Complex geometry of moment-angle manifolds

joint with Yuri Ustinovsky, Misha Verbitsky, Hiroaki Ishida and Roman Krutowski

Taras Panov

Moscow State University & NRU Higher School of Economics

Topology Seminar of AMSS, Beijing

15 November 2023

The moment-angle complex

$\mathcal{K}$ an abstract simplicial complex on the set $[m] = \{1, 2, \ldots, m\}$
$I = \{i_1, \ldots, i_k\} \in \mathcal{K}$ a simplex; always assume $\varnothing \in \mathcal{K}$.

Consider the m-dimensional unit polydisc:

$$\mathbb{D}^m = \{(z_1, ..., z_m) \in \mathbb{C}^m : |z_i|^2 \leqslant 1 \text{ for } i = 1, ..., m\}.$$

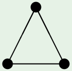
The moment-angle complex is

$$\mathcal{Z}_{\mathcal{K}} := \bigcup_{I \in \mathcal{K}} \Big(\prod_{i \in I} \mathbb{D} \times \prod_{i \notin I} \mathbb{S} \Big) \subset \mathbb{D}^m,$$

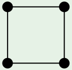
where $\mathbb{S}$ is the boundary of the unit disk $\mathbb{D}$.

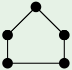
$\mathcal{Z}_{\mathcal{K}}$ has a natural action of the torus T^m.
When $\mathcal{K}$ is a simplicial subdivision of sphere (e.g., the boundary of a simplicial polytope), $\mathcal{Z}_{\mathcal{K}}$ is a topological manifold, called the moment-angle manifold.

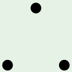
Example

1. Let $\mathcal{K} =$  (the boundary of a triangle). Then

$$\mathcal{Z}_\mathcal{K} = (\mathbb{D} \times \mathbb{D} \times \mathbb{S}) \cup (\mathbb{D} \times \mathbb{S} \times \mathbb{D}) \cup (\mathbb{S} \times \mathbb{D} \times \mathbb{D}) = \partial(\mathbb{D}^3) \cong S^5.$$

2. Let $\mathcal{K} =$  (the boundary of a square). Then $\mathcal{Z}_\mathcal{K} \cong S^3 \times S^3$.

3. Let $\mathcal{K} =$  Then $\mathcal{Z}_\mathcal{K} \cong (S^3 \times S^4) \# \cdots \# (S^3 \times S^4)$ (5 times).

4. Let $\mathcal{K} =$  (three disjoint points). Then

$$\mathcal{Z}_\mathcal{K} = (\mathbb{D} \times \mathbb{S} \times \mathbb{S}) \cup (\mathbb{S} \times \mathbb{D} \times \mathbb{S}) \cup (\mathbb{S} \times \mathbb{S} \times \mathbb{D}) \simeq S^3 \vee S^3 \vee S^3 \vee S^4 \vee S^4$$

(not a manifold).

We define an open submanifold $U(\mathcal{K}) \subset \mathbb{C}^m$ in a similar way:

$$U(\mathcal{K}) := \bigcup_{I \in \mathcal{K}} \Big(\prod_{i \in I} \mathbb{C} \times \prod_{i \notin I} \mathbb{C}^\times \Big), \qquad \mathbb{C}^\times = \mathbb{C} \setminus \{0\}.$$

$U(\mathcal{K})$ is a toric variety with the corresponding fan given by

$$\Sigma_{\mathcal{K}} = \{\mathbb{R}_\geqslant \langle e_i \colon i \in I \rangle \colon I \in \mathcal{K}\},$$

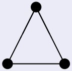
where e_i denotes the i-th standard basis vector of $\mathbb{R}^m$.

Theorem

(a) $U(\mathcal{K}) = \mathbb{C}^m \setminus \bigcup_{\{i_1,\ldots,i_k\} \notin \mathcal{K}} \{z_{i_1} = \cdots = z_{i_k} = 0\}$

(the complement to an arrangement of coordinate subspaces);

(b) *There is a T^m-equivariant deformation retraction $U(\mathcal{K}) \xrightarrow{\simeq} \mathcal{Z}_{\mathcal{K}}$.*

E.g., $\mathcal{K} =$  Then $U(\mathcal{K}) = \mathbb{C}^3 \setminus \{z_1 = z_2 = z_3 = 0\} \xrightarrow{\simeq} S^5 = \mathcal{Z}_{\mathcal{K}}$.

