v_1,\dots,v_m]}(\mathbb{C}[\mathcal{K}],\mathbb{C}) \\
&\cong H\big(\Lambda[u_1,\dots,u_m]\otimes\mathbb{C}[\mathcal{K}],d\big) \qquad du_i=v_i,\ dv_i=0 \\
&\cong H\big(\Lambda[t_1,\dots,t_{m-n}]\otimes H^*(X_\Sigma),d\big) \qquad \Lambda[t_1,\dots,t_{m-n}]=H^*(T_L) \\
&\cong \bigoplus_{I\subset[m]} \widetilde{H}^{*-|I|-1}(\mathcal{K}_I).
\end{aligned}
$$

Here $H\big(\Lambda[t_1,\dots,t_{m-n}]\otimes H^*(X_\Sigma),d\big)$ is the E_3 term of the Serre spectral sequence of the principal bundle $\mathcal{Z}_{\mathcal{K}}\to X_\Sigma$ with fibre T^{m-n}.

Theorem (P.-Ustinovsky)

Let Σ be a rational fan, $\mathcal{Z}_{\mathcal{K}} \longrightarrow X_\Sigma$ a holomorphic principal bundle with fibre $\widetilde{T}_L = \widetilde{V}^*/L^* \cong T^{2\ell}$.

Then the Dolbeault cohomology ring of $\mathcal{Z}_{\mathcal{K}}$ is given by

$$H^{*,*}_{\bar\partial}(\mathcal{Z}_{\mathcal{K}}) \cong H\big(H^{*,*}_{\bar\partial}(\widetilde{T}_L) \otimes H^{*,*}_{\bar\partial}(X_\Sigma), d\big),$$

where

$$H^{*,*}_{\bar\partial}(\widetilde{T}_L) = \Lambda\big(\widetilde{V} \oplus \overline{\widetilde{V}}\big) = \Lambda[\xi_1, \dots, \xi_\ell, \eta_1, \dots, \eta_\ell]$$
$$H^{*,*}_{\bar\partial}(X_\Sigma) \cong \mathbb{C}[v_1, \dots, v_m]/(I_{\mathcal{K}} + J_\Sigma),$$

$\xi_j \in H^{1,0}_{\bar\partial}(\widetilde{T}_L) = \widetilde{V}$, $\eta_j \in H^{0,1}_{\bar\partial}(\widetilde{T}_L) = \overline{\widetilde{V}}$, $dv_j = d\eta_j = 0$, $d\xi_j = c(\xi_j)$,
$c\colon H^{1,0}_{\bar\partial}(\widetilde{T}_L) \to H^{1,1}_{\bar\partial}(X_\Sigma)$ *is the holomorphic first Chern class map.*

Corollary

(a) *The Borel spectral sequence of the holomorphic fibration* $\mathcal{Z}_{\mathcal{K}} \xrightarrow{\widetilde{T}_L} X_\Sigma$ *(converging to Dolbeault cohomology of* $\mathcal{Z}_{\mathcal{K}}$*) collapses at the* E_3 *page;*

(b) *The Frölicher spectral sequence (with* $E_1 = H^{*,*}_{\bar\partial}(\mathcal{Z}_{\mathcal{K}})$*, converging to* $H^*(\mathcal{Z}_{\mathcal{K}})$*) collapses at* E_2*.*

Basic cohomology

M a manifold with an action of a connected Lie group G, $\mathfrak{g} = \operatorname{Lie} G$.

$$\Omega(M)_{\mathrm{bas},\,G} = \{\omega \in \Omega(M) \colon \iota_\xi \omega = L_\xi \omega = 0 \text{ for any } \xi \in \mathfrak{g}\},$$

$H^*_{\mathrm{bas},\,G}(M) = H(\Omega(M)_{\mathrm{bas},\,G}, d)$ the basic cohomology of M.

$S(\mathfrak{g}^*)$ the symmetric algebra on $\mathfrak{g}^*$ with generators of degree 2.
The Cartan model is

$$\mathcal{C}_{\mathfrak{g}}(\Omega(M)) = ((S(\mathfrak{g}^*) \otimes \Omega(M))^{\mathfrak{g}}, d_{\mathfrak{g}}),$$

where $(S(\mathfrak{g}^*) \otimes \Omega(M))^{\mathfrak{g}}$ denotes the $\mathfrak{g}$-invariant subalgebra.

