Permutohedral complex and complements of diagonal subspace arrangements

joint with Vsevolod Tril

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Coordinate and diagonal arrangements

An arrangement is a finite set $\mathcal{A} = \{L_1, \ldots, L_r\}$ of affine subspaces in some affine space (either real or complex).

An arrangement $\mathcal{A} = \{L_1, \ldots, L_r\}$ is coordinate, if every L_i, $i = 1, \ldots, r$, is a coordinate subspace

$$C_I = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \colon x_{i_1} = \cdots = x_{i_k} = 0\},$$

where $I = \{i_1, \ldots, i_k\}$ is a subset in $[m] = \{1, 2, \ldots, m\}$.

Given a simplicial complex $\mathcal{K}$ on $[m]$, define the real coordinate arrangement

$$\mathcal{CA}(\mathcal{K}) = \{C_I \colon I \notin \mathcal{K}\}$$

and its complement

$$U_{\mathbb{R}}(\mathcal{K}) = \mathbb{R}^m \setminus \bigcup_{I \notin \mathcal{K}} C_I.$$

Given $I = \{i_1, \dots, i_k\} \subset [m]$, the corresponding diagonal subspace is

$$D_I = \{(x_1, \dots, x_m) \in \mathbb{R}^m \colon x_{i_1} = \dots = x_{i_k}\}.$$

Every simplicial complex $\mathcal{K}$ on $[m]$ defines a real diagonal arrangement and its complement

$$\mathcal{DA}(\mathcal{K}) = \{D_I \colon I \notin \mathcal{K}\}, \qquad D_{\mathbb{R}}(\mathcal{K}) = \mathbb{R}^m \setminus \bigcup_{I \notin \mathcal{K}} D_I.$$

Proposition

$\mathcal{K} \mapsto U_{\mathbb{R}}(\mathcal{K})$ (respectively, $\mathcal{K} \mapsto D_{\mathbb{R}}(\mathcal{K})$) is a one-to-one order preserving correspondence between simplicial complexes on $[m]$ and coordinate (respectively, diagonal) arrangement complements in $\mathbb{R}^m$.

Complex diagonal subspaces $D_I^{\mathbb{C}} \subset \mathbb{C}^m$, diagonal arrangements $\mathcal{DA}_{\mathbb{C}}(\mathcal{K})$, and their complements $D_{\mathbb{C}}(\mathcal{K})$ are defined similarly.

Real moment-angle complexes and the Cai diagonal

The real moment-angle complex corresponding to $\mathcal{K}$ is a subcomplex in the cube $I^m = [-1,1]^m$:

$$\mathcal{R}_{\mathcal{K}} = \bigcup_{\sigma\in\mathcal{K}} (D^1, S^0)^\sigma = \bigcup_{\sigma\in\mathcal{K}} \Bigl(\prod_{i\in\sigma} D^1 \times \prod_{i\notin\sigma} S^0\Bigr),$$

where $(D^1, S^0) = ([-1,1], \{-1,1\})$.

Theorem

There is a deformation retraction $U_{\mathbb{R}}(\mathcal{K}) \xrightarrow{\simeq} \mathcal{R}_{\mathcal{K}}$.

Each factor $[-1,1]_i \subset I^m$ is a simplicial complex with vertices $\underline{t}_i = \{-1\}_i$, $t_i = \{1\}_i$ and 1-simplex $u_i = [-1,1]_i$. A cell in I^m is given by

$$u_\sigma t_\tau \underline{t}_{[m]\setminus(\sigma\cup\tau)} := \prod_{i\in\sigma} u_i \times \prod_{i\in\tau} t_i \times \prod_{i\notin(\sigma\cup\tau)} \underline{t}_i,$$

where σ, τ are non-intersecting subsets in $[m]$. Cells of $\mathcal{R}_{\mathcal{K}} \subset I^m$ are specified by the condition $\sigma \in \mathcal{K}$.

We dentify cells with the corresponding cellular chains.
Let $\varepsilon_i := \partial u_i = t_i - \underline{t}_i$.
Then the cellular chains $u_\sigma\varepsilon_\tau := u_\sigma\varepsilon_\tau\underline{t}_{[m]\setminus(\sigma\cup\tau)}$ form a basis in $C_*(I^m) = \bigotimes_{i=1}^m C_*(I)$.

