Holomorphic foliations on complex manifolds with torus action

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Vector configurations and exponential actions

$V \cong \mathbb{R}^k$ a k-dimensional real vector space
$\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ a configuration of m vectors in the dual space V^*.
Allow repetitions, but assume that $\gamma_1, \ldots, \gamma_m$ span V^*.

The exponential action of V on $\mathbb{R}^m$:

$$V \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$$
$$(\mathbf{v}, \mathbf{x}) \mapsto \mathbf{v} \cdot \mathbf{x} = \left(e^{\langle \gamma_1, \mathbf{v} \rangle} x_1, \ldots, e^{\langle \gamma_m, \mathbf{v} \rangle} x_m \right).$$

A classical dynamical system taking its origin in the works of Poincaré.
The linear properties of Γ determine the topology of the foliation of $\mathbb{R}^m$ by the orbits of the action.

Applications

Exponential actions $V \times \mathbb{R}^m \to \mathbb{R}^m$ and their holomorphic versions arise in several important constructions of algebraic geometry and topology:

- Leaf spaces of holomophic foliations & topology intersections of real and Hermitian quadrics
  (topology & holomorphic dynamics)

- The quotient construction of toric varieties (the Batyrev–Cox construction)
  (toric geometry)

- Smooth and complex-analytic structures on moment-angle manifolds and their partial quotients
  (complex geometry and toric topology)

Linear Gale duality

$\Gamma = (\gamma_1, \ldots, \gamma_m)$ defines a linear map $\Gamma \colon \mathbb{R}^m \to V^*$, $\mathbf{e}_i \mapsto \gamma_i$.
Let $W := \operatorname{Ker}\Gamma$, and consisder the dual exact sequences

$$0 \longrightarrow W \longrightarrow \mathbb{R}^m \overset{\Gamma}{\longrightarrow} V^* \longrightarrow 0,$$

$$0 \longrightarrow V \overset{\Gamma^*}{\longrightarrow} \mathbb{R}^m \overset{A}{\longrightarrow} W^* \longrightarrow 0,$$

where $\Gamma^* = (\langle \gamma_1, \mathbf{v} \rangle, \ldots, \langle \gamma_m, \mathbf{v} \rangle)$. Let $\mathbf{a}_i := A(\mathbf{e}_i)$.

The vector configuration $\mathrm{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ in W^* is the Gale dual
of $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$. The Gale dual of A is Γ.

If we choose bases V and W, then Γ becomes a $k \times m$-matrix with
columns $\gamma_1, \ldots, \gamma_m$ and A becomes an $(m - k) \times m$-matrix with columns
$\mathbf{a}_1, \ldots, \mathbf{a}_m$. Then $A\Gamma^* = 0$, i.e. the rows of A form a basis in the space of
linear relations between $\gamma_1, \ldots, \gamma_m$.

When the action is free and proper: the fan condition

For $I \subset [m]$, the subconfiguration $A_I = \{\mathbf{a}_i \colon i \in I\}$ is linearly independent $\iff \Gamma_{\widehat{I}} = \{\gamma_i \colon i \notin I\}$ spans V^*.

The independence complex

$$\mathcal{K}(\Gamma) := \{I \subset [m] \colon A_I \text{ is linearly independent}\} = \{I \subset [m] \colon \Gamma_{\widehat{I}} \text{ spans } V^*\}.$$

A simplicial complex $\mathcal{K}$ on $[m] = \{1, \ldots, m\}$ defines the complement of an arrangement of coordinate subspaces:

$$U(\mathcal{K}) = \mathbb{R}^m \setminus \bigcup_{\{i_1, \ldots, i_p\} \notin \mathcal{K}} \{\mathbf{x} \colon x_{i_1} = \cdots = x_{i_p} = 0\}.$$

Proposition

Then for any subcomplex $\mathcal{K} \subset \mathcal{K}(\Gamma)$, the restriction of the exponential action $V \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ to $U(\mathcal{K})$ is free.

A simplicial fan Σ in $W^* \cong \mathbb{R}^n$ is determined by two pieces of data:

- a simplicial complex $\mathcal{K}$ on $[m]$;
- a configuration of vectors $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ in W^* such that for any simplex $I \in \mathcal{K}$ the subset $A_I = \{\mathbf{a}_i : i \in I\}$ is linearly independent.