Simplicial fans, complex-analytic structures

Suppose $\mathcal{K}$ is the underlying complex of a complete simplicial (not necessarily rational) fan Σ in an n-dimensional space V.

Then the deformation retraction $U(\mathcal{K}) \to \mathcal{Z}_{\mathcal{K}}$ can be realised as the projection onto the orbit space of a smooth free and proper action of a non-compact subgroup $R \subset (\mathbb{C}^{\times})^m$ isomorphic to $\mathbb{R}^{m-n}$, as described next.

Choose generators $a_1, \dots, a_m$ of the one-dimensional cones of Σ (a marked fan). Consider the linear projection

$$q\colon \mathbb{R}^m \to V, \quad e_i \mapsto a_i.$$

Set
$$\mathfrak{r} = \operatorname{Ker} q,$$

$$R = \exp(\mathfrak{r}) = \{e^r\colon r \in \mathfrak{r}\} \subset (\mathbb{R}^\times)^m, \quad H' = \exp(i\mathfrak{r}) \subset T^m.$$

The subgrp $H' \subset T^m$ is *not* closed unless $\mathfrak{r} \subset \mathbb{R}^m$ is a rational subspace.

Theorem

The action of R on $U(\mathcal{K})$ free and proper, and the quotient $U(\mathcal{K})/R$ is T^m-equivariantly homeomorphic to the moment-angle manifold $\mathcal{Z}_\mathcal{K}$.

The subgroup $H' \subset T^m$ acts on $\mathcal{Z}_\mathcal{K} = U(\mathcal{K})/R$ by restriction.
The H'-action on $\mathcal{Z}_\mathcal{K}$ is almost free (finite stabilisers).
We therefore obtain a smooth foliation of $\mathcal{Z}_\mathcal{K}$ by the orbits of H'.

Assume that $\dim \mathcal{Z}_\mathcal{K} = m + n$ is even (otherwise add ghost vertices to $\mathcal{K}$). A T^m-invariant complex structure on $\mathcal{Z}_\mathcal{K}$ is defined by two pieces of data:

- a marked complete simplicial fan $\Sigma = \{\mathcal{K}; a_1, \ldots, a_m\}$ in V with underlying simplicial complex $\mathcal{K}$ and generators $a_1, \ldots, a_m$;
- a choice of a complex structure on the kernel of $q\colon \mathbb{R}^m \to V$, $e_i \mapsto a_i$.

A choice of a complex structure on $\operatorname{Ker} q$ is equivalent to a choice of an $\frac{m-n}{2}$-dimensional complex subspace $\mathfrak{h} \subset \mathbb{C}^m$ satisfying the two conditions:

(a) the composite $\mathfrak{h} \hookrightarrow \mathbb{C}^m \xrightarrow{\mathrm{Re}} \mathbb{R}^m$ is injective;

(b) the composite $\mathfrak{h} \hookrightarrow \mathbb{C}^m \xrightarrow{\mathrm{Re}} \mathbb{R}^m \xrightarrow{q} V$ is zero.

Consider the $\frac{m-n}{2}$-dimensional complex-analytic subgroup

$$H = \exp(\mathfrak{h}) \subset (\mathbb{C}^\times)^m.$$

It acts on $U(\mathcal{K})$ holomorphically.

Theorem

Let Σ be a marked complete simplicial fan in $V \cong \mathbb{R}^n$ with m one-dimensional cones, and let $\mathcal{K} = \mathcal{K}_\Sigma$ be its underlying simplicial complex. Assume that $m - n = 2\ell$. Then

(a) *the holomorphic action of the group $H \cong \mathbb{C}^\ell$ on $U(\mathcal{K})$ is free and proper, so the quotient $U(\mathcal{K})/H$ is a compact complex manifold;*

(b) *there is a T^m-equivariant diffeomorphism $U(\mathcal{K})/H \cong \mathcal{Z}_\mathcal{K}$ defining a complex structure on $\mathcal{Z}_\mathcal{K}$ in which T^m acts by holomorphic transformations.*

Example (holomorphic tori)

Let $\mathcal{K}$ be empty on 2 elements (that is, $\mathcal{K}$ has two ghost vertices). We therefore have $n = 0$, $m = 2$, $\ell = 1$, and $q\colon \mathbb{R}^2 \to 0$ is a zero map.