The dual basis consists of cochains of the form

$$u^\sigma t^\tau := u^\sigma t^\tau \delta^{[m]\setminus(\sigma\cup\tau)} = \bigotimes_{i\in\sigma} u_i^* \otimes \bigotimes_{i\in\tau} t_i^* \otimes \bigotimes_{i\notin\sigma\cup\tau} \delta_i^* \in C^{|\sigma|}(I^m),$$

where u_i^*, t_i^*, $\underline{t}_i^*$ are dual to u_i, t_i, $\underline{t}_i$ respectively, and $\delta_i^* = t_i^* + \underline{t}_i^*$.

The cellular differential is given by

$$du_i^* = 0, \quad dt_i^* = u_i^*, \quad d\delta_i^* = 0.$$

The standard $\smile$-product in $C^*(I) = \mathbb{Z}\langle t_i^*, \delta_i^*, u_i^* \rangle$ is given by the relations

$$t_i^* \smile t_i^* = t_i^*, \quad t_i^* \smile u_i^* = 0, \quad u_i^* \smile t_i^* = u_i^*, \quad u_i^* \smile u_i^* = 0,$$
$$\delta_i^* \smile t_i^* = t_i^* \smile \delta_i^* = t_i^*, \quad \delta_i^* \smile u_i^* = u_i^* \smile \delta_i^* = u_i^*, \quad \delta_i^* \smile \delta_i^* = \delta_i^*.$$

This extends to a product in cellular cochains $C^*(I^m)$ and $C^*(\mathcal{R}_\mathcal{K})$:

$$u^\sigma t^\tau \smile u^{\sigma'} t^{\tau'} = (-1)^{(\sigma,\sigma')} u^{\sigma\cup\sigma'} t^{\tau\cup(\tau'\setminus\sigma)}$$

if $\sigma \cap \sigma' = \varnothing$ and $\tau \cap \sigma' = \varnothing$, otherwise the product is zero, and

$$(\sigma, \sigma') = |\{(i,j)\colon i \in \sigma,\ j \in \sigma',\ i > j\}|.$$

Theorem (Li Cai)

The dga above is a model for $\mathcal{R}_\mathcal{K}$. In particular, there is a ring isomorphism

$$H(C^*(\mathcal{R}_\mathcal{K}), d) \cong H^*(\mathcal{R}_\mathcal{K}).$$

The dual diagonal of the cellular chain coalgebra $C_*(I^m)$ is given on the basis chains $u_\sigma t_\tau \underline{t}_{[m]\setminus(\sigma\cup\tau)}$ by

$$\Delta_C(u_\sigma t_\tau \underline{t}_{[m]\setminus(\sigma\cup\tau)}) = \sum_{\sigma'\subset\sigma}(-1)^{(\sigma',\sigma\setminus\sigma')}u_{\sigma'}t_\tau\underline{t}_{[m]\setminus(\sigma'\cup\tau)}\otimes u_{\sigma\setminus\sigma'}t_{\sigma'\cup\tau}\underline{t}_{[m]\setminus(\sigma\cup\tau)}$$

For example, on the top-dimensional cell it is given by

$$\Delta_C(u_{[m]}t_\varnothing\underline{t}_\varnothing) = \sum_{\sigma\subset[m]}(-1)^{(\sigma,[m]\setminus\sigma)}u_\sigma t_\varnothing\underline{t}_{[m]\setminus\sigma}\otimes u_{[m]\setminus\sigma}t_\sigma\underline{t}_\varnothing.$$

Permutohedral complex $\mathrm{Perm}(\mathcal{K})$

The permutohedron is the polytope in $\mathbb{R}^m$ given by

$$\mathrm{Perm}^{m-1} = \mathrm{conv}\{(\sigma(1), \ldots, \sigma(m)) \in \mathbb{R}^m \colon \sigma \in S_m\}.$$

Theorem

Faces of Perm^{m-1} of dimension p are in one-to-one correspondence with ordered partitions of the set $[m]$ into $m - p$ non-empty parts. An inclusion of faces $G \subset F$ occurs whenever the ordered partition corresponding to G can be obtained by refining the ordered partition corresponding to F.

Let $F(U_1|\cdots|U_p)$ be the face Perm^{m-1} corresponding to the ordered $(U_1|\cdots|U_p)$ of $[m]$ into non-empty parts $U_1, \ldots, U_p$.

Assume that elements in every part are increasingly ordered.

For each simplicial complex $\mathcal{K}$ on the vertex set $[m]$ define the permutohedral complex

$$\mathrm{Perm}(\mathcal{K}) = \bigcup_{\substack{U_1,\ldots,U_p\in\mathcal{K},\\ U_1\sqcup\cdots\sqcup U_p=[m]}} F(U_1|\cdots|U_p).$$

Theorem

There is a deformation retraction $D_{\mathbb{R}}(\mathcal{K}) \xrightarrow{\simeq} \mathrm{Perm}(\mathcal{K})$ *from the complement of a diagonal subspace arrangement to* $\mathrm{Perm}(\mathcal{K})$.