The collection of cones $\{\operatorname{cone}(A_I) : I \in \mathcal{K}\}$ patches into a fan Σ whenever any two cones intersect at a common face. In this case, we say that the data $\{\mathcal{K}, A\}$ defines a fan Σ, or $\{\mathcal{K}, A\}$ is a triangulated configuration.

Theorem

Let $\mathcal{K}$ be a simplicial complex on $[m]$, and let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ and $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ be Gale dual configurations. The following conditions are equivalent:

(a) $\{\mathcal{K}, A\}$ *is a triangulated configuration (defines a fan Σ);*

(b) $\operatorname{relint} \operatorname{cone}(A_I) \cap \operatorname{relint} \operatorname{cone}(A_J) = \varnothing$ *for any $I, J \in \mathcal{K}$, $I \neq J$;*

(c) $\operatorname{relint} \operatorname{cone}(\Gamma_{\widehat{I}}) \cap \operatorname{relint} \operatorname{cone}(\Gamma_{\widehat{J}}) \neq \varnothing$ *for any $I, J \in \mathcal{K}$.*

For the exponential action

$$V \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$$
$$(\mathbf{v}, \mathbf{x}) \mapsto \mathbf{v} \cdot \mathbf{x} = \left(e^{\langle \gamma_1, \mathbf{v} \rangle} x_1, \ldots, e^{\langle \gamma_m, \mathbf{v} \rangle} x_m \right)$$

we have

Theorem

Let $\mathcal{K}$ b a simplicial complex on $[m]$, and let $\mathrm{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ and $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ be Gale dual configurations in V^* and W^*. Then

(1) *the action $V \times U(\mathcal{K}) \to U(\mathcal{K})$ is free if and only if A_I is linearly independent for any $I \in \mathcal{K}$;*

(2) *the action $V \times U(\mathcal{K}) \to U(\mathcal{K})$ is proper if and only if $\{\mathcal{K}, \mathrm{A}\}$ is a triangulated configuration.*

Completeness and compactness

Theorem

(1) *The orbit space $U(\mathcal{K})/V$ of a proper exponential action is compact if and only if the fan $\Sigma = \{\mathcal{K}, \mathrm{A}\}$ is complete, i.e. $|\Sigma| = W^*$.*

(2) *$U(\mathcal{K})/V$ is homeomorphic to the real moment-angle complex $\mathcal{R}_{\mathcal{K}} = (D^1, S^0)^{\mathcal{K}}$ (a subcomplex in the m-cube $[0,1]^m$).*

Corollary

Let $\mathcal{K}$ be the underlying complex of a complete simplicial fan (a star-shaped sphere triangulation). Then

(1) *the real moment-angle complex $\mathcal{R}_{\mathcal{K}} = (D^1, S^0)^{\mathcal{K}}$ is a smooth manifold;*

2) *$U(\mathcal{K})$ and $\mathcal{R}_{\mathcal{K}}$ have the same homotopy type.*

Normal fans and intersections of quadrics

A simple polytope in $W \cong \mathbb{R}^n$

$$P = P(\mathrm{A}, \mathbf{b}) = \{\mathbf{w} \in W \colon \langle \mathbf{a}_i, \mathbf{w} \rangle + b_i \geqslant 0, \quad i = 1, \ldots, m\}$$

The normal fan Σ_P:

- the normals $\mathrm{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ are generators of one-dimensional cones;
- $\operatorname{cone} \mathrm{A}_I \in \Sigma_P$ if the facets with normals $\mathbf{a}_i$, $i \in I$ have nonempty intersection.

Not every simplicial fan is the normal fan of a polytope!

Theorem

Let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ and $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ be Gale dual configurations. Suppose $\Sigma = \{\operatorname{cone} A_I : I \in \mathcal{K}\}$ is a complete fan. The following conditions are equivalent:

(a) Σ is the normal fan of a polytope;

(b) $\bigcap_{I \in \mathcal{K}} \operatorname{relint} \operatorname{cone}(\Gamma_{\widehat{I}}) \neq \varnothing$.

