

# Holomorphic foliations on complex moment-angle manifolds

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# Symplectic reduction and moment-angle manifolds

An  $m$ -torus  $T^m$  acts on  $\mathbb{C}^m$  coordinatewise. This is a Hamiltonian torus action with respect to  $\omega = i \sum_{k=1}^m dz_k \wedge d\bar{z}_k$ , with the moment map

$$\mu: \mathbb{C}^m \rightarrow \mathbb{R}^m = \text{Lie}(T^m)^*, \quad (z_1, \dots, z_m) \mapsto (|z_1|^2, \dots, |z_m|^2).$$

A **Hamiltonian toric manifold**  $M^{2n}$  is the symplectic quotient  $\mathbb{C}^m // K$  by an  $(m - n)$ -dimensional subtorus  $K \subset T^m$ . It has a residual Hamiltonian action of  $T^m/K \cong T^n$ .

In more detail, the moment map for the  $K$ -action on  $\mathbb{C}^m$  is the composite

$$\mu_K: \mathbb{C}^m \xrightarrow{\mu} \mathbb{R}^m \rightarrow \mathfrak{k}^*.$$

Take a regular value  $\delta \in \mathfrak{k}^* \cong \mathbb{R}^{m-n}$ . Then  $M^{2n} = \mu_K^{-1}(\delta)/K$ . It has a symplectic form  $\omega'$  satisfying  $p^*\omega' = i^*\omega$ , where  $p: \mu_K^{-1}(\delta) \rightarrow M^{2n}$  and  $i: \mu_K^{-1}(\delta) \hookrightarrow \mathbb{C}^m$ .

We refer to  $\mathcal{Z} := \mu_K^{-1}(\delta)$  as a (polytopal) **moment-angle manifold**.

It can be written as an intersection of  $(m - n)$  Hermitian quadrics in  $\mathbb{C}^m$ :

$$\mathcal{Z} = \left\{ (z_1, \dots, z_m) \in \mathbb{C}^m : \sum_{k=1}^m \gamma_{jk} |z_k|^2 = \delta_j, \quad j = 1, \dots, m - n \right\}.$$

The quotient  $\mathcal{Z}/T^m = M^{2n}/T^n$  is a convex polytope in  $\text{Lie}(T^n)^* \subset \mathbb{R}^m$  (the **moment polytope**) given by

$$P = \left\{ (y_1, \dots, y_m) \in \mathbb{R}_{\geq 0}^m : \sum_{k=1}^m \gamma_{jk} y_k = \delta_j, \quad j = 1, \dots, m - n \right\}.$$

Its facet normals  $\mathbf{a}_1, \dots, \mathbf{a}_m$  form the **Gale dual** configuration to  $\gamma_1, \dots, \gamma_m \in \mathfrak{k}^*$ . They satisfy the **Delzant condition**:  $\{\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_n}\}$  is a lattice basis whenever the facets  $F_{i_1}, \dots, F_{i_n}$  intersect at a vertex.

Now consider an arbitrary (not necessarily rational) polytope

$$P = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{a}_i, \mathbf{x} \rangle + b_i \geq 0 \text{ for } i = 1, \dots, m\}.$$

Define an affine map

$$i_P: \mathbb{R}^n \rightarrow \mathbb{R}^m, \quad i_P(\mathbf{x}) = (\langle \mathbf{a}_1, \mathbf{x} \rangle + b_1, \dots, \langle \mathbf{a}_m, \mathbf{x} \rangle + b_m).$$

Then  $i_P(P)$  is the intersection of an  $n$ -plane with

$$\mathbb{R}_{\geq}^m = \{\mathbf{y} = (y_1, \dots, y_m) : y_i \geq 0\}.$$

Define the space  $\mathcal{Z}_P$  from the diagram

$$\begin{array}{ccc} \mathcal{Z}_P & \xrightarrow{i_Z} & \mathbb{C}^m & (z_1, \dots, z_m) \\ \downarrow & & \downarrow \mu & \downarrow \\ P & \xrightarrow{i_P} & \mathbb{R}_{\geq}^m & (|z_1|^2, \dots, |z_m|^2) \end{array}$$

$\mathcal{Z}_P$  has a  $T^m$ -action,  $\mathcal{Z}_P/T^m = P$ , and  $i_Z$  is a  $T^m$ -equivariant inclusion.