Example

Let $\mathcal{K}$ be the complete graph on m vertices. Then $\mathrm{Perm}(\mathcal{K})$ is the complex of all cubic faces of the permutohedron.

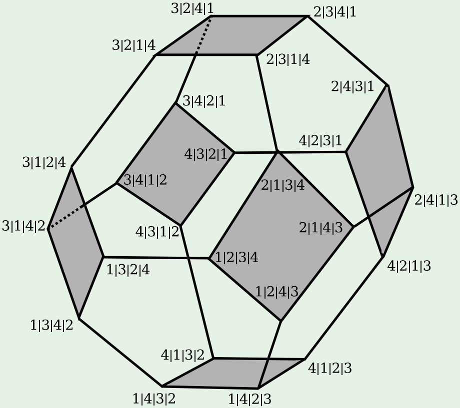

Figure: Complex $\mathrm{Perm}(\mathcal{K})$ for the complete graph on 4 vertices

There is also a cellular model for complex diagonal arrangement complements.
Namely, let Perm^{2m-1} be the standard permutohedron in $\mathbb{C}^m \cong \mathbb{R}^{2m}$. Denote the indices $(m+1,\ldots,2m)$ by $(1',\ldots,m')$, the faces of Perm^{2m-1} correspond to ordered partitions of the set

$$[m] \cup [m'] = \{1,2,\ldots,m,1',2',\ldots,m'\}.$$

Now consider the complex

$$\mathrm{Perm}_{\mathbb{C}}(\mathcal{K}) := \mathrm{Perm}^{2m-1} \setminus \bigcup_{I \notin \mathcal{K}} \bigcup_{\substack{(U_1|\cdots|U_p) \\ \exists j,k: I \subset U_j, I' \subset U_k}} \operatorname{relint} F(U_1|\cdots|U_p),$$

where each set $I' = \{i'_1,\ldots,i'_s\}$ of primed indices corresponds to the set $I = \{i_1,\ldots,i_s\}$ of same indices without primes.

Theorem

There is a homotopy equivalence $D_{\mathbb{C}}(\mathcal{K}) \simeq \mathrm{Perm}_{\mathbb{C}}(\mathcal{K})$.

Algebraic model for cellular cochains

Let k a commutative ring with unit, and A a graded k-algebra with unit.

Let $(\overline{B}(A), \overline{d})$ be the reduced bar construction of the ring k as a left A-module. We have

$$\mathrm{Tor}_A^{-n}(\mathrm{k},\mathrm{k}) = H^{-n}[\overline{B}(A), \overline{d}].$$

Now let A be the exterior Stanley–Reisner algebra

$$\Lambda[\mathcal{K}] = \Lambda[x_1, \ldots, x_m]/\mathcal{I}_{SR},$$

where $\mathcal{I}_{SR} = (x_{i_1} \cdots x_{i_k} \colon \{i_1, \ldots, i_k\} \notin \mathcal{K})$, with the standard $\mathbb{Z}^m_{\geq 0}$-grading.

The basis of $\overline{B}^{-n}(\Lambda[\mathcal{K}])$ consists of the elements $[X_1|\cdots|X_n]$, where each X_i is a monomial in $\Lambda[\mathcal{K}]$, $i = 1, \ldots, n$, with the multigrading $\mathrm{mdeg}([X_1|\cdots|X_n]) = \mathrm{mdeg}(X_1) + \cdots + \mathrm{mdeg}(X_n)$.

Theorem

The cellular cochain complex of $\mathrm{Perm}(\mathcal{K})$ *is isomorphic to the* $(1,\ldots,1)$*-component of the reduced bar construction of* $\Lambda[\mathcal{K}]$*:*

$$\big(C^p(\mathrm{Perm}(\mathcal{K});\mathrm{k}),d\big)\cong\big(\overline{B}^{p-m}(\Lambda[\mathcal{K}]),\overline{d}\big)_{(1,\ldots,1)}.$$

Proof.