If $\delta \in \bigcap_{I \in \mathcal{K}} \operatorname{relint} \operatorname{cone}(\Gamma_{\widehat{I}}) \neq \varnothing$, then Σ is the normal fan of
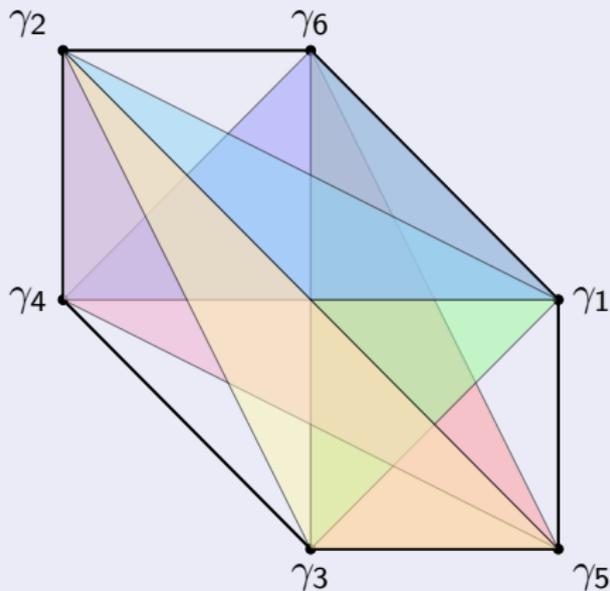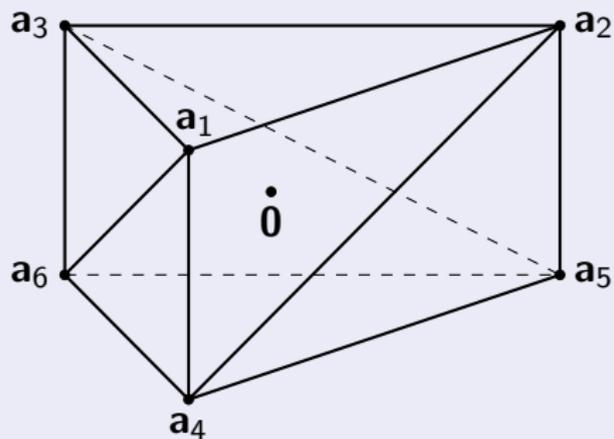
$$P = P(A, \mathbf{b}) = \{\mathbf{w} \in W : \langle \mathbf{a}_i, \mathbf{w} \rangle + b_i \geqslant 0, \quad i = 1, \ldots, m\},$$

where $\delta = b_1 \gamma_1 + \ldots + b_m \gamma_m$.

That is, the data $\{\mathcal{K}, A\}$ defines a fan Σ if the Gale dual cones $\operatorname{cone} \Gamma_{\widehat{I}}$, $I \in \mathcal{K}$ overlap pairwise, and Σ is a normal fan if the Gale dual cones have a common interior point.

Example

$$A = \begin{pmatrix} 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Intersections of quadrics

$$\mu_\Gamma \colon \mathbb{R}^m \to V^*, \quad (x_1, \dots, x_m) \mapsto x_1^2 \gamma_1 + \cdots + x_m^2 \gamma_m.$$

Theorem

Let $\Sigma = \{\operatorname{cone} A_I \colon I \in \mathcal{K}\}$ be the normal fan of a polytope $P = P(A, \mathbf{b})$, and $\Gamma = \{\gamma_1, \dots, \gamma_m\}$ the Gale dual configuration.
Let $\delta = \sum_{i=1}^m b_i \gamma_i$. Then

- *$\delta \in \bigcap_{I \in \mathcal{K}} \operatorname{relint} \operatorname{cone} \Gamma_{\widehat{I}}$;*

- *$\mu_\Gamma^{-1}(\delta) \subset U(\mathcal{K})$;*

- *δ is a regular value of μ_Γ;*

- *the exponential action $V \times U(\mathcal{K}) \to U(\mathcal{K})$ is free and proper, and the quotient $U(\mathcal{K}_P)/V$ is diffeomorphic to the level set $\mu_\Gamma^{-1}(\delta)$, which is the intersection of k quadrics:*

$$U(\mathcal{K}_P)/V \cong \{(x_1, \dots, x_m) \in \mathbb{R}^m \colon \gamma_1 x_1^2 + \cdots + \gamma_m x_m^2 = \delta\}.$$

Holomorphic exponential actions

$\widetilde{V} = (V, \mathcal{J}) \cong \mathbb{C}^\ell$, where $\mathcal{J} \colon V \to V$ is a complex structure on $V \cong \mathbb{R}^k$ (assuming $k = 2\ell$ is even).
$\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ a vector configuration in $\widetilde{V}^* = \mathrm{Hom}_{\mathbb{C}}(\widetilde{V}, \mathbb{C})$.