## Proposition

If  $P$  is a simple polytope (more generally, if the presentation of  $P$  by inequalities is generic), then  $\mathcal{Z}_P$  is a smooth manifold of dimension  $m + n$ .

## Proof.

Write  $i_P(\mathbb{R}^n)$  by  $m - n$  linear equations in  $(y_1, \dots, y_m) \in \mathbb{R}^m$ . Replace  $y_k$  by  $|z_k|^2$  to obtain a presentation of  $\mathcal{Z}_P$  by quadrics.  $\square$

$\mathcal{Z}_P$ : **polytopal moment-angle manifold** corresponding to  $P$ .

When  $P$  is a Delzant (in particular, rational) polytope,  $\mathcal{Z}_P$  is the level set  $\mu_K^{-1}(\delta)$  of the moment map for a subtorus  $K \subset T^m$  given by

$$K = \text{Ker}(q: T^m \rightarrow T^n), \quad q: \mathbb{R}^m \rightarrow \mathbb{R}^n, \quad \mathbf{e}_j \mapsto \mathbf{a}_j.$$

# The moment-angle complex (as a polyhedral product)

$\mathcal{K}$  an abstract simplicial complex on the set  $[m] = \{1, 2, \dots, m\}$   
 $I = \{i_1, \dots, i_k\} \in \mathcal{K}$  a **simplex**; always assume  $\emptyset \in \mathcal{K}$ .

Consider the unit  $m$ -dimensional polydisc:

$$\mathbb{D}^m = \{(z_1, \dots, z_m) \in \mathbb{C}^m : |z_i|^2 \leq 1 \text{ for } i = 1, \dots, m\}.$$

The **moment-angle complex** is


$$\mathcal{Z}_{\mathcal{K}} := \bigcup_{I \in \mathcal{K}} \left( \prod_{i \in I} \mathbb{D} \times \prod_{i \notin I} \mathbb{S} \right) \subset \mathbb{D}^m,$$

where  $\mathbb{S}$  is the boundary of the unit disk  $\mathbb{D}$ .


$\mathcal{Z}_{\mathcal{K}}$  has a natural action of the torus  $T^m$ .


When  $\mathcal{K}$  is simplicial subdivision of a sphere (e.g., the boundary of a simplicial polytope),  $\mathcal{Z}_{\mathcal{K}}$  is a topological manifold, called the **moment-angle manifold**.


## Example

1. Let  $\mathcal{K} =$   (the boundary of a triangle). Then

$$\mathcal{Z}_{\mathcal{K}} = (\mathbb{D} \times \mathbb{D} \times \mathbb{S}) \cup (\mathbb{D} \times \mathbb{S} \times \mathbb{D}) \cup (\mathbb{S} \times \mathbb{D} \times \mathbb{D}) = \partial(\mathbb{D}^3) \cong S^5.$$

2. Let  $\mathcal{K} =$   (the boundary of a square). Then  $\mathcal{Z}_{\mathcal{K}} \cong S^3 \times S^3$ .

3. Let  $\mathcal{K} =$   Then  $\mathcal{Z}_{\mathcal{K}} \cong (S^3 \times S^4) \# \dots \# (S^3 \times S^4)$  (5 times).

4. Let  $\mathcal{K} =$   (three disjoint points). Then

$$\mathcal{Z}_{\mathcal{K}} = (\mathbb{D} \times \mathbb{S} \times \mathbb{S}) \cup (\mathbb{S} \times \mathbb{D} \times \mathbb{S}) \cup (\mathbb{S} \times \mathbb{S} \times \mathbb{D}) \simeq S^3 \vee S^3 \vee S^3 \vee S^4 \vee S^4$$

(not a manifold).