An element $\omega \in \mathcal{C}_{\mathfrak{g}}(\Omega(M))$ is a "$\mathfrak{g}$-equivariant polynomial map from $\mathfrak{g}$ to $\Omega(M)$". The differential $d_{\mathfrak{g}}$ is given by

$$d_{\mathfrak{g}}(\omega)(\xi) = d(\omega(\xi)) - \iota_\xi(\omega(\xi)).$$

Theorem

$$H^*_{\mathrm{bas},\,G}(M) \cong H\big(\mathcal{C}_{\mathfrak{g}}(\Omega(M)), d_{\mathfrak{g}}\big).$$

If in addition G is a compact, then

$$H^*_{\mathrm{bas},\,G}(M) \cong H^*_G(M) = H^*(EG \times_G M) \quad \text{the equivariant cohomology.}$$

Basic de Rham and Dolbeault cohomology of the foliated manifold $(\mathcal{Z}_\mathcal{K}, \mathcal{F})$ (irrational toric variety) is like the cohomology of the toric quotient $X_\Sigma = \mathcal{Z}_\mathcal{K}/\widetilde{T}_L$ in the rational case:

Theorem (Ishida–Krutowski–P)

There is an isomorphism of algebras:

$$H^{*,*}_{\mathrm{bas},\bar\partial}(\mathcal{Z}_\mathcal{K}, \mathcal{F}) \cong \mathbb{C}[v_1, \ldots, v_m]/(I_\mathcal{K} + J_\Sigma),$$

where $v_i \in H^{1,1}_{\mathrm{bas},\bar\partial}(\mathcal{Z}_\mathcal{K}, \mathcal{F})$, $i = 1, \ldots, m$,
$I_\mathcal{K}$ *is the Stanley–Reisner ideal, generated by*

$$v_{i_1} \cdots v_{i_k} \quad \text{with } \{i_1, \ldots, i_k\} \notin \mathcal{K},$$

and J_Σ *is the ideal generated by the linear forms*

$$\sum_{i=1}^{m} \langle a_i, w\rangle v_i, \quad w \in W.$$

The proof of the theorem is based on the following formality result. Let $\mathfrak{t} = \mathrm{Lie}(T^m) \cong \mathbb{R}^m$ and consider the Cartan model

$$\mathcal{C}_{\mathfrak{t}}(\Omega(\mathcal{Z}_{\mathcal{K}})) = \big((S(\mathfrak{t}^*) \otimes \Omega(\mathcal{Z}_{\mathcal{K}}))^{T^m}, d_{\mathfrak{t}}\big).$$

Since T^m is compact, we get

$$H(\mathcal{C}_{\mathfrak{t}}(\Omega(\mathcal{Z}_{\mathcal{K}}))) = H^*_{T^m}(\mathcal{Z}_{\mathcal{K}}) = \mathbb{C}[v_1, ..., v_m]/I_{\mathcal{K}}.$$

Lemma

The DGA $\mathcal{C}_{\mathfrak{t}}(\Omega(\mathcal{Z}_{\mathcal{K}}))$ is formal. Furthermore, there is a zigzag of quasi-isomorphisms of DGAs between $\mathcal{C}_{\mathfrak{t}}(\Omega(\mathcal{Z}_{\mathcal{K}}))$ and $H_{T^m}(\mathcal{Z}_{\mathcal{K}})$ which respect the $S(\mathfrak{t}^*)$-module structure.

Dolbeault cohomology in the irrational case

Theorem (P–Ustinovsky, Krutowski–P)

The Dolbeault cohomology algebra of $\mathcal{Z}_{\mathcal{K}}$ is given by

$$H^{*,*}_{\bar\partial}(\mathcal{Z}_{\mathcal{K}}) \cong H\big(\Lambda(\widetilde{V} \oplus \overline{\widetilde{V}}) \otimes H^{*,*}_{\mathrm{bas},\bar\partial}(\mathcal{Z}_{\mathcal{K}}, \mathcal{F}), d\big),$$

where $d(\overline{\widetilde{V}}) = 0$ and $d\colon \widetilde{V} \to H^{1,1}_{\mathrm{bas},\bar\partial}(\mathcal{Z}_{\mathcal{K}}, \mathcal{F})$ is the "foliated Chern class".