Double cohomology of moment-angle complexes and bigraded persistence barcodes

Joint with Tony Bahri, Ivan Limonchenko, Jongbaek Song and Donald Stanley.

Taras Panov

Moscow State University & NRU HSE

Beijing Institute of Mathematical Sciences and Applications (BIMSA)
Topology Seminar, 13 November 2023

1. Preliminaries

$\mathcal{K}$ a simplicial complex on $[m] = \{1, 2, \dots, m\}$, $\varnothing \in \mathcal{K}$.
$I = \{i_1, \dots, i_k\} \in \mathcal{K}$ a face (or a simplex).
Assume $\varnothing \in \mathcal{K}$ and $\{i\} \in \mathcal{K}$ for each $i = 1, \dots, m$ (no ghost vertices).

$\mathrm{CAT}(\mathcal{K})$ the face category of $\mathcal{K}$, with objects $I \in \mathcal{K}$ and morphisms $I \subset J$.

For $I \in \mathcal{K}$, consider

$$(D^2, S^1)^I = \{(z_1, \dots, z_m) \in (D^2)^m \colon |z_j| = 1 \text{ if } j \notin I\} \subset (D^2)^m.$$

Note that $(D^2, S^1)^I \subset (D^2, S^1)^J$ whenever $I \subset J$. Have a diagram

$$\mathscr{D}_{\mathcal{K}} \colon \mathrm{CAT}(\mathcal{K}) \to \mathrm{TOP}$$

mapping $I \in \mathcal{K}$ to $(D^2, S^1)^I$.

The moment-angle complex corresponding to $\mathcal{K}$ is

$$\mathcal{Z}_{\mathcal{K}} := \operatorname{colim} \mathscr{D}_K = \bigcup_{I \in \mathcal{K}} (D^2, S^1)^I \subset (D^2)^m.$$

The face ring of $\mathcal{K}$ is

$$\mathbb{Z}[\mathcal{K}] := \mathbb{Z}[v_1, \ldots, v_m]/\mathcal{I}_{\mathcal{K}},$$

where $\mathcal{I}_{\mathcal{K}}$ is generated by $\prod_{i \in I} v_i$ for which $I \subset [m]$ is not a simplex of $\mathcal{K}$.

Theorem

There are isomorphisms of bigraded commutative algebras

$$\begin{aligned} H^*(\mathcal{Z}_{\mathcal{K}}) &\cong \operatorname{Tor}_{\mathbb{Z}[v_1,\ldots,v_m]}(\mathbb{Z}[\mathcal{K}], \mathbb{Z}) \\ &\cong H\big(\Lambda[u_1, \ldots, u_m] \otimes \mathbb{Z}[\mathcal{K}], d\big) \\ &\cong \bigoplus_{I \subset [m]} \widetilde{H}^*(\mathcal{K}_I). \end{aligned}$$

Here $\big(\Lambda[u_1, \ldots, u_m] \otimes \mathbb{Z}[\mathcal{K}], d\big)$ *is the Koszul complex with* $\mathrm{bideg}\, u_i = (-1, 2)$, $\mathrm{bideg}\, v_i = (0, 2)$ *and* $du_i = v_i$, $dv_i = 0$.
$\widetilde{H}^*(\mathcal{K}_I)$ *denotes the reduced simplicial cohomology of the full subcomplex* $\mathcal{K}_I \subset \mathcal{K}$ *(the restriction of* $\mathcal{K}$ *to* $I \subset [m]$*).*

The bigraded components of the cohomology of $\mathcal{Z}_\mathcal{K}$ are given by

$$H^{-k,2\ell}(\mathcal{Z}_\mathcal{K}) \cong \bigoplus_{I\subset[m]\colon |I|=\ell} \widetilde{H}^{\ell-k-1}(\mathcal{K}_I), \quad H^p(\mathcal{Z}_\mathcal{K}) = \bigoplus_{-k+2\ell=p} H^{-k,2\ell}(\mathcal{Z}_\mathcal{K}).$$

Consider the following quotient of the Koszul ring $\Lambda[u_1,\ldots,u_m]\otimes\mathbb{Z}[\mathcal{K}]$:

$$R^*(\mathcal{K}) = \Lambda[u_1,\ldots,u_m]\otimes\mathbb{Z}[\mathcal{K}]\big/(v_i^2 = u_iv_i = 0,\ 1\leqslant i\leqslant m).$$

Then $R^*(\mathcal{K})$ has finite rank as an abelian group, with a basis of monomials u_Jv_I where $J\subset[m]$, $I\in\mathcal{K}$ and $J\cap I=\varnothing$.