A 1-dim complex subspace $\mathfrak{h} \hookrightarrow \mathbb{C}^2$ is given by $z \mapsto (\gamma_1 z, \gamma_2 z)$ for some $\gamma_1, \gamma_2 \in \mathbb{C}$, so that

$$H = \{(e^{\gamma_1 z}, e^{\gamma_2 z})\} \subset (\mathbb{C}^\times)^2.$$

Condition (b) above is void, while (a) is equivalent to that γ_1, γ_2 are linerly independent over $\mathbb{R}$. This implies that $\exp \mathfrak{h} = H \to (\mathbb{C}^\times)^2$ is an inclusion of a closed subgroup, and the quotient $(\mathbb{C}^\times)^2/H$ is a complex torus T^2:

$$(\mathbb{C}^\times)^2/H \cong \mathbb{C}/(\gamma_1\mathbb{Z} \oplus \gamma_2\mathbb{Z}) \cong T^2.$$

Similarly, if $\mathcal{K}$ is empty on 2ℓ elements (so that $n = 0$, $m = 2\ell$), we can obtain any complex torus $T^{2\ell}$ as the quotient $(\mathbb{C}^\times)^{2\ell}/H$.

Conversely, suppose $\mathcal{Z}_\mathcal{K}$ admits a T^m-invariant complex structure. Then the T^m-action extends to a holomorphic action of $(\mathbb{C}^\times)^m$ on $\mathcal{Z}_\mathcal{K}$. Have a complex-analytic subgroup of global stabilisers

$$H = \{g \in (\mathbb{C}^\times)^m \colon g \cdot x = x \text{ for all } x \in \mathcal{Z}_\mathcal{K}\}.$$

$\mathfrak{h} = \mathrm{Lie}(H)$ is a complex subalgebra of $\mathrm{Lie}(\mathbb{C}^\times)^m = \mathbb{C}^m$ and satisfies

(a) the composite $\mathfrak{h} \hookrightarrow \mathbb{C}^m \xrightarrow{\mathrm{Re}} \mathbb{R}^m$ is injective;

(b) the quotient map $q \colon \mathbb{R}^m \to \mathbb{R}^m / \mathrm{Re}(\mathfrak{h})$ sends the fan $\Sigma_\mathcal{K}$ to a complete fan $q(\Sigma_\mathcal{K})$ in $\mathbb{R}^m / \mathrm{Re}(\mathfrak{h})$.

Theorem (Ishida)

Every complex moment-angle manifold $\mathcal{Z}_\mathcal{K}$ is T^m-equivariantly biholomorphic to the quotient manifold $U(\mathcal{K})/H$.

Thus, $\mathcal{Z}_\mathcal{K}$ admits a complex structure if and only if $\mathcal{K}$ is the underlying complex of a complete simplicial fan (i. e., a star-shaped sphere).

Canonical holomorphic foliation on $\mathcal{Z}_\mathcal{K}$

Recall $q\colon \mathbb{R}^m \to V$, $e_i \mapsto a_i$, $\qquad \mathfrak{r} = \operatorname{Ker} q$,

$$R = \exp(\mathfrak{r}) = \{e^r\colon r \in \mathfrak{r}\} \subset (\mathbb{R}^\times)^m, \quad H' = \exp(i\mathfrak{r}) \subset T^m.$$

Consider the complexification $\mathfrak{r}_\mathbb{C} = \operatorname{Ker}(q_\mathbb{C}\colon \mathbb{C}^m \to V_\mathbb{C})$ and

$$R_\mathbb{C} = \exp(\mathfrak{r}_\mathbb{C}) \subset (\mathbb{C}^\times)^m, \quad R_\mathbb{C}/H \cong H'.$$

Holomorphic foliation $\mathcal{F}$ on $\mathcal{Z}_\mathcal{K} = U(\mathcal{K})/H$ by the orbits of $R_\mathbb{C}/H \cong H'$.