The holomorphic exponential action of $\widetilde{V}$ on $\mathbb{C}^m$
$$\widetilde{V} \times \mathbb{C}^m \longrightarrow \mathbb{C}^m$$
$$(\mathbf{v}, \mathbf{z}) \mapsto \mathbf{v} \cdot \mathbf{z} = \left(e^{\langle \gamma_1, \mathbf{v} \rangle_{\mathbb{C}}} z_1, \ldots, e^{\langle \gamma_m, \mathbf{v} \rangle_{\mathbb{C}}} z_m \right).$$

Restrict to the complement of a complex coordinate subspace arrangement
$$U_{\mathbb{C}}(\mathcal{K}) = (\mathbb{C}, \mathbb{C}^\times)^{\mathcal{K}} = \mathbb{C}^m \setminus \bigcup_{\{i_1, \ldots, i_p\} \notin \mathcal{K}} \{\mathbf{z} \colon z_{i_1} = \cdots = z_{i_p} = 0\}$$

Theorem

The holomorphic exponential action $\widetilde{V} \times U_{\mathbb{C}}(\mathcal{K}) \to U_{\mathbb{C}}(\mathcal{K})$ is free and proper if and only if $\{\mathcal{K}, \mathrm{A}\}$ is a triangulated configuration.

Theorem

(1) *The orbit space $U_{\mathbb{C}}(\mathcal{K})/\widetilde{V}$ of a proper holomorphic exponential action is compact if and only if the fan $\Sigma = \{\mathcal{K}, A\}$ is complete.*

(2) *$U_{\mathbb{C}}(\mathcal{K})/\widetilde{V}$ is homeomorphic to the* moment-angle complex *$\mathcal{Z}_{\mathcal{K}} = (D^2, S^1)^{\mathcal{K}}$.*

Corollary

Let $\mathcal{K}$ be the underlying complex of a complete simplicial fan (a star-shaped sphere triangulation*). Then*

(1) *the moment-angle complex $\mathcal{Z}_{\mathcal{K}} = (D^2, S^1)^{\mathcal{K}}$ has a structure of a compact complex manifold;*

2) *$U_{\mathbb{C}}(\mathcal{K})$ and $\mathcal{Z}_{\mathcal{K}}$ have the same homotopy type.*

Example (holomorphic torus)

Exponential action of $\widetilde{V} \cong \mathbb{C}$ on $\mathbb{C}^2$ defined by a configuration $\{\gamma_1, \gamma_2\}$:

$$\widetilde{V} \times \mathbb{C}^2 \to \mathbb{C}^2, \quad (v, z_1, z_2) \mapsto (e^{\gamma_1 v} z_1, e^{\gamma_2 v} z_2).$$

Let $\mathcal{K} = \{\varnothing\}$ on $[2] = \{1, 2\}$, so that $U_{\mathbb{C}}(\mathcal{K}) = (\mathbb{C}^\times)^2$. If $\mathbb{R}\langle\gamma_1, \gamma_2\rangle = \mathbb{R}^2$, then the restriction of the action to $(\mathbb{C}^\times)^2$ is free and proper. The quotient

$$(\mathbb{C}^\times)^2 / \widetilde{V} \cong \mathbb{C} / (\mathbb{Z}\langle\gamma_1, \gamma_2\rangle) \cong T^2$$

is a one-dimensional complex torus.

More generally, let $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ be a real basis in $\widetilde{V} \cong \mathbb{C}^\ell$. Then A is a configuration of m zero vectors in $W^* = \{\mathbf{0}\}$, $\mathcal{K} = \{\varnothing\}$ on $[m] = [2\ell]$ and $U_{\mathbb{C}}(\mathcal{K}) = (\mathbb{C}^\times)^m$. The quotient $(\mathbb{C}^\times)^m / \widetilde{V} \cong T^{2\ell}$ is a holomorphic torus. Any holomorphic torus can be obtained in this way.

Proposition

A complex moment-angle manifold $\mathcal{Z}_K = U_{\mathbb{C}}(\mathcal{K})/\widetilde{V}$ is non-Kähler, unless it is a holomorphic torus.