Furthermore, $R^*(\mathcal{K})$ can be identified with the cellular cochains $C^*(\mathcal{Z}_\mathcal{K})$ of $\mathcal{Z}_\mathcal{K}$ with the standard cell decomposition, the quotient ideal $(v_i^2 = u_iv_i = 0,\ 1\leqslant i\leqslant m)$ is d-invariant and acyclic, and there is a ring isomorphism

$$H^*(\mathcal{Z}_\mathcal{K}) \cong H\big(R^*(\mathcal{K}), d\big).$$

2. Double (co)homology

We have

$$H_p(\mathcal{Z}_\mathcal{K}) \cong \bigoplus_{I\subset[m]} \widetilde{H}_{p-|I|-1}(\mathcal{K}_I),$$

Given $j \in [m] \setminus I$, consider the homomorphism

$$\phi_{p;I,j}\colon \widetilde{H}_p(\mathcal{K}_I) \to \widetilde{H}_p(\mathcal{K}_{I\cup\{j\}})$$

induced by the inclusion $\mathcal{K}_I \hookrightarrow \mathcal{K}_{I\cup\{j\}}$.Then, we define

$$\partial'_p = (-1)^{p+1} \bigoplus_{I\subset[m],\, j\in[m]\setminus I} \varepsilon(j,I)\,\phi_{p;I,j},$$

where

$$\varepsilon(j,I) = (-1)^{\#\{i\in I\colon\, i<j\}}.$$

Lemma

$\partial'_p\colon \bigoplus_{I\subset[m]} \widetilde{H}_p(\mathcal{K}_I) \to \bigoplus_{I\subset[m]} \widetilde{H}_p(\mathcal{K}_I)$ *satisfies* $(\partial'_p)^2 = 0$.

We therefore have a chain complex

$$CH_*(\mathcal{Z}_\mathcal{K}) := (H_*(\mathcal{Z}_\mathcal{K}), \partial'),$$

where

$$\partial' \colon \widetilde{H}_{-k,2\ell}(\mathcal{Z}_\mathcal{K}) \to \widetilde{H}_{-k-1,2\ell+2}(\mathcal{Z}_\mathcal{K})$$

with respect to the following bigraded decomposition of $H_p(\mathcal{Z}_\mathcal{K})$

$$H_p(\mathcal{Z}_\mathcal{K}) = \bigoplus_{-k+2\ell=p} H_{-k,2\ell}(\mathcal{Z}_\mathcal{K}), \quad H_{-k,2\ell}(\mathcal{Z}_\mathcal{K}) \cong \bigoplus_{I\subset[m]\colon |I|=\ell} \widetilde{H}_{\ell-k-1}(\mathcal{K}_I).$$

We define the bigraded double homology of $\mathcal{Z}_\mathcal{K}$ by

$$HH_*(\mathcal{Z}_\mathcal{K}) = H(H_*(\mathcal{Z}_\mathcal{K}), \partial').$$

For the cohomological version, given $i \in I$, consider the homomorphism

$$\psi_{p;i,I} \colon \widetilde{H}^p(\mathcal{K}_I) \to \widetilde{H}^p(\mathcal{K}_{I\setminus\{i\}})$$

induced by the inclusion $\mathcal{K}_{I\setminus\{i\}} \hookrightarrow \mathcal{K}_I$, and

$$d'_p = (-1)^{p+1} \sum_{i \in I} \varepsilon(i,I)\psi_{p;i,I}.$$

We define $d' \colon H^*(\mathcal{Z}_\mathcal{K}) \to H^*(\mathcal{Z}_\mathcal{K})$ using $H^*(\mathcal{Z}_\mathcal{K}) = \bigoplus_{I\subset[m]} \widetilde{H}^*(\mathcal{K}_I)$:

$$d' \colon H^{-k,2\ell}(\mathcal{Z}_\mathcal{K}) \to H^{-k+1,2\ell-2}(\mathcal{Z}_\mathcal{K}).$$

Similarly, have $(d')^2 = 0$, which turns $H^*(\mathcal{Z}_\mathcal{K})$ into a cochain complex

$$CH^*(\mathcal{Z}_\mathcal{K}) := (H^*(\mathcal{Z}_\mathcal{K}), d').$$

We define the bigraded double cohomology of $\mathcal{Z}_\mathcal{K}$ by

$$HH^*(\mathcal{Z}_\mathcal{K}) = H\big(H^*(\mathcal{Z}_\mathcal{K}), d'\big).$$

3. The bicomplexes

Given $I \subset [m]$, let $C^p(\mathcal{K}_I)$ be the pth simplicial cochain group of $\mathcal{K}_I$.

Denote by $\alpha_{L,I} \in C^{q-1}(\mathcal{K}_I)$ the basis cochain corresponding to an oriented simplex $L = (\ell_1, \ldots, \ell_q) \in \mathcal{K}_I$; it takes value 1 on L and vanishes on all other simplices.