If the subspace $\mathfrak{r} \subset \mathbb{R}^m$ is rational, then $R_\mathbb{C} \subset (\mathbb{C}^\times)^m$ is closed (and algebraic), and the complete simplicial fan $\Sigma := q(\Sigma_\mathcal{K})$ is rational.
The rational fan Σ defines a toric variety

$$X_\Sigma = \mathcal{Z}_\mathcal{K}/H' = U(\mathcal{K})/R_\mathbb{C}.$$

The canonical holomorphic foliation becomes a holomorphic Seifert fibration over the toric orbifold X_Σ with fibres complex tori $R_\mathbb{C}/H \cong T^{m-n}$.

The rational case:

$$\mathbb{C}^m \supset U(\mathcal{K})$$

$U(\mathcal{K}) \xrightarrow{H \cong \mathbb{C}^\ell} \mathcal{Z}_\mathcal{K}$ moment-angle manifold

$U(\mathcal{K}) \xrightarrow{R_\mathbb{C} \cong (\mathbb{C}^\times)^{2\ell}} X_\Sigma$

$\mathcal{Z}_\mathcal{K} \xrightarrow{R_\mathbb{C}/H \cong T^{2\ell}} X_\Sigma$

toric variety X_Σ

The non-rational case:

Have $U(\mathcal{K}) \xrightarrow{H} \mathcal{Z}_\mathcal{K}$,

and a holomorphic foliation $\mathcal{F}$ of $\mathcal{Z}_\mathcal{K}$ by the orbits of $R_\mathbb{C}/H = H' \subset T^m$.

The holomorphic foliated manifold $(\mathcal{Z}_\mathcal{K}, \mathcal{F})$ is a model for an 'irrational toric variety'.

De Rham and Dolbeault cohomology

The face ring (the Stanley–Reisner ring) of $\mathcal{K}$ is

$$\mathbb{C}[\mathcal{K}] := \mathbb{C}[v_1, \ldots, v_m]/I_{\mathcal{K}} = \mathbb{C}[v_1, \ldots, v_m]/(v_{i_1} \cdots v_{i_k} \colon \{i_1, \ldots, i_k\} \notin \mathcal{K}),$$

where $\mathbb{C}[v_1, \ldots, v_m]$ is the polynomial algebra, $\deg v_i = 2$, and $I_{\mathcal{K}}$ is the Stanley–Reisner ideal.

Proposition

The T^m-equivariant cohomology is given by

$$H^*_{T^m}(\mathcal{Z}_{\mathcal{K}}) = H^*_{T^m}(U(\mathcal{K})) \cong \mathbb{C}[\mathcal{K}].$$

The toric variety X_Σ is Kähler (equivalently, projective) if and only if Σ is the normal fan of a nonsingular (Delzant) polytope P.

Theorem (Danilov)

The Dolbeault cohomology of complete nonsingular X_Σ is given by

$$H^{*,*}_{\bar\partial}(X_\Sigma) \cong \mathbb{C}[v_1, ..., v_m]/(I_\mathcal{K} + J_\Sigma),$$

where $v_i \in H^{1,1}_{\bar\partial}(X_\Sigma)$, $I_\mathcal{K}$ is the Stanley–Reisner ideal, J_Σ is the ideal generated by the linear forms $\sum_{k=1}^m \langle a_k, u\rangle v_k$, $a_k = q(e_k)$ are the generators of 1-dim cones of Σ, $u \in V^*$.

The nonzero Hodge numbers are given by $h^{p,p}(X_\Sigma) = h_p$, where $h(\Sigma) = (h_0, h_1, \ldots, h_n)$ is the h-vector of Σ.

Theorem (Buchstaber-P.)