We define an open submanifold  $U(\mathcal{K}) \subset \mathbb{C}^m$  in a similar way:

$$U(\mathcal{K}) := \bigcup_{I \in \mathcal{K}} \left( \prod_{i \in I} \mathbb{C} \times \prod_{i \notin I} \mathbb{C}^\times \right), \quad \mathbb{C}^\times = \mathbb{C} \setminus \{0\}.$$

$U(\mathcal{K})$  is a toric variety with the corresponding fan given by

$$\Sigma_{\mathcal{K}} = \{\mathbb{R}_{\geq 0} \langle \mathbf{e}_i : i \in I \rangle : I \in \mathcal{K}\},$$

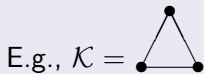
where  $\mathbf{e}_i$  denotes the  $i$ -th standard basis vector of  $\mathbb{R}^m$ .


## Theorem

$$(a) \quad U(\mathcal{K}) = \mathbb{C}^m \setminus \bigcup_{\{i_1, \dots, i_k\} \notin \mathcal{K}} \{z_{i_1} = \dots = z_{i_k} = 0\}$$

(the complement to a coordinate subspace arrangement);

$$(b) \quad \text{There is a } T^m\text{-equivariant deformation retraction } U(\mathcal{K}) \xrightarrow{\simeq} \mathcal{Z}_{\mathcal{K}}.$$



E.g.,  $\mathcal{K} =$   Then  $U(\mathcal{K}) = \mathbb{C}^3 \setminus \{z_1 = z_2 = z_3 = 0\} \xrightarrow{\simeq} S^5 = \mathcal{Z}_{\mathcal{K}}.$



# Complex-analytic structures on moment-angle manifolds

**General approach:** realise the deformation retraction  $U(\mathcal{K}) \rightarrow \mathcal{Z}_{\mathcal{K}}$  as the orbit quotient map for a holomorphic, free and proper action of a complex-analytic subgroup  $H \subset (\mathbb{C}^{\times})^m$ , i. e.  $\mathcal{Z}_{\mathcal{K}} = U(\mathcal{K})/H$ . This will make  $\mathcal{Z}_{\mathcal{K}}$  into a compact complex manifold.

Let  $\mathcal{K}$  be a sphere triangulation, i.e.  $|\mathcal{K}| \cong S^{n-1}$ .

A realisation  $|\mathcal{K}| \subset \mathbb{R}^n$  is **starshaped** if there is a point  $\mathbf{x} \notin |\mathcal{K}|$  such that any ray from  $\mathbf{x}$  intersects  $|\mathcal{K}|$  in exactly one point.

A convex triangulation  $\mathcal{K}_{\mathcal{P}}$  is starshaped, but not vice versa!

$\mathcal{K}$  has a starshaped realisation if and only if it is the underlying complex of a **complete simplicial fan**  $\Sigma$ .

$\mathbf{a}_1, \dots, \mathbf{a}_m \in \mathbb{R}^n$  the generators of the 1-dim cones of  $\Sigma$ . Define a map

$$q: \mathbb{R}^m \rightarrow \mathbb{R}^n, \quad \mathbf{e}_j \mapsto \mathbf{a}_j.$$

Set  $\mathbb{R}_{>}^m = \{(y_1, \dots, y_m) \in \mathbb{R}^m : y_i > 0\}$  and define

$$R := \exp(\text{Ker } q) = \{(y_1, \dots, y_m) \in \mathbb{R}_{>}^m : \prod_{i=1}^m y_i^{\langle \mathbf{a}_i, \mathbf{u} \rangle} = 1 \text{ for all } \mathbf{u} \in \mathbb{R}^n\},$$

$R \subset \mathbb{R}_{>}^m$  acts on  $U(\mathcal{K}) \subset \mathbb{C}^m$  by coordinatewise multiplications.