The simplicial coboundary map (differential) $d\colon C^p(\mathcal{K}_I) \to C^{p+1}(\mathcal{K}_I)$ is

$$d\alpha_{L,I} = \sum_{j\in I\setminus L,\; L\cup\{j\}\in\mathcal{K}} \varepsilon(j, L)\alpha_{L\cup\{j\},I}.$$

Consider $\psi_{p;i,I}\colon C^p(\mathcal{K}_I) \to C^p(\mathcal{K}_{I\setminus\{i\}})$ induced by the inclusion $\mathcal{K}_{I\setminus\{i\}} \hookrightarrow \mathcal{K}_I$, and define

$$d'_p = (-1)^{p+1}\sum_{i\in I}\varepsilon(i, I)\psi_{p;i,I}.$$

Recall that the differential d on the Koszul complex $\Lambda[u_1,\ldots,u_m]\otimes\mathbb{Z}[\mathcal{K}]$ has bidegree $(1,0)$ and satisfies

$$du_j=v_j,\quad dv_j=0,\qquad \text{for } j=1,\ldots,m.$$

We introduce the second differential d' of bidegree $(1,-2)$ on $\Lambda[u_1,\ldots,u_m]\otimes\mathbb{Z}[\mathcal{K}]$ by setting

$$d'u_j=1,\quad d'v_j=0,\qquad \text{for } j=1,\ldots,m,$$

and extending by the Leibniz rule. Explicitly, the differential d' is defined on square-free monomials u_Jv_I by

$$d'(u_Jv_I)=\sum_{j\in J}\varepsilon(j,J)u_{J\setminus\{j\}}v_I,\quad d'(v_I)=0.$$

The differential d' is also defined by the same formula on the submodule $R^*(\mathcal{K})\subset\Lambda[u_1,\ldots,u_m]\otimes\mathbb{Z}[\mathcal{K}]$ generated by the monomials u_Jv_I with $J\cap I=\varnothing$. However, the ideal $\bigl(v_i^2=u_iv_i=0,\ 1\leqslant i\leqslant m\bigr)$ is not d'-invariant, so $(R^*(\mathcal{K}),d')$ is not a differential graded algebra.

Lemma

With d and d' defined above, $\big(\bigoplus_{I\subset[m]} C^*(\mathcal{K}_I), d, d'\big)$, $\big(\Lambda[u_1,\ldots,u_m]\otimes\mathbb{Z}[\mathcal{K}], d, d'\big)$ and $\big(R^*(\mathcal{K}), d, d'\big)$ are bicomplexes, that is, d and d' satisfy $dd' = -d'd$.

By construction, $HH^*(\mathcal{Z}_\mathcal{K})$ is the first double cohomology of the bicomplex $\big(\bigoplus_{I\subset[m]} C^*(\mathcal{K}_I), d, d'\big)$:

$$HH^*(\mathcal{Z}_\mathcal{K}) = H\big(H\big(\bigoplus_{I\subset[m]} C^*(\mathcal{K}_I), d\big), d'\big).$$

Theorem

The bicomplexes $\left(\bigoplus_{I\subset[m]} C^*(\mathcal{K}_I), d, d'\right)$ *and* $\left(R^*(\mathcal{K}), d, d'\right)$ *are isomorphic. Therefore,* $HH^*(\mathcal{Z}_\mathcal{K})$ *is isomorphic to the first double cohomology of the bicomplex* $\left(\Lambda[u_1,\ldots,u_m]\otimes\mathbb{Z}[\mathcal{K}], d, d'\right)$*:*

$$HH^*(\mathcal{Z}_\mathcal{K}) \cong H\big(H\big(\Lambda[u_1,\ldots,u_m]\otimes\mathbb{Z}[\mathcal{K}], d\big), d'\big).$$

Proof (sketch).

Define a homomorphism

$$f\colon C^{q-1}(\mathcal{K}_I) \longrightarrow R^{q-|I|,2|I|}(\mathcal{K}),$$
$$\alpha_{L,I} \longmapsto \varepsilon(L,I)\, u_{I\setminus L} v_L,$$

where $\varepsilon(L,I) = \prod_{i\in L}\varepsilon(i,I) = (-1)^{\sum_{\ell\in L}\#\{i\in I\colon\, i<\ell\}}$.
Then f is an isomorphism of free abelian groups commuting with d and d'.
That is, have an isomorphism of bicomplexes

$$f\colon \Big(\bigoplus_{I\subset[m]} C^*(\mathcal{K}_I), d, d'\Big) \longrightarrow \big(R^*(\mathcal{K}), d, d'\big).$$

□

Corollary

The double cohomology $HH^*(\mathcal{Z}_\mathcal{K})$ is a graded commutative algebra, with the product induced from the cohomology product on $H^*(\mathcal{Z}_\mathcal{K})$.