The de Rham cohomology ring of $\mathcal{Z}_{\mathcal{K}}$ is given by

$$\begin{aligned} H^*(\mathcal{Z}_{\mathcal{K}}) &\cong \mathrm{Tor}_{\mathbb{C}[v_1,\ldots,v_m]}(\mathbb{C}[\mathcal{K}],\mathbb{C}) \\ &\cong H\bigl(\Lambda[u_1,\ldots,u_m]\otimes\mathbb{C}[\mathcal{K}],d\bigr) \qquad du_i=v_i,\ dv_i=0 \\ &\cong H\bigl(\Lambda[t_1,\ldots,t_{m-n}]\otimes H^*(X_\Sigma),d\bigr) \qquad \Lambda[t_1,\ldots,t_{m-n}]=H^*(H') \\ &\cong \bigoplus_{I\subset[m]}\widetilde{H}^{*-|I|-1}(\mathcal{K}_I). \end{aligned}$$

Theorem (P.-Ustinovsky)

Let Σ be a rational fan, $\mathcal{Z}_{\mathcal{K}} \xrightarrow{T^{2\ell}} X_\Sigma$ a holomorphic torus fibration. Then the Dolbeault cohomology ring of $\mathcal{Z}_{\mathcal{K}}$ is given by

$$H^{*,*}_{\bar\partial}(\mathcal{Z}_{\mathcal{K}}) \cong H\bigl(\Lambda[\xi_1,\dots,\xi_\ell,\eta_1,\dots,\eta_\ell] \otimes H^{*,*}_{\bar\partial}(X_\Sigma), d\bigr),$$

where $\Lambda[\xi_1,\dots,\xi_\ell,\eta_1,\dots,\eta_\ell] = H^{*,*}_{\bar\partial}(T^{2\ell})$, $\xi_j \in H^{1,0}_{\bar\partial}(T^{2\ell})$, $\eta_j \in H^{0,1}_{\bar\partial}(T^{2\ell})$,
$dv_j = d\eta_j = 0$, $d\xi_j = c(\xi_j)$,
$c\colon H^{1,0}_{\bar\partial}(T^{2\ell}) \to H^{1,1}_{\bar\partial}(X_\Sigma)$ *is the first Chern class map.*

Corollary

(a) *The Borel spectral sequence of the holomorphic fibration* $\mathcal{Z}_{\mathcal{K}} \xrightarrow{T^{2\ell}} X_\Sigma$ *(converging to Dolbeault cohomology of* $\mathcal{Z}_{\mathcal{K}}$*) collapses at the* E_3 *page;*

(b) *The Frölicher spectral sequence (with* $E_1 = H^{*,*}_{\bar\partial}(\mathcal{Z}_{\mathcal{K}})$*, converging to* $H^*(\mathcal{Z}_{\mathcal{K}})$*) collapses at* E_2*.*

Basic cohomology

M a manifold with an action of a connected Lie group G, $\mathfrak{g} = \mathrm{Lie}\, G$.

$$\Omega(M)_{\mathrm{bas},\, G} = \{\omega \in \Omega(M) \colon \iota_\xi \omega = L_\xi \omega = 0 \text{ for any } \xi \in \mathfrak{g}\},$$

$H^*_{\mathrm{bas},\, G}(M) = H(\Omega(M)_{\mathrm{bas},\, G}, d)$ the basic cohomology of M.

$S(\mathfrak{g}^*)$ the symmetric algebra on $\mathfrak{g}^*$ with generators of degree 2.
The Cartan model is

$$\mathcal{C}_{\mathfrak{g}}(\Omega(M)) = ((S(\mathfrak{g}^*) \otimes \Omega(M))^{\mathfrak{g}}, d_{\mathfrak{g}}),$$

where $(S(\mathfrak{g}^*) \otimes \Omega(M))^{\mathfrak{g}}$ denotes the $\mathfrak{g}$-invariant subalgebra.

An element $\omega \in \mathcal{C}_{\mathfrak{g}}(\Omega(M))$ is a "$\mathfrak{g}$-equivariant polynomial map from $\mathfrak{g}$ to $\Omega(M)$". The differential $d_{\mathfrak{g}}$ is given by

$$d_{\mathfrak{g}}(\omega)(\xi) = d(\omega(\xi)) - \iota_\xi(\omega(\xi)).$$

Theorem

$$H^*_{\mathrm{bas},\,G}(M) \cong H\big(\mathcal{C}_{\mathfrak{g}}(\Omega(M)), d_{\mathfrak{g}}\big).$$

If in addition G is a compact, then

$$H^*_{\mathrm{bas},\,G}(M) \cong H^*_G(M) = H^*(EG \times_G M) \quad \text{the equivariant cohomology.}$$

Now consider $\mathcal{Z}_\mathcal{K}$ with the action of H' (a holomorphic foliation $\mathcal{F}$).