Consider the k-module homomorphism

$$\varphi\colon C^*(\mathrm{Perm}(\mathcal{K}))\to\overline{B}(\Lambda[\mathcal{K}])_{(1,\ldots,1)},$$

defined on the generators by

$$\varphi(F(U_1|\cdots|U_{m-p})^*)=[X_1|\cdots|X_{m-p}],$$

where $X_j=\prod_{i\in U_j}x_i$. The map φ is an isomorphism of k-modules, since it is bijective on generators.
Use Milgram's description of the boundary in the cellular chain complex of permutohedron to show that φ is an isomorphism of chain complexes. □

Corollary

For any commutative ring k *with unit we have and isomorphism of* k*-modules*

$$H^p(D_{\mathbb{R}}(\mathcal{K});\mathsf{k}) \cong \mathrm{Tor}^{p-m}_{\Lambda[\mathcal{K}]}(\mathsf{k},\mathsf{k})_{(1,\dots,1)}.$$

Since $\Lambda[\mathcal{K}]$ is a graded commutative algebra, there is a natural graded commutative product in the k-module $\mathrm{Tor}^*_{\Lambda[\mathcal{K}]}(\mathsf{k},\mathsf{k})$ that arises from the bar construction. However, this product does not preserve the grading $(1,\dots,1)$, so it cannot be used for description of the product in the cohomology ring $H^*(\mathrm{Perm}(\mathcal{K});\mathsf{k})$, which is additively isomorphic to $\mathrm{Tor}^*_{\Lambda[\mathcal{K}]}(\mathsf{k},\mathsf{k})_{(1,\dots,1)}$.

A product of cellular cochains $C^*(\mathrm{Perm}(\mathcal{K}))$ can be defined using a cellular diagonal approximation $\widetilde{\Delta}\colon \mathrm{Perm}^{m-1}\to \mathrm{Perm}^{m-1}\times\mathrm{Perm}^{m-1}$ of the standard diagonal in permutohedron.

Saneblidze–Umble diagonal

A $q \times p$ matrix $O = (o_{i,j})$ is ordered, if:

1) $\{o_{i,j}\} = \{0, 1, \ldots, q + p - 1\}$;
2) each row and column of O is non-zero;
3) non-zero entries in O are distinct and increase in each row and column.

The set of ordered $q \times p$ matrices is denoted by $\mathcal{O}^{q \times p}$.

Given an ordered matrix O, we consider two partitions of $[q + p - 1]$:
$c(O) = (O_1 | \cdots | O_q)$, where O_j is the j-th column of O;
$r(O) = (O^p | \cdots | O^1)$, where O^i is the i-th row of O (note the reversed order of rows).

Here we assume that all zero entries of O_j and O^i are removed.

An ordered matrix $E = (e_{i,j})$ is a step matrix if:

1) non-zero entries in each row of E appear in consecutive columns;
2) non-zero entries in each column of E appear in consecutive rows;
3) the sub, main and super diagonals of E contain a single non-zero entry (i. e., there is a single non-zero entry among $e_{i,j}$ with fixed $j - i$).

The set of $q \times p$ step matrices is denoted by $\mathcal{E}^{q\times p}$.

Example

The matrix $E = \begin{pmatrix} 0 & 2 & 3 \\ 1 & 5 & 0 \\ 4 & 0 & 0 \end{pmatrix}$ is a step matrix. The corresponding partitions of $\{1, 2, 3, 4, 5\}$ are $c(E) = (14|25|3)$, $r(E) = (4|15|23)$.

For any $(i,j) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ define the down-shift operators $D_{i,j}\colon \mathcal{O} \to \mathcal{O}$ and the right-shift operators $R_{i,j}\colon \mathcal{O} \to \mathcal{O}$ on $O \in \mathcal{O}^{q\times p}$ by:

1) If $o_{i,j} > 0$, $o_{i+1,j} = 0$, $o_{i+1,l} < o_{i,j}$ whenever $l < j$, $o_{i+1,l} > o_{i,j}$ whenever $l > j$ and $o_{i+1,l} \neq 0$, and there is $o_{i,k} \neq 0$ for some $k \neq i$, then $D_{i,j}O$ is obtained from O by transposing $o_{i,j}$ and $o_{i+1,j}$. Otherwise $D_{i,j}O = O$.
2) If $o_{i,j} > 0$, $o_{i,j+1} = 0$, $o_{l,j+1} < o_{i,j}$ whenever $l < i$, $o_{l,j+1} > o_{i,j}$ whenever $l > i$ and $o_{l,j+1} \neq 0$, and there is $o_{k,j} \neq 0$ for some $k \neq j$, then $R_{i,j}O$ is obtained from O by transposing $o_{i,j}$ and $o_{i,j+1}$. Otherwise $R_{i,j}O = O$.