Proposition

(a) *For any $\mathcal{K}$, the d'-cohomology of $\Lambda[u_1,\ldots,u_m]\otimes\mathbb{Z}[\mathcal{K}]$ is zero:*

$$H\bigl(\Lambda[u_1,\ldots,u_m]\otimes\mathbb{Z}[\mathcal{K}],d'\bigr)=0.$$

(b) *If $\mathcal{K}\neq\Delta^{m-1}$ (the full simplex on $[m]$), then the d'-cohomology of the bicomplexes $\bigoplus_{I\subset[m]}C^*(\mathcal{K}_I)$ and $R^*(\mathcal{K})$ is zero:*

$$H\Bigl(\bigoplus_{I\subset[m]}C^*(\mathcal{K}_I),d'\Bigr)=H\bigl(R^*(\mathcal{K}),d'\bigr)=0.$$

Therefore, the second double cohomology and the total cohomology of the bicomplexes $\bigl(\bigoplus_{I\subset[m]}C^*(\mathcal{K}_I),d,d'\bigr)$ and $\bigl(R^*(\mathcal{K}),d,d'\bigr)$ is zero unless $\mathcal{K}=\Delta^{m-1}$.

(c) *If $\mathcal{K}=\Delta^{m-1}$, then the only nonzero d'-cohomology group of $\bigoplus_{I\subset[m]}C^*(\mathcal{K}_I)$ and $R^*(\mathcal{K})$ is $H^{2m}\cong\mathbb{Z}$, represented by $\alpha_{[m],[m]}$ and $v_1\cdots v_m$, respectively.*

4. Relation to the torus action

Given a circle action $S^1 \times X \to X$ on a space X, the induced map in cohomology has the form

$$H^*(X) \to H^*(S^1 \times X) = \Lambda[u] \otimes H^*(X), \quad \alpha \mapsto 1 \otimes \alpha + u \otimes \iota(\alpha),$$

where $u \in H^1(S^1)$ is a generator and $\iota\colon H^*(X) \to H^{*-1}(X)$ is a derivation.

Proposition

The derivation corresponding to the i^{th} coordinate circle action $S^1_i \times \mathcal{Z}_\mathcal{K} \to \mathcal{Z}_\mathcal{K}$ is induced by the derivation ι_i of the Koszul complex $(\Lambda[u_1, \ldots, u_m] \otimes \mathbb{Z}[\mathcal{K}], d)$ given on the generators by

$$\iota_i(u_j) = \delta_{ij}, \quad \iota_i(v_j) = 0, \qquad \text{for } j = 1, \ldots, m,$$

where δ_{ij} is the Kronecker delta.

The derivation corresponding to the diagonal circle action $S^1_d \times \mathcal{Z}_\mathcal{K} \to \mathcal{Z}_\mathcal{K}$ coincides with the differential d'.

Summary of 3 definitions of $HH^*(\mathcal{Z}_\mathcal{K})$

The bigraded double cohomology $HH^*(\mathcal{Z}_\mathcal{K})$ can be defined as

- the cohomology of the cochain complex

$$CH^*(\mathcal{Z}_\mathcal{K}) := (H^*(\mathcal{Z}_\mathcal{K}), d'),$$

  where d' is defined on $H^*(\mathcal{Z}_\mathcal{K}) = \bigoplus_{I\subset[m]} \widetilde{H}^*(\mathcal{K}_I)$ via alternating the homomorphisms $H^p(\mathcal{K}_I) \to \widetilde{H}^p(\mathcal{K}_{I\setminus\{i\}})$ induced by $\mathcal{K}_{I\setminus\{i\}} \hookrightarrow \mathcal{K}_I$;

- the first double cohomology of the bicomplex

$$\Big(\Lambda[u_1, \ldots, u_m] \otimes \mathbb{Z}[\mathcal{K}], d, d'\Big)$$

  with $du_j = v_j$, $dv_j = 0$, $d'u_j = 1$, $d'v_j = 0$.

- the cohomology of $H^*(\mathcal{Z}_\mathcal{K})$ with respect to the derivation defined by the diagonal circle action $S^1_d \times \mathcal{Z}_\mathcal{K} \to \mathcal{Z}_\mathcal{K}$.