Theorem (Ishida–Krutowski–P.)

There is an isomorphism of algebras:

$$H^*_{\mathrm{bas},\,H'}(\mathcal{Z}_\mathcal{K}) \cong \mathbb{C}[v_1, \ldots, v_m]/(I_\mathcal{K} + J_\Sigma),$$

where $I_\mathcal{K}$ is the Stanley–Reisner ideal of $\mathcal{K}$, generated by the monomials

$$v_{i_1} \cdots v_{i_k} \quad \textit{with } \{i_1, \ldots, i_k\} \notin \mathcal{K},$$

and J_Σ is the ideal generated by the linear forms

$$\sum_{i=1}^{m} \langle a_i, u \rangle v_i \quad \textit{with } u \in V^*.$$

This settles a conjecture by [Battaglia and Zaffran] (arXiv:1108.1637).

If H' is a compact torus (the fan Σ is rational), then we get

$$H^*_{\mathrm{bas},\,H'}(\mathcal{Z}_\mathcal{K}) = H^*(\mathcal{Z}_\mathcal{K}/H') = H^*(X_\Sigma)$$

and we recover well-known description of the cohomology of toric manifolds, due to [Danilov and Jurkiewicz].

The proof of the theorem is based on the following formality result. Let $\mathfrak{t} = \mathrm{Lie}(T^m) \cong \mathbb{R}^m$ and consider the Cartan model

$$\mathcal{C}_\mathfrak{t}(\Omega(\mathcal{Z}_\mathcal{K})) = \big((S(\mathfrak{t}^*) \otimes \Omega(\mathcal{Z}_\mathcal{K}))^{T^m}, d_\mathfrak{t}\big).$$

Since T^m is compact, we get

$$H(\mathcal{C}_\mathfrak{t}(\Omega(\mathcal{Z}_\mathcal{K}))) = H^*_{T^m}(\mathcal{Z}_\mathcal{K}) = \mathbb{C}[v_1, \dots, v_m]/I_\mathcal{K}.$$

Lemma

The DGA $\mathcal{C}_\mathfrak{t}(\Omega(\mathcal{Z}_\mathcal{K}))$ is formal. Furthermore, there is a zigzag of quasi-isomorphisms of DGAs between $\mathcal{C}_\mathfrak{t}(\Omega(\mathcal{Z}_\mathcal{K}))$ and $H_{T^m}(\mathcal{Z}_\mathcal{K})$ which respect the $S(\mathfrak{t}^*)$-module structure.

Generalisation: maximal torus actions

M a connected complex manifold with an effective action of a compact torus T by holomorphic transformations.

The T-action on M is maximal if there is $x \in M$ such that

$$\dim T + \dim T_x = \dim M.$$

If the T-action is maximal, then T is a maximal compact torus in the group of diffeomorphisms on M.

Examples of maximal torus actions include the half-dimensional torus action on a smooth toric variety and the T^m-action on a complex moment-angle manifold $\mathcal{Z}_{\mathcal{K}}$.

Let $\mathfrak{t} = \mathrm{Lie}\, T$ and $\exp_T\colon \mathfrak{t} \to T$ the exponential map.
Let $\mathfrak{t}^{\mathbb{C}} = \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{t} \oplus i\mathfrak{t}$ and $p\colon \mathfrak{t}^{\mathbb{C}} \to \mathfrak{t}$ the first projection.

To a maximal torus action (M, T) one assigns the fan data $(\Sigma, \mathfrak{h})$, where

- Σ is a nonsingular fan in $\mathfrak{t}$ with respect to the lattice $\mathrm{Ker}\exp_T$;
- $\mathfrak{h} \subset \mathfrak{t}^{\mathbb{C}}$ is a complex subspace such that $p|_{\mathfrak{h}}\colon \mathfrak{h} \to \mathfrak{t}$ is injective; we denote by $q\colon \mathfrak{t} \to \mathfrak{t}/p(\mathfrak{h})$ the quotient projection;
- $\widetilde{\Sigma} := q(\Sigma) = \{q(\sigma) \subset \mathfrak{t}/p(\mathfrak{h})\colon \sigma \in \Sigma\}$ is a complete fan.