Holomorphic torus-exponential action and partial quotients

Have Gale dual $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\} \subset W^*$ and $\Gamma = \{\gamma_1, \ldots, \gamma_m\} \subset V^*$. Then V is the space of linear relations between $\mathbf{a}_1, \ldots, \mathbf{a}_m$, and $\gamma_k \in V^*$ maps a linear relation to its kth coefficient. This gives

$$\Gamma^*\colon V \to \mathbb{R}^m, \qquad \mathbf{v} \mapsto (\langle \gamma_1, \mathbf{v} \rangle, \ldots, \langle \gamma_m, \mathbf{v} \rangle).$$

A subspace $Q \subset V$ is rational (with respect to Γ) if it is generated by relations with integer coefficients, that is, if $Q \subset \mathbb{R}\langle (\Gamma^*)^{-1}(\mathbb{Z}^m) \rangle$. Here $(\Gamma^*)^{-1}(\mathbb{Z}^m)$ is a lattice in V of rank between 0 and $k = \dim V$, and $\mathbb{R}\langle (\Gamma^*)^{-1}(\mathbb{Z}^m) \rangle$ is the largest rational subspace in V.

A rational subspace $Q \subset V$ contains a full-rank lattice

$$L = (\Gamma^*)^{-1}(\mathbb{Z}^m) \cap Q = \{\mathbf{q} \in Q \colon \langle \gamma_k, \mathbf{q} \rangle \in \mathbb{Z} \text{ for } k = 1, \ldots, m\}.$$

The image of Γ under the dual projection $V^* \to Q^*$ is a rational vector configuration $\widehat{\Gamma} = \{\widehat{\gamma}_1, \ldots, \widehat{\gamma}_m\}$ in Q^* with $\mathbb{Z}\langle \widehat{\Gamma} \rangle = L^*$.

The lattice L in a rational subspace $Q \subset V$ defines an algebraic torus
$$\mathbb{C}_L^\times = L \otimes_{\mathbb{Z}} \mathbb{C}^\times \cong Q_{\mathbb{C}}/(2\pi i L),$$
where $Q_{\mathbb{C}} = Q \otimes_{\mathbb{R}} \mathbb{C} = Q \oplus iQ$, and a compact torus
$$T_L = L \otimes_{\mathbb{Z}} S^1 \cong Q/(2\pi L).$$
Have $\mathbb{C}_L^\times = Q \times T_L$ as (real) Lie groups.

The algebraic torus $\mathbb{C}_L^\times$ embeds as a closed subgroup in $(\mathbb{C}^\times)^m$:
$$(\Gamma^*|_L \otimes_{\mathbb{Z}} \mathbb{C}^\times)(\mathbb{C}_L^\times) = \exp \Gamma_{\mathbb{C}}^*(Q_{\mathbb{C}})$$
$$= \left\{ \left(e^{\langle \gamma_1, \mathbf{v} \rangle + i\langle \gamma_1, \mathbf{u} \rangle}, \ldots, e^{\langle \gamma_m, \mathbf{v} \rangle + i\langle \gamma_m, \mathbf{u} \rangle} \right) \in (\mathbb{C}^\times)^m \colon \mathbf{v} + i\mathbf{u} \in Q_{\mathbb{C}} \right\}.$$

Let $\mathcal{J}$ be a complex structre on V, $\widetilde{V} = (V, \mathcal{J})$ and $\widetilde{Q} = (Q, \mathcal{J})$ a rational complex subspace. The holomorphic torus-exponential action
$$(\mathbb{C}_L^\times \times_{\widetilde{Q}} \widetilde{V}) \times \mathbb{C}^m \to \mathbb{C}^m$$
is the product of holomorphic actions of $\mathbb{C}_L^\times$ and $\widetilde{V}$, which agree on $\widetilde{Q}$.

Proposition

Let Γ and A be a pair of Gale dual vector configurations, $Q \subset V$ a rational subspace, $\widehat{\Gamma}$ and $\widehat{\mathrm{A}}$ the corresponding rational configurations in Q^* and U^*, and $\mathcal{K}$ a simplicial complex on $[m]$. Suppose that

(a) $\{\mathcal{K}, \mathrm{A}\}$ is a triangulated configuration defining a simplicial fan Σ;

(b) $\{\mathcal{K}, \widehat{\mathrm{A}}\}$ is a nonsingular triangulated configuration defining a nonsingular simplicial fan $\widehat{\Sigma}$ in $U^* \subset W^*$;

(c) $\widetilde{Q} = (Q, \mathcal{J}) \subset \widetilde{V} = (V, \mathcal{J})$ is a complex subspace.