5. Techniques for computing $HH^*(\mathcal{Z}_\mathcal{K})$

Proposition

Let $\mathcal{K} = \partial\Delta^{m-1}$*, the boundary of an* $(m-1)$*-simplex. Then,*

$$HH^{-k,2\ell}(\mathcal{Z}_\mathcal{K}) = \begin{cases} \mathbb{Z} & \text{for } (-k, 2\ell) = (0,0),\ (-1, 2m); \\ 0 & \text{otherwise.} \end{cases}$$

Theorem

For two simplicial complexes $\mathcal{K}$ *and* $\mathcal{L}$*, if either* $H^*(\mathcal{Z}_\mathcal{K})$ *or* $H^*(\mathcal{Z}_\mathcal{L})$ *is free, then there is an isomorphism of chain complexes*

$$CH^*(\mathcal{Z}_{\mathcal{K}*\mathcal{L}}) \cong CH^*(\mathcal{Z}_\mathcal{K}) \otimes CH^*(\mathcal{Z}_\mathcal{L}).$$

In particular, we have $HH^*(\mathcal{Z}_{\mathcal{K}*\mathcal{L}}; k) \cong HH^*(\mathcal{Z}_\mathcal{K}; k) \otimes HH^*(\mathcal{Z}_\mathcal{L}; k)$ *with field coefficients.*

In the previous examples $HH^*(\mathcal{Z}_\mathcal{K})$ behaved like $H^*(\mathcal{Z}_\mathcal{K})$. Here is an example of a major difference.

Theorem

Let $\mathcal{K} = \mathcal{K}' \sqcup pt$ be the disjoint union of a nonempty simplicial complex $\mathcal{K}'$ and a point. Then,

$$HH^{-k,2\ell}(\mathcal{Z}_\mathcal{K}) = \begin{cases} \mathbb{Z} & \text{for } (-k, 2\ell) = (0,0),\ (-1,4); \\ 0 & \text{otherwise.} \end{cases}$$

More generally,

Theorem

Let $\mathcal{K} = \mathcal{K}' \cup_\sigma \Delta^n$ be a simplicial complex obtained from a nonempty simplicial complex $\mathcal{K}'$ by gluing an n-simplex along a proper, possibly empty, face $\sigma \in \mathcal{K}$. Then either $\mathcal{K}$ is a simplex, or

$$HH^{-k,2\ell}(\mathcal{Z}_\mathcal{K}) = \begin{cases} \mathbb{Z} & \text{for } (-k, 2\ell) = (0,0),\ (-1,4); \\ 0 & \text{otherwise.} \end{cases}$$

6. m-cycles and Poincaré duality

Let $\mathcal{Z}_\mathcal{L}$ be the moment-angle complex corresponding to an m-cycle $\mathcal{L}$. By a result of McGavran, $\mathcal{Z}_\mathcal{L}$ is homeomorphic to connected sum of sphere products:

$$\mathcal{Z}_\mathcal{L} \cong \mathop{\#}_{k=3}^{m-1} \left(S^k \times S^{m+2-k}\right)^{\#(k-2)\binom{m-2}{k-1}}.$$

Theorem

Let $\mathcal{L}$ be an m-cycle for $m \geq 5$. Then $HH^{-k,2\ell}(\mathcal{Z}_\mathcal{L})$ is $\mathbb{Z}$ in bidegrees $(-k, 2\ell) = (0,0)$, $(-1,4)$, $(-m+3, 2(m-2))$, $(-m+2, 2m)$, and is 0 otherwise.

Example

For $m = 5$, the (singly graded) Betti vector of $H^*(\mathcal{Z}_\mathcal{K})$ is (10055001), while for $HH^*(\mathcal{Z}_\mathcal{K})$ it is (10011001).

Theorem

Suppose $\mathcal{K}$ is a Gorenstein complex of dimension $n-1$ (in particular, a triangulated sphere). Then the double cohomology $HH^*(\mathcal{Z}_\mathcal{K})$ is a Poincaré duality algebra. In particular,*

$$\operatorname{rank} HH^{-k,2\ell}(\mathcal{Z}_\mathcal{K}) = \operatorname{rank} HH^{-(m-n)+k,2(m-\ell)}(\mathcal{Z}_\mathcal{K}).$$

The converse does not hold, unlike the situation with the ordinary cohomology $H^*(\mathcal{Z}_\mathcal{K})$. For example, if $\mathcal{K}$ is m disjoint points, then $HH^*(\mathcal{Z}_\mathcal{K})$ is a Poincaré algebra, but $\mathcal{K}$ is not Gorenstein if $m > 2$.

Question

Characterise simplicial complexes $\mathcal{K}$ for which $HH^*(\mathcal{Z}_\mathcal{K})$ is a Poincaré algebra.

7. Bigraded persistence and barcodes

$\mathbb{R}_{\geq 0}$ nonnegative real numbers, a poset category with respect $\leq$.
A persistence module is a (covariant) functor

$$\mathcal{M}\colon \mathbb{R}_{\geq 0} \to \mathrm{k}\text{-}\textsc{mod}$$

to the category of modules over a principal ideal domain k.