## Theorem

Let  $\Sigma$  be a complete simplicial fan in  $\mathbb{R}^n$  with  $m$  one-dimensional cones, and let  $\mathcal{K} = \mathcal{K}_\Sigma$  be its underlying simplicial complex. Then

- (a) the group  $R \cong \mathbb{R}^{m-n}$  acts on  $U(\mathcal{K})$  freely and properly, so the quotient  $U(\mathcal{K})/R$  is a smooth  $(m+n)$ -dimensional manifold;
- (b)  $U(\mathcal{K})/R$  is  $T^m$ -equivariantly homeomorphic to  $\mathcal{Z}_{\mathcal{K}}$ .

Therefore,  $\mathcal{Z}_{\mathcal{K}}$  can be smoothed canonically.

Assume  $m - n$  is even and set  $\ell = \frac{m-n}{2}$ .

Choose a linear map  $\psi: \mathbb{C}^\ell \rightarrow \mathbb{C}^m$  satisfying the two conditions:

- (a)  $\operatorname{Re} \circ \psi: \mathbb{C}^\ell \rightarrow \mathbb{R}^m$  is a monomorphism;
- (b)  $q \circ \operatorname{Re} \circ \psi = 0$ .

$$\begin{array}{ccccccc}
 \mathbb{C}^\ell & \xrightarrow{\psi} & \mathbb{C}^m & \xrightarrow{\operatorname{Re}} & \mathbb{R}^m & \xrightarrow{q} & \mathbb{R}^n \\
 & & \downarrow \operatorname{exp} & & \downarrow \operatorname{exp} & & \downarrow \operatorname{exp} \\
 & & (\mathbb{C}^\times)^m & \xrightarrow{|\cdot|} & \mathbb{R}_{>}^m & \xrightarrow{\operatorname{exp} q} & \mathbb{R}_{>}^n
 \end{array}$$

here  $|\cdot|$  denotes the map  $(z_1, \dots, z_m) \mapsto (|z_1|, \dots, |z_m|)$ . Now set

$$H = \operatorname{exp} \psi(\mathbb{C}^\ell) = \{ (e^{\langle \psi_1, \mathbf{w} \rangle}, \dots, e^{\langle \psi_m, \mathbf{w} \rangle}) \in (\mathbb{C}^\times)^m \}$$

where  $\mathbf{w} = (w_1, \dots, w_\ell) \in \mathbb{C}^\ell$ .

Then  $H \cong \mathbb{C}^\ell$  is a complex-analytic (but not algebraic) subgroup of  $(\mathbb{C}^\times)^m$ . It acts on  $U(\mathcal{K})$  by holomorphic transformations.

## Example (holomorphic tori)

Let  $\mathcal{K}$  be empty on 2 elements (that is,  $\mathcal{K}$  has two ghost vertices). We therefore have  $n = 0$ ,  $m = 2$ ,  $\ell = 1$ , and  $q: \mathbb{R}^2 \rightarrow 0$  is a zero map.

Let  $\psi: \mathbb{C} \rightarrow \mathbb{C}^2$  be given by  $z \mapsto (z, \alpha z)$  for some  $\alpha \in \mathbb{C}$ , so that

$$H = \{(e^z, e^{\alpha z})\} \subset (\mathbb{C}^\times)^2.$$

Condition (b) above is void, while (a) is equivalent to  $\alpha \notin \mathbb{R}$ . Then  $\exp \psi: H \rightarrow (\mathbb{C}^\times)^2$  is an embedding, and the quotient  $(\mathbb{C}^\times)^2/H$  is a complex torus  $T_{\mathbb{C}}^2$  with parameter  $\alpha \in \mathbb{C}$ :

$$(\mathbb{C}^\times)^2/H \cong \mathbb{C}/(\mathbb{Z} \oplus \alpha\mathbb{Z}) = T_{\mathbb{C}}^2(\alpha).$$

Similarly, if  $\mathcal{K}$  is empty on  $2\ell$  elements (so that  $n = 0$ ,  $m = 2\ell$ ), we can obtain any complex torus  $T_{\mathbb{C}}^{2\ell}$  as the quotient  $(\mathbb{C}^\times)^{2\ell}/H$ .

