New families of moment-angle manifolds diffeomorphic to connected sums of products of spheres

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The motivating problem

- A **moment-angle complex** $\mathcal{Z}_\mathcal{K}$ is a cell complex with a torus action, determined by a simplicial complex $\mathcal{K}$ on $[m] = \{1, \dots, m\}$.
- If $\mathcal{K}$ is a triangulation of a $(n-1)$-sphere S^{n-1}, then $\mathcal{Z}_\mathcal{K}$ is a topological $(m+n)$-manifold: a **moment-angle manifold**.
- If $\mathcal{K}$ is *starshaped* (arises from a complete simplicial fan), $\mathcal{Z}_\mathcal{K}$ is smooth.
- If $\mathcal{K} = \mathcal{K}_P$ is the dual complex of a simple n-polytope P, we write $\mathcal{Z}_P$.

Question

For which simplicial spheres $\mathcal{K}$ (or simple polytopes P) is $\mathcal{Z}_\mathcal{K}$ diffeomorphic to a connected sum of products of spheres?

Historical context

- **López de Medrano (1989):** early examples from intersections of quadrics.
- **McGavran (1979):** polygons $\Rightarrow$ connected sums of $S^k \times S^{m+2-k}$.
- **Bosio–Meersseman (2006):** dual-stacked polytopes; complete classification for 2-dim spheres (3-dim polytopes).
- **Gitler–López de Medrano (2013):** simply connected surgery + h-cobordism; dual-neighbourly even-dimensional polytopes.
- **Chen–Fan–Wang (2020):** removed the restriction $m < 3n$.
- **Fan–Chen–Ma–Wang (2016):** first example with a product of *three* spheres.
- **Iriye (2018):** the FCMW example is in fact a diffeomorphism.

The starting point: a cohomological criterion

Theorem (Kovyrshina–P)

Let $\mathcal{K}$ be a 3-dimensional simplicial sphere. There is a ring isomorphism

$$H^*(\mathcal{Z}_{\mathcal{K}}) \cong H^*(M_1 \# \cdots \# M_k),$$

with each M_i a product of spheres, iff one of:

(a) $\mathcal{K} = S^0 * S^0 * S^0 * S^0$ *(boundary of the 4-cross-polytope);*

(b) $\mathcal{K}^1$ *is a* **chordal** *graph;*

(c) $\mathcal{K}^1$ *has exactly two missing edges, not adjacent (a chordless 4-cycle).*

In case (c), one summand is a product of three spheres.

Goal of the present work

Upgrade this cohomological criterion to a **diffeomorphism**, whenever $\mathcal{Z}_{\mathcal{K}}$ is smooth (i.e. $\mathcal{K}$ is starshaped; in particular polytopal).

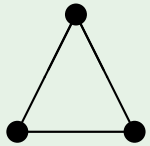
Moment-angle complexes

Let $\mathcal{K}$ be a simplicial complex on $[m]$. The **moment-angle complex** is

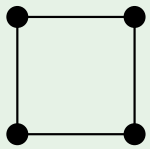
$$\mathcal{Z}_{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} \Big(\prod_{i \in I} D^2 \times \prod_{i \notin I} S^1 \Big) \subset \prod_{i=1}^{m} D^2.$$

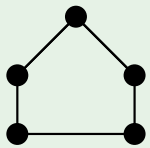
- If $\mathcal{K}$ is a simplicial $(n-1)$-sphere with m vertices, then $\mathcal{Z}_{\mathcal{K}}$ is a closed topological manifold of dimension $m+n$.
- If $\mathcal{K} = \mathcal{K}_P$ for a simple n-polytope P with m facets, then $\mathcal{Z}_P$ is smooth of dimension $m+n$.
- More generally, if $\mathcal{K}$ is a **starshaped sphere** (comes from a complete simplicial fan), then $\mathcal{Z}_{\mathcal{K}}$ is smooth.

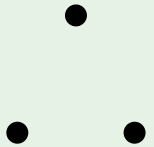
Example

1. $\mathcal{K} =$  (the boundary of a triangle). Then

$$\mathcal{Z}_{\mathcal{K}} = (D^2 \times D^2 \times S^1) \cup (D^2 \times S^1 \times D^2) \cup (S^1 \times D^2 \times D^2) = \partial((D^2)^3) \cong S^5.$$

2. $\mathcal{K} =$  (the boundary of a square). Then $\mathcal{Z}_{\mathcal{K}} \cong S^3 \times S^3$.