That is, a family of k-modules $\{M_s\}_{s\in\mathbb{R}_{\geq 0}}$ together with morphisms $\{\phi_{s_1,s_2}\colon M_{s_1} \to M_{s_2}\}_{s_1\leq s_2}$ such that $\phi_{s,s}$ is the identity on M_s and $\phi_{s_2,s_3}\circ\phi_{s_1,s_2} = \phi_{s_1,s_3}$ whenever $s_1 \leq s_2 \leq s_3$ in $\mathbb{R}_{\geq 0}$.

Example

Given an interval $I \subset \mathbb{R}_{\geq 0}$, define the interval module

$$\mathrm{k}(I)\colon \mathbb{R}_{\geq 0} \to \mathrm{k}\text{-}\mathrm{MOD}, \quad s \mapsto \mathrm{k}^I_s := \begin{cases} \mathrm{k} & \text{if } s \in I; \\ 0 & \text{otherwise.} \end{cases}$$

Theorem (interval decomposition)

Let $\mathcal{M} = \{M_s\}_{s \in \mathbb{R}_{\geq 0}}$ be a persistence module. If k is a field and all M_s are finite dimensional k-vector spaces, then

$$\mathcal{M} = \bigoplus_{I \in B(\mathcal{M})} \mathrm{k}(I)$$

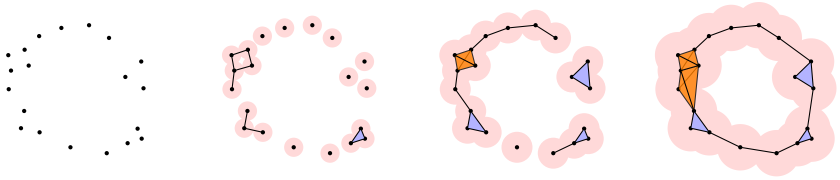
for some multiset $B(\mathcal{M})$ of intervals in $\mathbb{R}_{\geq 0}$.

The multiset of intervals $B(\mathcal{M})$ is called the barcode of $\mathcal{M}$.

(X, d_X) a finite pseudo-metric space (a point cloud).

The Vietoris–Rips filtration $\{R(X,t)\}_{t\geq 0}$ associated with (X, d_X) consists of the Vietoris–Rips simplicial complexes $R(X,t)$.

$R(X,t)$ is the clique complex of the graph whose vertex set is X and two vertices x and y are connected by an edge if $d_X(x,y) \leq t$.
Have a simplicial inclusion $R(X,t_1) \hookrightarrow R(X,t_2)$ whenever $t_1 \leq t_2$.



$$X = R(X,0) \hookrightarrow \cdots \hookrightarrow R(X,t_1) \hookrightarrow \cdots \hookrightarrow R(X,t_2) \hookrightarrow \cdots \hookrightarrow R(X,t_3) \hookrightarrow \cdots$$

Figure: A point cloud and the corresponding Vietoris–Rips filtration.

The n-dimensional persistent homology module

$$\mathcal{PH}_n(X)\colon \mathbb{R}_{\ge 0} \to \text{k-}\mathrm{MOD}, \quad t \mapsto \widetilde{H}_n(R(X,t)).$$

$B(X) = B(\mathcal{PH}(X))$ the barcode of $\mathcal{PH}(X) = \bigoplus_{n\geqslant 0} \mathcal{PH}_n(X)$.

A homology class $\alpha \in \widetilde{H}_n(R(X,t))$ is said to

(1) be born at r if
 (i) $\alpha \in \operatorname{im}\big(\widetilde{H}_n(R(X,r)) \to \widetilde{H}_n(R(X,t))\big)$;
 (ii) $\alpha \notin \operatorname{im}\big(\widetilde{H}_n(R(X,p)) \to \widetilde{H}_n(R(X,t))\big)$ for $p < r$,

(2) die at s if
 (i) $\alpha \in \ker\big(\widetilde{H}_n(R(X,t)) \to \widetilde{H}_n(R(X,s))\big)$;
 (ii) $\alpha \notin \ker\big(\widetilde{H}_n(R(X,t)) \to \widetilde{H}_n(R(X,q))\big)$ for $q < s$.

If $\alpha \in \widetilde{H}_n(R(X,t))$ is born at r and dies at s, then $[r,s)$ is the persistence interval of α. For $t \in \mathbb{R}_{\ge 0}$, the dimension of $\widetilde{H}_n(R(X,t))$ is the number of n-dimensional persistence intervals containing t.

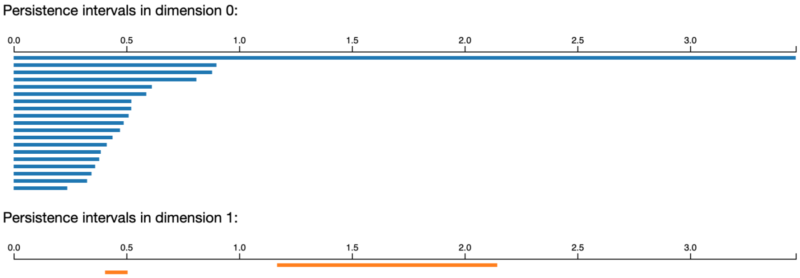


Figure: The barcode corresponding to the Vietoris–Rips complex.