## Theorem (P.-Ustinovsky)

Let  $\Sigma$  be a complete simplicial fan in  $\mathbb{R}^n$  with  $m$  one-dimensional cones, and let  $\mathcal{K} = \mathcal{K}_\Sigma$  be its underlying simplicial complex. Assume that  $m - n = 2\ell$ . Then

- (a) the holomorphic action of the group  $H \cong \mathbb{C}^\ell$  on  $U(\mathcal{K})$  is free and proper, so the quotient  $U(\mathcal{K})/H$  is a compact complex  $(m - \ell)$ -manifold;
- (b) there is a  $T^m$ -equivariant diffeomorphism  $U(\mathcal{K})/H \cong \mathcal{Z}_\mathcal{K}$  defining a complex structure on  $\mathcal{Z}_\mathcal{K}$  in which  $T^m$  acts by holomorphic transformations.

Conversely, assume  $\mathcal{Z}_{\mathcal{K}}$  admits a  $T^m$ -invariant complex structure. Then the  $T^m$ -action extends to a holomorphic action of  $(\mathbb{C}^\times)^m$  on  $\mathcal{Z}_{\mathcal{K}}$ . Have a complex-analytic subgroup of global stabilisers

$$H = \{g \in (\mathbb{C}^\times)^m : g \cdot x = x \text{ for all } x \in \mathcal{Z}_{\mathcal{K}}\}.$$

$\mathfrak{h} = \text{Lie}(H)$  is a complex subalgebra of  $\text{Lie}(\mathbb{C}^\times)^m = \mathbb{C}^m$  and satisfies

- (a) the composite  $\mathfrak{h} \hookrightarrow \mathbb{C}^m \xrightarrow{\text{Re}} \mathbb{R}^m$  is injective;
- (b) the quotient map  $q: \mathbb{R}^m \rightarrow \mathbb{R}^m / \text{Re}(\mathfrak{h})$  sends the fan  $\Sigma_{\mathcal{K}}$  to a complete fan  $q(\Sigma_{\mathcal{K}})$  in  $\mathbb{R}^m / \text{Re}(\mathfrak{h})$ .

## Theorem (Ishida)

*Every complex moment-angle manifold  $\mathcal{Z}_{\mathcal{K}}$  is  $T^m$ -equivariantly biholomorphic to the quotient manifold  $U(\mathcal{K})/H$ .*

Thus,  $\mathcal{Z}_{\mathcal{K}}$  admits a complex structure if and only if  $\mathcal{K}$  is the underlying complex of a complete simplicial fan (i. e., a star-shaped sphere).

## Example (Hopf manifold)

Let  $\Sigma$  be a complete fan in  $\mathbb{R}^n$  whose cones are generated by all proper subsets of  $n + 1$  vectors  $\mathbf{e}_1, \dots, \mathbf{e}_n, -\mathbf{e}_1 - \dots - \mathbf{e}_n$ .

Add one 'empty' 1-cone to make  $m - n$  even:  $m = n + 2$ ,  $\ell = 1$ .

Then  $q: \mathbb{R}^{n+2} \rightarrow \mathbb{R}^n$  is given by the matrix  $(\mathbf{0} \mid -\mathbf{1})$ , where  $I$  is the unit  $n \times n$  matrix, and  $\mathbf{0}$ ,  $\mathbf{1}$  are the  $n$ -columns of zeros and units respectively.