The category of holomorphic maximal torus actions (M, T) is equivalent to the category of pairs $(\Sigma, \mathfrak{h})$ with appropriate morphisms [Ishida].

To recover the maximal torus action from $(\Sigma, \mathfrak{h})$ one takes $M := X_\Sigma / H$, where X_Σ is the toric variety associated with Σ and H is the subgroup of the algebraic torus $T^{\mathbb{C}}$ corresponding to $\mathfrak{h} \subset \mathfrak{t}^{\mathbb{C}}$.

In particular, if Σ is a subfan of the standard fan in $\mathfrak{t} = \mathbb{R}^m$ defining $\mathbb{C}^m$, then X_Σ / H is T-equivariantly homeomorphic to the moment-angle manifold $\mathcal{Z}_{\mathcal{K}}$, where $\mathcal{K}$ is the underlying simplicial complex of Σ.

Transverse equivalence

Given a maximal torus action (M, T) with the fan data $(\Sigma, \mathfrak{h})$ and $p\colon \mathfrak{t}^{\mathbb{C}} \to \mathfrak{t}$, let $\mathfrak{h}' := p(\mathfrak{h}) \subset \mathfrak{t}$ and consider the corresponding Lie subgroup $H' \subset T$. The action of H' on M is almost free.
Get the canonical foliation $\mathcal{F}_M$ of M by H'-orbits.

Two smooth (or complex) foliated manifolds $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ are transversely equivalent if there exist a foliated manifold $(M_0, \mathcal{F}_0)$ and surjective submersions $f_i\colon M_0 \to M_i$ for $i = 1, 2$ such that

- $f_i^{-1}(x_i)$ is connected for all $x_i \in M_i$, and
- the preimage under f_i of every leaf of $\mathcal{F}_i$ is a leaf of $\mathcal{F}_0$

Proposition

If foliated manifolds $(M_1, \mathcal{F}_1)$, $(M_2, \mathcal{F}_2)$ are transversely equivalent, then there is a DGA isomorphism $\Omega^*_{\mathrm{bas}}(M_1) \cong \Omega^*_{\mathrm{bas}}(M_2)$.

Lemma

Every complex manifold M with a maximal torus action and canonical holomorphic foliation $\mathcal{F}_M$ is transversely equivalent to a complex moment-angle manifold $\mathcal{Z}_{\mathcal{K}}$.

Theorem (Ishida–Krutowski–P.)

The basic cohomology of a maximal torus action (M, T) with the fan data $(\Sigma, \mathfrak{h})$ and the canonical foliation $\mathcal{F}_M$ is given by

$$H^*_{\mathrm{bas}}(M) \cong \mathbb{C}[v_1, ..., v_m]/(I_{\mathcal{K}} + J_\Sigma),$$

where $I_{\mathcal{K}}$ is the Stanley–Reisner ideal of the complete fan $\widetilde{\Sigma} = q(\Sigma)$, and J_Σ is the ideal generated by the linear forms

$$\sum_{i=1}^{m} \langle a_i, u \rangle v_i \quad \text{with } u \in V^*.$$

Here $V = \mathfrak{t}/\mathfrak{h}'$ and $a_i = q(e_i)$, where e_i is the primitive generator of the ith cone of Σ.

References

[1] Hiroaki Ishida, Roman Krutowski and Taras Panov. *Basic cohomology of canonical holomorphic foliations on complex moment-angle manifolds.* Internat. Math. Research Notices 2022 (2022), no. 7, 5541–5563.

[2] Roman Krutowski and Taras Panov. *Dolbeault cohomology of complex manifolds with torus action.* In "Topology, Geometry, and Dynamics: Rokhlin Memorial". Contemp. Math., vol. 772; American Mathematical Society, Providence, RI, 2021, pp. 173–187.