Recall $\mathcal{Z}_{\mathcal{K}} = \bigcup_{I \in K} (D^2, S^1)^I \subset (D^2)^m$ the moment-angle complex.

$$H_p(\mathcal{Z}_K) = \bigoplus_{-i+2j=p} H_{-i,2j}(\mathcal{Z}_K) \cong \bigoplus_{J \subset [m]} \widetilde{H}_{p-|J|-1}(K_J).$$

Bigraded Betti numbers of K (with coefficients in k):

$$\beta_{-i,2j}(K) := \dim H_{-i,2j}(\mathcal{Z}_K) = \sum_{J \subset [m]\colon\, |J|=j} \dim \widetilde{H}_{j-i-1}(K_J).$$

For $j = m$, we get $\beta_{-i,2m}(K) = \dim \widetilde{H}_{m-i-1}(K)$.

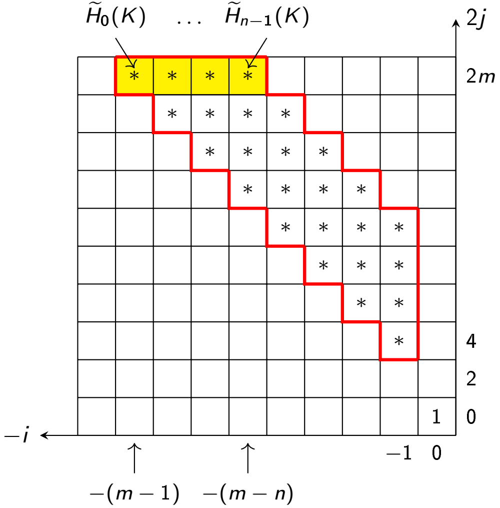


Figure: Bigraded Betti numbers of $(n-1)$-dimensional K with m vertices.

(X, d_X) a finite pseudo-metric space
$\{R(X,t)\}_{t\geq 0}$ its associated Vietoris–Rips filtration.

The bigraded persistent homology module of bidegree $(-i, 2j)$ as

$$\mathcal{PHZ}_{-i,2j}(X)\colon \mathbb{R}_{\geq 0} \to \mathrm{k}\text{-}\textsc{mod}, \quad t \mapsto H_{-i,2j}(\mathcal{Z}_{R(X,t)}).$$

The bigraded barcode $BB(X)$ is the collection of persistence intervals of generators of the bigraded homology groups $H_{-i,2j}(\mathcal{Z}_{R(X,t)})$.
For each $t \in \mathbb{R}_{\geq 0}$, the dimension of $H_{-i,2j}(\mathcal{Z}_{R(X,t)})$ is equal to the number of persistence intervals of bidegree $(-i, 2j)$ containing t.

The bigraded barcode of X is a diagram in 3-dimensional space.
It contains the original barcode of X in its top level.

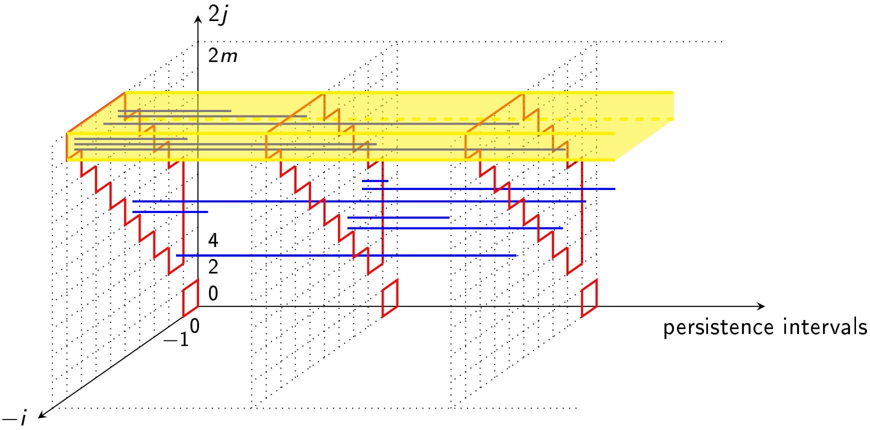


Figure: A bigraded barcode.