The underlying complex  $\mathcal{K} = \partial\Delta^n$  with  $n + 1$  vertices and 1 ghost vertex,  $\mathcal{Z}_{\mathcal{K}} \cong S^1 \times S^{2n+1}$ , and  $U(\mathcal{K}) = \mathbb{C}^\times \times (\mathbb{C}^{n+1} \setminus \{0\})$ .

Take  $\psi: \mathbb{C} \rightarrow \mathbb{C}^{n+2}$ ,  $z \mapsto (z, \alpha z, \dots, \alpha z)$  for some  $\alpha \in \mathbb{C}$ ,  $\alpha \notin \mathbb{R}$ . Then

$$H = \{(e^z, e^{\alpha z}, \dots, e^{\alpha z}) : z \in \mathbb{C}\} \subset (\mathbb{C}^\times)^{n+2},$$

and  $\mathcal{Z}_{\mathcal{K}}$  acquires a complex structure as the quotient  $U(\mathcal{K})/H$ :

$$\mathbb{C}^\times \times (\mathbb{C}^{n+1} \setminus \{0\}) / \{(t, \mathbf{w}) \sim (e^z t, e^{\alpha z} \mathbf{w})\} \cong (\mathbb{C}^{n+1} \setminus \{0\}) / \{\mathbf{w} \sim e^{2\pi i \alpha} \mathbf{w}\},$$

where  $t \in \mathbb{C}^\times$ ,  $\mathbf{w} \in \mathbb{C}^{n+1} \setminus \{0\}$ . The **Hopf manifold**.

# A holomorphic foliation on $\mathcal{Z}_K$

Define the Lie subalgebra and the corresponding Lie group

$$\mathfrak{k} = \operatorname{Re}(\mathfrak{h}) \subset \mathbb{R}^m = \operatorname{Lie}(T^m), \quad K = \exp(\mathfrak{k}) \subset T^m.$$

The restriction of the  $T^m$ -action on  $U(K)/H$  to  $K \subset T^m$  is almost free. We obtain a *holomorphic* foliation  $\mathcal{F}$  on  $\mathcal{Z}_K$  by the orbits of  $K$ .

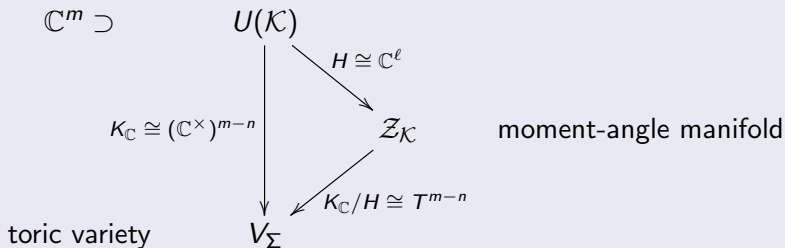
If the subspace  $\mathfrak{k} \subset \mathbb{R}^m$  is rational (i. e., generated by integer vectors), then  $K$  is a subtorus of  $T^m$  and the complete simplicial fan  $\Sigma := q(\Sigma_K)$  is rational. The rational fan  $\Sigma$  defines a toric variety

$$V_\Sigma = \mathcal{Z}_K/K = U(K)/K_{\mathbb{C}}.$$

The holomorphic foliation of  $\mathcal{Z}_K$  by the orbits of  $K$  becomes a holomorphic **Seifert fibration** over the toric orbifold  $V_\Sigma$  with fibres compact complex tori  $K_{\mathbb{C}}/H \cong T^{m-n}$ .



The rational case:



The non-rational case:

Have  $U(\mathcal{K}) \xrightarrow{H} \mathcal{Z}_{\mathcal{K}}$ ,

and a holomorphic foliation  $\mathcal{F}$  of  $\mathcal{Z}_{\mathcal{K}}$  by the orbits of  $K \subset T^m$ .

The holomorphic foliated manifold  $(\mathcal{Z}_{\mathcal{K}}, \mathcal{F})$  is a model for 'non-commutative' toric varieties in the sense of [Katzarkov, Lupercio, Meersseman, Verjovsky] (arXiv:1308.2774) and [Ratiu, Zung] (arXiv:1705.11110).