Then the partial quotient $\mathcal{Z}_{\mathcal{K}}/T_L$ has a structure of a compact complex manifold as the quotient $U_{\mathbb{C}}(\mathcal{K})/(\mathbb{C}_L^\times \times_{\widetilde{Q}} \widetilde{V})$.

The compact torus T_L acquires a holomorphic structure as the quotient $\widetilde{T}_L = \mathbb{C}_L^\times / \widetilde{Q}$, as in example above.

Have a commutative diagram of holomorpic quotient maps:

$$
\begin{array}{ccc}
U_{\mathbb{C}}(\mathcal{K}) & \xrightarrow{\;/\mathbb{C}_L^\times\;} & X_{\widehat{\Sigma}} \\
\Big\downarrow{\scriptstyle /\widetilde{V}} & & \Big\downarrow{\scriptstyle /(\widetilde{V/Q})} \\
\mathcal{Z}_{\mathcal{K}} & \xrightarrow{\;/\widetilde{T}_L\;} & \mathcal{Z}_{\mathcal{K}}/\widetilde{T}_L \cong X_{\widehat{\Sigma}}/(\widetilde{V/Q})
\end{array}
$$

where $X_{\widehat{\Sigma}}$ is the toric variety corresponding to the rational fan $\widehat{\Sigma}$.

If the rational triangulated configuration $\{\mathcal{K}, \widehat{A}\}$ is nonsingular, then the actions of $\mathbb{C}_L^\times$ and $\widetilde{T}_L$ are free, and $\mathcal{Z}_{\mathcal{K}} \to \mathcal{Z}_{\mathcal{K}}/\widetilde{T}_L$ is a holomorphic torus principal bundle over the holomorphic partial quotient.

More generally, $\mathcal{Z}_{\mathcal{K}} \to \mathcal{Z}_{\mathcal{K}}/\widetilde{T}_L$ is a holomorphic Seifert principal bundle over the orbifold $\mathcal{Z}_{\mathcal{K}}/\widetilde{T}_L \cong X_{\widehat{\Sigma}}/(\widetilde{V/Q})$, where the latter the quotient of the toric orbifold $X_{\widehat{\Sigma}}$ by a free proper action of $\widetilde{V/Q}$.

Irrational toric varieties

If the whole space V is rational with respect to $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$, then

$$L = \{\mathbf{v} \in V \colon \langle \gamma_k, \mathbf{v} \rangle \in \mathbb{Z} \ \text{ for } k = 1, \ldots, m\}.$$

is a full rank lattice in V, and $L^* = \mathbb{Z}\langle \gamma_1, \ldots, \gamma_m \rangle$.

The Gale dual rational vector configuration $\mathrm{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ spans a full rank lattice $N = M^*$ in W^*.

The torus $\mathbb{C}_L^\times$ embeds in $(\mathbb{C}^\times)^m$ as the closed algebraic subgroup

$$G = (\Gamma^*|_L \otimes_{\mathbb{Z}} \mathbb{C}^\times)(\mathbb{C}_L^\times) = \exp \Gamma_{\mathbb{C}}^*(V_{\mathbb{C}}) = \exp(\operatorname{Ker} A_{\mathbb{C}} \colon \mathbb{C}^m \to W_{\mathbb{C}}^*)$$

$$= \Big\{ (t_1, \ldots, t_m) \in (\mathbb{C}^\times)^m \colon \prod_{i=1}^m t_i^{\langle \mathbf{a}_i, \mathbf{w} \rangle} = 1, \quad \mathbf{w} \in M \Big\}.$$

The torus-exponential action $(\mathbb{C}_L^\times \times_{\widetilde{Q}} \widetilde{V}) \times \mathbb{C}^m \to \mathbb{C}^m$ becomes the algebraic action of $\mathbb{C}_L^\times$ on $\mathbb{C}^m$:

$$
\begin{array}{ccc}
 & U_{\mathbb{C}}(\mathcal{K}) & \\
\swarrow^{/\widetilde{V}} & & \downarrow^{/\mathbb{C}_L^\times} \\
\mathcal{Z}_{\mathcal{K}} & & \\
 & \searrow^{/\widetilde{T}_L} & \\
 & & X_\Sigma
\end{array}
$$

A perturbation of the vector configuration A destroys the rationality of the fan $\{\mathcal{K}, \mathrm{A}\}$, the subgroup $G \subset (\mathbb{C}^\times)^m$ ceases to be closed, the closed holomorphic tori in the fibres of the bundle $\mathcal{Z}_{\mathcal{K}} \to X_\Sigma$ "open up", and the fibre bundle turns into a holomorphic foliation $\mathcal{F}$ of $\mathcal{Z}_{\mathcal{K}}$ with noncompact leaves $G/\widetilde{V}$.