3. $\mathcal{K} =$  Then $\mathcal{Z}_{\mathcal{K}} \cong (S^3 \times S^4) \# \cdots \# (S^3 \times S^4)$ (5 summands).

4. $\mathcal{K} =$  (three disjoint vertices). Then

$$\mathcal{Z}_{\mathcal{K}} = (D^2 \times S^1 \times S^1) \cup (S^1 \times D^2 \times S^1) \cup (S^1 \times S^1 \times D^2) \simeq S^3 \vee S^3 \vee S^3 \vee S^4 \vee S^4$$

(not a manifold).

Multigraded cohomology

Theorem

There is a ring isomorphism

$$H^*(\mathcal{Z}_\mathcal{K}) \cong \bigoplus_{J\subset[m]} \widetilde{H}^*(\mathcal{K}_J), \quad \text{where} \quad H^i(\mathcal{Z}_\mathcal{K}) \cong \bigoplus_{J\subset[m]} \widetilde{H}^{i-|J|-1}(\mathcal{K}_J),$$

and the cohomology product is induced by the maps $\mathcal{K}_{I\sqcup J} \to \mathcal{K}_I * \mathcal{K}_J$ for $I \cap J = \varnothing$, and zero otherwise.

Notation:

$$\mathcal{H}^{\ell,J} = \widetilde{H}^\ell(\mathcal{K}_J), \qquad \mathcal{H}^{*,J} = \widetilde{H}^*(\mathcal{K}_J), \qquad \mathcal{H}^{\ell,*}(\mathcal{K}) = \bigoplus_{J\subset[m]} \widetilde{H}^\ell(\mathcal{K}_J).$$

The product is

$$\mathcal{H}^{k,I} \otimes \mathcal{H}^{\ell,J} \longrightarrow \mathcal{H}^{k+\ell+1,I\sqcup J}, \qquad I \cap J = \varnothing.$$

Poincaré duality and missing faces

For a simplicial $(n-1)$-sphere $\mathcal{K}$, multigraded Poincaré duality reads

$$\mathcal{H}^{I,J} \cong \mathcal{H}_{n-2-I,[m]\setminus J}.$$

A **missing face** $I \subset [m]$ is a minimal non-face: $I \notin \mathcal{K}$, but every proper subset is in $\mathcal{K}$. Then $\mathcal{K}_I = \partial\Delta_I$, and $\partial\Delta_I$ represents a generator of

$$\mathcal{H}_{|I|-2,I} = \widetilde{H}_{|I|-2}(\partial\Delta_I) \cong \mathbb{Z},$$

hence a nonzero class in $H_{2|I|-1}(\mathcal{Z}_\mathcal{K})$.

Moreover, for every $J \subset [m] \setminus I$ the simplicial cycle $\partial\Delta_I$ represents a class in $\mathcal{H}_{|I|-2,I\sqcup J} = \widetilde{H}_{|I|-2}(\mathcal{K}_{I\sqcup J})$, hence a class in $H_{2|I|+|J|-1}(\mathcal{Z}_\mathcal{K})$.
These classes of $H_*(\mathcal{Z}_\mathcal{K})$ are called **corresponding to missing faces** of $\mathcal{K}$.

Key idea

To prove a diffeomorphism of $\mathcal{Z}_\mathcal{K}$ with a connected sum of products of spheres, we must represent classes corresponding to missing faces by embedded spheres with **trivial normal bundles**.

The h-cobordism criterion

Theorem (Gitler–López de Medrano, Iriye)

Let $\mathcal{Q}$ be a compact smooth $(d+1)$-manifold, $d+1 \geq 6$, with boundary $\partial\mathcal{Q}$, such that

1. *$\mathcal{Q}$ and $\partial\mathcal{Q}$ are simply connected and $H_i(\mathcal{Q}) = 0$ for $i \geq d-1$;*
2. *there are disjointly embedded closed smooth submanifolds $X_j \subset \mathcal{Q}$ with trivial normal bundles, such that $\bigsqcup_j X_j \to \mathcal{Q}$ induces isomorphisms on integral homology in positive dimensions.*