Example

Consider $E = \begin{pmatrix} 0 & 2 & 3 \\ 1 & 5 & 0 \\ 4 & 0 & 0 \end{pmatrix}$. Then

$$R_{2,2}E = \begin{pmatrix} 0 & 2 & 3 \\ 1 & 0 & 5 \\ 4 & 0 & 0 \end{pmatrix}, \quad D_{2,2}E = \begin{pmatrix} 0 & 2 & 3 \\ 1 & 0 & 0 \\ 4 & 5 & 0 \end{pmatrix}, \quad D_{2,3}R_{2,2}E = \begin{pmatrix} 0 & 2 & 3 \\ 1 & 0 & 0 \\ 4 & 0 & 5 \end{pmatrix}.$$

A matrix $A \in \mathcal{O}$ is a configuration matrix, if there is a step matrix E and a sequence of shift operators $G_1, \dots, G_s$ such that

1) $A = G_s \cdots G_1 E$,
2) if $G_s \cdots G_1 = \cdots D_{i_2,j_2} \cdots D_{i_1,j_1} \cdots$, then $i_1 \le i_2$,
3) if $G_s \cdots G_1 = \cdots R_{i_2,j_2} \cdots R_{i_1,j_1} \cdots$, then $j_1 \le j_2$.

When this occurs, we say that A is derived from E.

The set of $q \times p$ configuration matrices is denoted by $\mathcal{C}^{q\times p}$.

All matrices of the previous Example are configuration matrices.

For each m, define

$$\Delta_{SU}(F(1,2,\dots,m)) = \sum_{q=1}^{m} \sum_{A\in\mathcal{C}^{q\times(m-q+1)}} csgn(A)\, F(c(A)) \otimes F(r(A)),$$

where $csgn(A)$ is a certain sign.

Then extend Δ_{SU} to proper faces $F(U_1|\cdots|U_p)$ via the standard comultiplicative extension:

$$\Delta_{SU}\big(F(U_1|\cdots|U_p)\big) = F\big(\Delta_{SU}(F(U_1))\big|\cdots\big|\Delta_{SU}(F(U_p))\big).$$

Example

$$\Delta(F(12)) = F(12)\otimes F(2|1) + F(1|2)\otimes F(12),$$

$$\begin{aligned}\Delta(F(123)) = {} & F(1|2|3)\otimes F(123) + F(123)\otimes F(3|2|1) - \\ & - F(1|23)\otimes F(13|2) + F(2|13)\otimes F(23|1) - F(13|2)\otimes F(3|12) + \\ & + F(12|3)\otimes F(2|13) - F(1|23)\otimes F(3|12) + F(12|3)\otimes F(23|1).\end{aligned}$$

Connection between Cai and Saneblidze–Umble diagonals

The piecewise linear projection $\rho\colon \mathrm{Perm}^{m-1} \to I^{m-1}$ defined on vertices by

$$\rho(F(U_1|\cdots|U_m)) = \prod_{i\in\tau}\{1\} \times \prod_{i\notin\tau}\{-1\},$$

where $\tau = \{i\colon U_j = \{i+1\},\ U_k = \{i\} \text{ for some } j < k\}$.

The image of any face is given by

$$\rho\,(F(U_1|\cdots|U_p)) = \prod_{i\in\sigma} D^1 \times \prod_{i\in\tau}\{1\} \times \prod_{i\notin\sigma\cup\tau}\{-1\},$$

where $\sigma = \{i \mid \exists j : \{i, i+1\} \subset U_j\}$, $\tau = \{i \mid \exists j < k : i+1 \in U_j,\ i \subset U_k\}$.

Theorem

For any face $F(U_1|\cdots|U_p)$ of Perm^{m-1} we have

$$(\rho_* \otimes \rho_*)\Delta_{SU} F(U_1|\cdots|U_p) = \Delta_C(\rho_* F(U_1|\cdots|U_p)).$$

Theorem

Let $\mathcal{K}$ be a simplicial complex on the vertex set $[m]$. Then the image of $\mathrm{Perm}(\mathcal{K})$ *under the projection* $\rho\colon \mathrm{Perm}^{m-1} \to I^{m-1}$ *is the real moment-angle complex* $\mathcal{R}_{\mathcal{L}}$*, where* $\mathcal{L} = \mathcal{L}(\mathcal{K})$ *is the simplicial complex on the set* $[m-1]$ *defined below.*

A set $J \subset [m-1]$ belongs to $\mathcal{L}(\mathcal{K})$ if and only if each subset formed by consecutive elements $\{j, j+1, \ldots, j+k\}$ of J together with the element $\{j+k+1\}$ forms a simplex of $\mathcal{K}$.

References

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