The holomorphic foliated manifolds $(\mathcal{Z}_{\mathcal{K}}, \mathcal{F})$ model irrational deformations of toric varieties.

Basic de Rham and Dolbeault cohomology of the foliated manifold $(\mathcal{Z}_\mathcal{K}, \mathcal{F})$ is described in the same way as the cohomology of the toric quotient $X_\Sigma = \mathcal{Z}_\mathcal{K}/\widetilde{T}_L$ in the rational case:

Theorem (Ishida–Krutowski–P)

There is an isomorphism of algebras:

$$H_{\mathrm{bas},\bar\partial}^{*,*}(\mathcal{Z}_\mathcal{K}, \mathcal{F}) \cong \mathbb{C}[v_1, \ldots, v_m]/(I_\mathcal{K} + J_\Sigma),$$

where $v_i \in H_{\mathrm{bas},\bar\partial}^{1,1}(\mathcal{Z}_\mathcal{K}, \mathcal{F})$, $i = 1, \ldots, m$,
$I_\mathcal{K}$ is the Stanley–Reisner ideal, generated by

$$v_{i_1} \cdots v_{i_k} \quad \text{with } \{i_1, \ldots, i_k\} \notin \mathcal{K},$$

and J_Σ is the ideal generated by the linear forms

$$\sum_{i=1}^m \langle \mathbf{a}_i, \mathbf{w} \rangle v_i, \quad \mathbf{w} \in W.$$

In the rational case, the E_3 term of the Borel spectral sequence of the holomorphic principal bundle $\mathcal{Z}_\mathcal{K} \to X_\Sigma$ with fibre $\widetilde{T}_L \cong T^{2\ell}$ gives a dga model for the Dolbeault cohomology $H_{\bar{\partial}}^{*,*}(\mathcal{Z}_\mathcal{K})$:

$$\left(H_{\bar{\partial}}^{*,*}(\widetilde{T}_L) \otimes H_{\bar{\partial}}^{*,*}(X_\Sigma), d \right),$$

where $H_{\bar{\partial}}^{*,*}(X_\Sigma) \cong \mathbb{C}[v_1, \ldots, v_m]/(I_\mathcal{K} + J_\Sigma)$ (Danilov),

$$H_{\bar{\partial}}^{*,*}(\widetilde{T}_L) = \Lambda(\widetilde{V} \oplus \overline{\widetilde{V}}) = \Lambda[\xi_1, \ldots, \xi_\ell, \eta_1, \ldots, \eta_\ell]$$

$\xi_j \in H_{\bar{\partial}}^{1,0}(\widetilde{T}_L) = \widetilde{V}$, $\eta_j \in H_{\bar{\partial}}^{0,1}(\widetilde{T}_L) = \overline{\widetilde{V}}$, $dv_j = d\eta_j = 0$, $d\xi_j = c(\xi_j)$,

$c \colon H_{\bar{\partial}}^{1,0}(\widetilde{T}_L) \to H_{\bar{\partial}}^{1,1}(X_\Sigma)$ is the holomorphic first Chern class map (the transgression). Similarly, in the irrational case,

Theorem (P–Ustinovsky, Krutowski–P)

The Dolbeault cohomology algebra of $\mathcal{Z}_\mathcal{K}$ is given by

$$H_{\bar{\partial}}^{*,*}(\mathcal{Z}_\mathcal{K}) \cong H\left(\Lambda(\widetilde{V} \oplus \overline{\widetilde{V}}) \otimes H_{\mathrm{bas},\bar{\partial}}^{*,*}(\mathcal{Z}_\mathcal{K}, \mathcal{F}), d \right),$$

where $d(\overline{\widetilde{V}}) = 0$ and $d \colon \widetilde{V} \to H_{\mathrm{bas},\bar{\partial}}^{1,1}(\mathcal{Z}_\mathcal{K}, \mathcal{F})$ is the foliated Chern class.

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