Then $\mathcal{Q}$ is diffeomorphic to the boundary connected sum $\coprod_j (X_j \times D^{d+1-\dim X_j})$, and

$$\partial\mathcal{Q} \;\cong\; \#_j \big(X_j \times S^{d-\dim X_j}\big).$$

$\mathcal{Z}_P$ as a nondegenerate intersection of quadrics

Let

$$P = P(\mathrm{A}, b) = \{w \in W : \langle a_i, w\rangle + b_i \geq 0,\ i = 1, \ldots, m\}.$$

be a simple n-polytope with m facets, and let $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ be the Gale dual configuration to $\mathrm{A} = \{a_1, \ldots, a_m\}$. Then

$$\mathcal{Z}_P \ \cong\ \left\{z \in \mathbb{C}^m : \gamma_1|z_1|^2 + \cdots + \gamma_m|z_m|^2 = \delta\right\},$$

a nondegenerate intersection of $m - n$ Hermitian quadrics. In particular:

- $\mathcal{Z}_P \hookrightarrow \mathbb{C}^m$ is a smooth embedding with **trivial normal bundle**.
- The T^m-action has quotient P, and the moment map is $\mu(z) = (|z_1|^2, \ldots, |z_m|^2)$.

Submanifolds with trivial normal bundles I: faces

Proposition (Face embedding)

Let G be a face of a simple polytope $P = P(\mathrm{A}, b)$. Then $\mathcal{Z}_G \hookrightarrow \mathcal{Z}_P$ is a smooth embedding with trivial normal bundle.

Proof (idea).

Write $G = P(\mathrm{A}_{\widehat{I}}, b_{\widehat{I}})$ by setting some inequalities to equalities. The quadric system defining $\mathcal{Z}_G$ is obtained from that of $\mathcal{Z}_P$ by adding the equations $z_i = 0$ for $i \in I$. Adding a nondegenerate equation in $\mathbb{C}^m$ gives a submanifold with trivial normal bundle. □

Submanifolds with trivial normal bundles II: transverse planes

Proposition (Transverse-plane embedding)

Let L be an affine plane transverse to P (i.e. L meets no $(k-1)$-face, where $k = \operatorname{codim} L$) and $Q = L \cap P \neq \varnothing$. Then $\mathcal{Z}_Q \hookrightarrow \mathcal{Z}_P$ is a smooth embedding with trivial normal bundle.

Proof (idea).

Induction on $\operatorname{codim} L$. For $k = 1$, use the face embedding for Q as a facet of $P' = P \cap H^+$; then compare the two quadric systems. □

Representing missing-face classes by spheres

Lemma (Polytopal case)

Let I be a missing face of $\mathcal{K}_P$ with $|I| = k$. Then there is a smooth embedding

$$S^{2k-1} \times T^{[m]\setminus I} \hookrightarrow \mathcal{Z}_P$$

with trivial normal bundle, whose restriction to $S^{2k-1} \times T^J$ for any $J \subset [m] \setminus I$ represents the homology class in $H_{2k-1+|J|}(\mathcal{Z}_\mathcal{K})$ corresponding to the missing face I.

Proof (idea).

Choose interior points w_j on the faces $F_{I\setminus j}$, and let L be the $(k-1)$-plane through them. Then $Q = L \cap P$ is a $(k-1)$-simplex, and $\mathcal{Z}_Q \cong S^{2k-1}$. Apply the transverse-plane embedding. □

From tori to spheres: the gyration trick

Problem

We get $S^{2k-1} \times T^J$, not a sphere.

Solution: gyration

Define the **gyration**

$$\mathcal{G}(X) = \big((X \setminus D^d) \times S^1\big) \cup_{S^{d-1}\times S^1} (S^{d-1} \times D^2),$$

of dimension $d+1$. Then $\mathcal{G}(S^d) \cong S^{d+1}$.