The bigraded persistent double homology module of bidegree $(-i,2j)$ is

$$\mathcal{PHHZ}_{-i,2j}(X)\colon \mathbb{R}_{\geq 0}\to \mathrm{k}\text{-}\mathrm{MOD},\quad t\mapsto HH_{-i,2j}(\mathcal{Z}_{R(X,t)}).$$

One can view the bigraded persistent homology module as a functor to differential bigraded k-modules,

$$\mathcal{PHZ}(X)\colon \mathbb{R}_{\geq 0}\to \mathrm{DG}(\mathrm{k}\text{-}\mathrm{MOD}),\quad t\mapsto (H_{*,*}(\mathcal{Z}_{R(X,t)}),\partial').$$

Then

$$\mathcal{PHHZ}(X)=\mathcal{H}\circ\mathcal{PHZ}(X),$$

where $\mathcal{H}\colon \mathrm{DG}(\mathrm{k}\text{-}\mathrm{MOD})\to \mathrm{k}\text{-}\mathrm{MOD}$ is the homology functor.
This is convenient for comparing the interleaving distances.

$\mathbb{BB}(X)$: the double barcode corresponding to the bigraded persistence module $\mathcal{PHHZ}(X)$.

8. Isometry and stability

The stability theorem asserts that the persistent homology barcodes are stable under perturbations of the data sets in the Gromov–Hausdorff metric. It is a key result justifying the use of persistent homology in data science.

Theorem (stability theorem)

Let (X, d_X) and (Y, d_Y) be two finite pseudo-metric spaces, and let $B(X)$ and $B(Y)$ be the barcodes corresponding to the persistence modules $\mathcal{PH}(X)$ and $\mathcal{PH}(Y)$. Then,

$$W_\infty(B(X), B(Y)) \leqslant 2\, d_{GH}(X, Y).$$

The Hausdorff distance between two nonempty subsets A and B in a finite pseudo-metric space (Z, d) is

$$d_H(A, B) := \max\Big\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\Big\}.$$

The Gromov–Hausdorff distance between two finite pseudo-metric spaces (X, d_X) and (Y, d_Y) is

$$d_{GH}(X, Y) := \inf_{Z,f,g} d_H(f(X), g(Y)),$$

where the infimum is taken over all isometric embeddings $f\colon X \to Z$ and $g\colon Y \to Z$ into a pseudo-metric space Z. Equivalently,

$$d_{GH}(X, Y) = \frac{1}{2}\min_{C} \max_{(x_1,y_1),(x_2,y_2)\in C} |d_X(x_1, x_2) - d_Y(y_1, y_2)|,$$

where the minimum is taken over all correspondences between X and Y.

Let B and B' be finite multisets of intervals of the form $[a, b)$.
Define the multiset $\overline{B} = B \cup \varnothing^{|B'|}$, obtained by adding to B the multiset containing the empty interval $\varnothing$ with cardinality $|B'|$.
Similarly, define $\overline{B'} = B' \cup \varnothing^{|B|}$. Now $\overline{B}$ and $\overline{B'}$ have the same cardinality

The distance function $\pi\colon \overline{B} \times \overline{B'} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is given by

$$\pi\left([a,b),[a',b')\right) = \max\{|a'-a|, |b'-b|\}, \quad \pi([a,\infty),[a',\infty)) = |a'-a|,$$
$$\pi([a,b),\varnothing) = \frac{b-a}{2}, \quad \pi(\varnothing,[a',b')) = \frac{b'-a'}{2}, \quad \pi(\varnothing,\varnothing) = 0,$$
$$\pi([a,\infty),[a',b')) = \pi([a,b),[a',\infty)) = \pi([a,\infty),\varnothing) = \pi(\varnothing,[a',\infty)) = \infty$$

Denote by $\mathcal{D}\left(\overline{B}, \overline{B'}\right)$ the set of all bijections $\theta\colon \overline{B} \to \overline{B'}$.
Then the ∞-Wasserstein distance, or the bottleneck distance, is

$$W_\infty(B, B') = \min_{\theta \in \mathcal{D}(\overline{B}, \overline{B'})} \max_{I \in \overline{B}} \pi(I, \theta(I)).$$

Bigraded persistent homology does not satisfy the stability property, but bigraded persistent *double* homology does:

Theorem (Bahri–Limonchenko-P-Song-Stanley)

Let $\mathbb{BB}(X)$ and $\mathbb{BB}(Y)$ be the bigraded barcodes corresponding to the persistence modules $\mathcal{PHHZ}(X)$ and $\mathcal{PHHZ}(Y)$, respectively. Then, we have

$$W_\infty(\mathbb{BB}(X), \mathbb{BB}(Y)) \leq 2d_{GH}(X, Y).$$

References

[1] Ivan Limonchenko, Taras Panov, Jongbaek Song and Donald Stanley. *Double cohomology of moment-angle complexes.* Advances in Math. 432 (2023), Paper no. 109274, 34 pp.

[2] Anthony Bahri, Ivan Limonchenko, Taras Panov, Jongbaek Song and Donald Stanley. *A stability theorem for bigraded persistence barcodes.* Preprint (2023); arXiv:2303.14694.