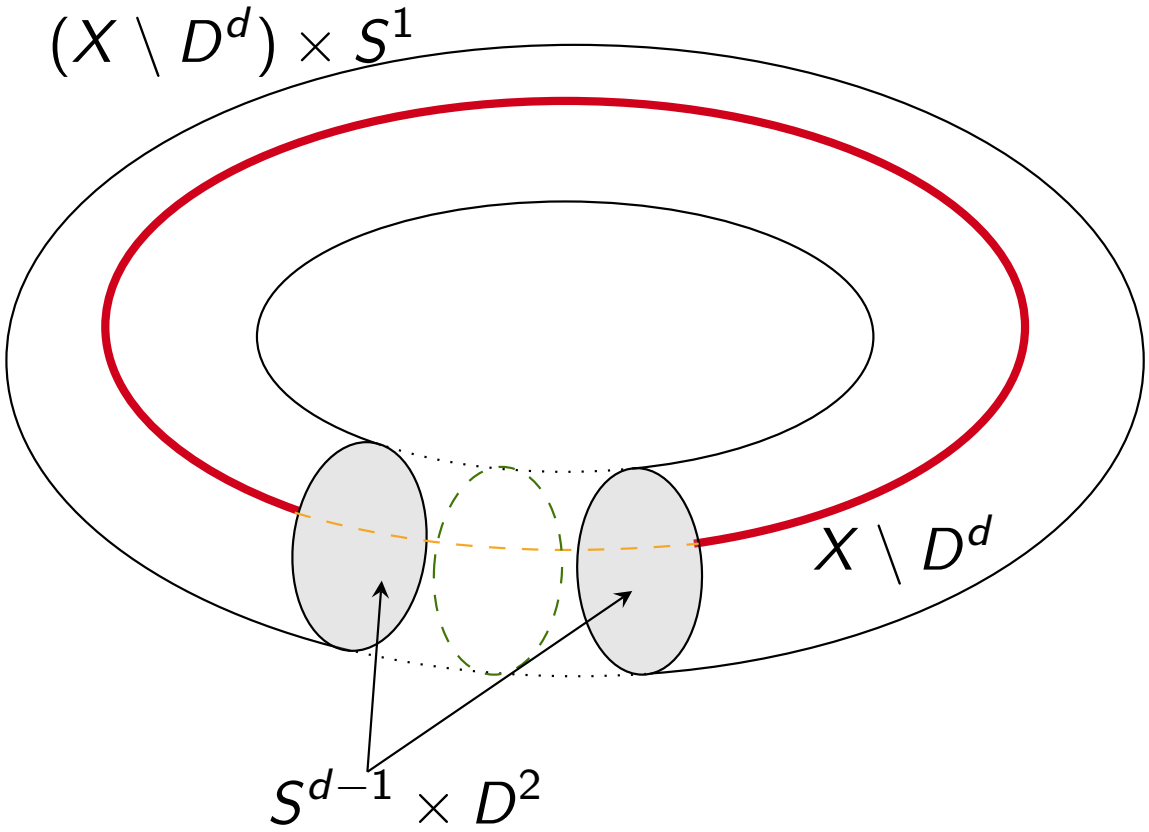
Lemma (Polytopal case)

The embedding $S^{2k-1} \times T^J \hookrightarrow \mathcal{Z}_P$ is bordant to an embedding of the $|J|$-fold gyration

$$g\colon \mathcal{G}^{|J|}(S^{2k-1}) \hookrightarrow \mathcal{Z}_P, \qquad \mathcal{G}^{|J|}(S^{2k-1}) \cong S^{2k-1+|J|},$$

with trivial normal bundle.

Gyration $\mathcal{G}(X) = \big((X \setminus D^d) \times S^1\big) \cup_{S^{d-1}\times S^1} (S^{d-1} \times D^2)$



Case (c): two non-adjacent missing edges

Theorem (Polytopal case (c))

Let P be a 4-dimensional simple polytope such that $\mathcal{K}_P$ has exactly two missing edges, not adjacent. Then

$$\mathcal{Z}_P \cong M_1 \# \cdots \# M_k,$$

where each M_i is a product of spheres, and one M_i is a product of **three** *spheres.*

In terms of P

P has exactly two pairs of non-adjacent facets forming a prismatic 4-circuit. The first such example is due to Fan–Chen–Ma–Wang.

Proof sketch of the polytopal case (c)

1. Build the bounding manifold

$$\mathcal{Q} = \{(x, z) \in \mathbb{R}_{\geq} \times \mathbb{C}^m : \gamma_1(x^2 + |z_1|^2) + |z_2|^2 + \cdots + \gamma_m |z_m|^2 = \delta\}.$$

   Then $\partial\mathcal{Q} = \mathcal{Z}_P$, and $\mathcal{Q} \simeq \mathcal{Z}_{\mathcal{K}_{[m]\setminus 1}}$.
2. Check simple connectivity and $H_i(\mathcal{Q}) = 0$ for $i \geq d - 1$.
3. Identify $\widetilde{H}_*(\mathcal{Q})$:

$$\widetilde{H}_*(\mathcal{Q}) = \mathbb{Z}\langle a_1, a_2, c\rangle \oplus \mathbb{Z}\langle b_\alpha : \alpha \in A\rangle \cong \widetilde{H}_*(S^3 \times S^3) \oplus \widetilde{H}_*\Big(\bigvee_\alpha S^{n_\alpha}\Big).$$

4. The classes a_1, a_2, c come from the chordless 4-cycle C: they are represented by $\mathcal{Z}_C \cong S^3 \times S^3$ with trivial normal bundle.
5. The classes b_α correspond to missing 2-faces, represented by gyrated spheres.
6. Apply the h-cobordism criterion.

Case (b): chordal graph

A **starshaped** sphere is the underlying complex of a complete simplicial fan.
A polytopal sphere $\mathcal{K}_P$ is starshaped (comes from the **normal fan** of P), but not vice versa.

Theorem (Starshaped, chordal + a dominating vertex)

Let $\mathcal{K}$ be a 3-dimensional starshaped sphere such that $\mathcal{K}^1$ is chordal and there is a vertex adjacent to all others. Then

$$\mathcal{Z}_{\mathcal{K}} \cong M_1 \# \cdots \# M_k,$$

with each M_i a product of **two** *spheres.*

In particular, this applies to 3-dimensional starshaped **neighbourly** spheres.

Case (c), starshaped version

Theorem (Starshaped case (c))

Let $\mathcal{K}$ be a 3-dimensional **starshaped** *simplicial sphere with exactly two missing edges, not adjacent. Then*

$$\mathcal{Z}_{\mathcal{K}} \cong M_1 \# \cdots \# M_k,$$

where $M_1 \cong S^3 \times S^3 \times S^{m-2}$, and each M_i $(i \geq 2)$ is a product of two spheres.

Key point

For non-polytopal starshaped spheres, we cannot use the quadric model. Instead, we use the quotient construction by the exponential action:

$$V \times \mathbb{C}^m \to \mathbb{C}^m, \qquad (v, z) \mapsto (e^{\langle \gamma_1, v \rangle} z_1, \ldots, e^{\langle \gamma_m, v \rangle} z_m),$$

and the homeomorphism $U(\mathcal{K})/V \cong \mathcal{Z}_{\mathcal{K}}$.

Odd-dimensional neighbourly spheres

Theorem (Odd-dimensional neighbourly starshaped)

Let $\mathcal{K}$ be a starshaped neighbourly simplicial sphere of odd dimension $n-1$, $\mathcal{K} \neq \partial\Delta^n$. Then

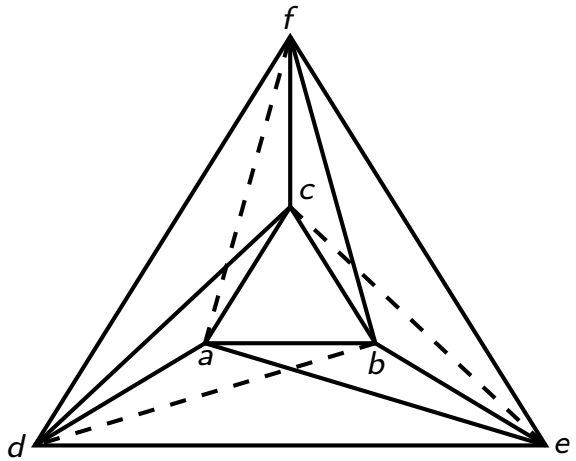
$$\mathcal{Z}_{\mathcal{K}} \cong M_1 \# \cdots \# M_k,$$

with each M_i a product of **two** *spheres.*

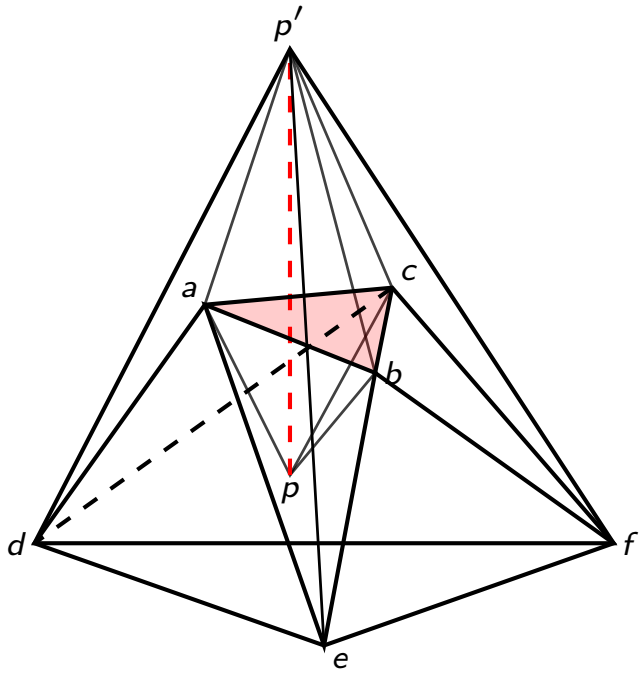
Idea of proof.

Set $q = (n-2)/2$. Neighbourliness gives $\mathcal{H}_{k,I} = 0$ for $k < q$, and Poincaré duality gives $\mathcal{H}_{k,I} = 0$ for $q < k < n-1$. Thus only $\mathcal{H}_{q,*}$ survives, and its generators correspond to missing faces. Represent them by spheres via the starshaped gyration lemma, then apply the h-cobordism criterion. □

Barnette and Brückner spheres

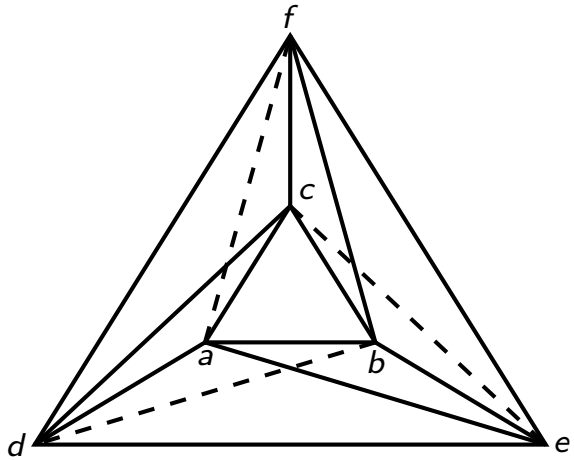


Starting configuration:
three tetrahedra *abde*, *acdf*, *bcef*.

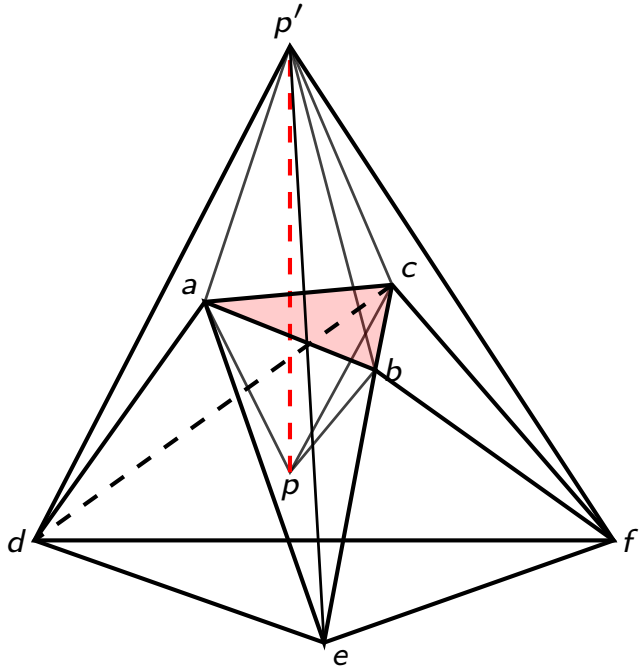


$\mathcal{K}_{\mathrm{Ba}}$: no edge pp' (dashed red)
$\mathcal{K}_{\mathrm{Br}}$: pp' added by a bistellar flip

- Adding vertex p (inside) and vertex p' (outside) to the starting configuration yields $\mathcal{K}_{\mathrm{Ba}}$: 8 vertices, 19 tetrahedra, non-polytopal, starshaped.
- A bistellar flip along the edge pp' replaces $pabc$ and $p'abc$ by $pp'ab$, $pp'bc$, $pp'ac$, giving the neighbourly sphere $\mathcal{K}_{\mathrm{Br}}$: 8 vertices, 20 tetrahedra.



Starting configuration:
three tetrahedra *abde*, *acdf*, *bcef*.



$\mathcal{K}_{\mathrm{Ba}}$: no edge pp' (dashed red)

$\mathcal{K}_{\mathrm{Br}}$: pp' added by a bistellar flip

- Both have 8 vertices; $\mathcal{K}_{\mathrm{Ba}}$ has 19 tetrahedra, $\mathcal{K}_{\mathrm{Br}}$ has 20.
- Both are *non-polytopal* but *starshaped*, so $\mathcal{Z}_{\mathcal{K}_{\mathrm{Ba}}}$ and $\mathcal{Z}_{\mathcal{K}_{\mathrm{Br}}}$ are smooth.
- $\mathcal{K}_{\mathrm{Br}}$ is *neighbourly*; $\mathcal{K}_{\mathrm{Ba}}$ has exactly one missing edge.
- They differ by a single bistellar flip along pp'.

Proposition (Explicit diffeomorphism types)

$$\mathcal{Z}_{\mathcal{K}_{\mathrm{Ba}}} \cong (S^3 \times S^9) \# (S^5 \times S^7)^{\#12} \# (S^6 \times S^6)^{\#12},$$
$$\mathcal{Z}_{\mathcal{K}_{\mathrm{Br}}} \cong (S^5 \times S^7)^{\#16} \# (S^6 \times S^6)^{\#15}.$$

Idea of proof.

Both spheres satisfy Case (b): the 1-skeleton is chordal and the vertex f is adjacent to all others. The counts come from the missing faces:

- Missing edges $\Rightarrow$ summands $S^3 \times S^9$ (Ba) or none (Br).
- Missing triangles $\Rightarrow$ 12 (Ba) resp. 16 (Br) summands $S^5 \times S^7$.
- $H_{1,I} \cong \mathbb{Z}$ for some 4-element sets I
  $\Rightarrow$ 12 (Ba) resp. 15 (Br) summands $S^6 \times S^6$.

□

Open problems

- Remove the *starshaped* assumption: can one prove diffeomorphism for all 3-dimensional simplicial spheres satisfying (b) or (c)?
- Are there non-starshaped spheres satisfying the criteria? (Non-starshaped examples exist, but their moment-angle manifolds are only topological.)
- Non-starshaped odd-dimensional neighbourly spheres: can one prove diffeomorphism or homeomorphism? (**Membrillo Solis and Theriault** proved homotopy equivalence to a connected sum of sphere products for arbitrary odd-dim neighbourly spheres.)
- Even-dimensional neighbourly spheres: the torsion-free condition on $\mathcal{H}^{\frac{n-1}{2},*}(\mathcal{K})$ is essential for the cohomological statement; do spheres with this torsion exist?
- Higher-dimensional analogues of the chordal/two-missing-edges classification.

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Backup: the gyration construction in detail

- $Y_j := \mu^{-1}([x_j, y_j]) \cong D^2 \times T^{m-1}$ is a smooth submanifold with boundary in $\mathcal{Z}_P$.
- $X \cap Y_j = \mu^{-1}(x_j) \cong T^m$.
- Construct an embedding

$$b\colon (S^{2k-1} \times S^1_j \cup_{D^{2k-1}\times S^1} D^{2k-1} \times D^2_j) \times T^{J\setminus j} \hookrightarrow \mathcal{Z}_P,$$

  whose restriction to $S^{2k-1} \times T^J$ coincides with i.
- Restricting b to the gyration $\mathcal{G}(S^{2k-1})$ and smoothing corners gives the required sphere.

Backup: starshaped case, normal bundles

- Use the smooth embedding $\mathcal{Z}_\mathcal{K} \hookrightarrow \mathbb{C}^m$ from the triviality of the principal V-bundle $U(\mathcal{K}) \to U(\mathcal{K})/V \cong \mathcal{Z}_\mathcal{K}$.
- For a missing face I, construct $S^{2k-1} \times T^{[m]\setminus I} \hookrightarrow \mathcal{Z}_\mathcal{K}$ with trivial normal bundle by considering two cases (whether the vectors A_I are linearly dependent or independent) and taking smooth sections of trivial bundles.
- The gyration argument then goes through verbatim.