The **face ring** (the **Stanley–Reisner ring**) of  $\mathcal{K}$  is

$$\mathbb{C}[\mathcal{K}] := \mathbb{C}[v_1, \dots, v_m] / I_{\mathcal{K}} = \mathbb{C}[v_1, \dots, v_m] / (v_{i_1} \cdots v_{i_k} : \{i_1, \dots, i_k\} \notin \mathcal{K}),$$

where  $\mathbb{C}[v_1, \dots, v_m]$  is the polynomial algebra,  $\deg v_i = 2$ , and  $I_{\mathcal{K}}$  is the **Stanley–Reisner ideal**.

## Proposition

*The  $T^m$ -equivariant cohomology is given by*

$$H_{T^m}^*(\mathcal{Z}_{\mathcal{K}}) = H_{T^m}^*(U(\mathcal{K})) \cong \mathbb{C}[\mathcal{K}].$$

The toric variety  $V_\Sigma$  is Kähler (equivalently, projective) if and only if  $\Sigma$  is the normal fan of lattice (Delzant) polytope  $P$ .

## Theorem (Danilov)

The Dolbeault cohomology of  $V_\Sigma$  is given by

$$H_{\bar{\partial}}^{*,*}(V_\Sigma) \cong \mathbb{C}[v_1, \dots, v_m]/(I_\Sigma + J_\Sigma),$$

where  $v_i \in H_{\bar{\partial}}^{1,1}(V_\Sigma)$ ,  $I_\Sigma$  is the Stanley–Reisner ideal,  $J_\Sigma$  is the ideal generated by the linear forms  $\sum_{k=1}^m \langle \mathbf{a}_k, \mathbf{u} \rangle v_k$ ,  $\mathbf{a}_k = q(\mathbf{e}_k)$  are the generators of 1-dim cones of  $\Sigma$ ,  $\mathbf{u} \in (\mathbb{R}^m/\mathfrak{k})^*$ .

The nonzero Hodge numbers are given by  $h^{p,p}(V_\Sigma) = h_p$ , where  $h(\Sigma) = (h_0, h_1, \dots, h_n)$  is the ***h*-vector** of  $\Sigma$ .

## Theorem (Buchstaber-P.)

The de Rham cohomology ring of  $Z_{\mathcal{K}}$  is given by

$$\begin{aligned} H^*(Z_{\mathcal{K}}) &\cong \operatorname{Tor}_{\mathbb{C}[v_1, \dots, v_m]}(\mathbb{C}[\mathcal{K}], \mathbb{C}) \\ &\cong H(\Lambda[u_1, \dots, u_m] \otimes \mathbb{C}[\mathcal{K}], d) \quad du_i = v_i, \quad dv_i = 0 \\ &\cong H(\Lambda[t_1, \dots, t_{m-n}] \otimes H^*(V_{\Sigma}), d) \quad \Lambda[t_1, \dots, t_{m-n}] = H^*(K) \\ &\cong \bigoplus_{I \subset [m]} \tilde{H}^{*-|I|-1}(\mathcal{K}_I). \end{aligned}$$

## Theorem (P.-Ustinovsky)

Let  $\Sigma$  be a rational fan,  $\mathcal{Z}_{\mathcal{K}} \xrightarrow{K} V_{\Sigma}$  a holomorphic torus fibration. Then the Dolbeault cohomology ring of  $\mathcal{Z}_{\mathcal{K}}$  is given by

$$H_{\bar{\partial}}^{*,*}(\mathcal{Z}_{\mathcal{K}}) \cong H(\Lambda[\xi_1, \dots, \xi_{\ell}, \eta_1, \dots, \eta_{\ell}] \otimes H_{\bar{\partial}}^{*,*}(V_{\Sigma}), d),$$

where  $\Lambda[\xi_1, \dots, \xi_{\ell}, \eta_1, \dots, \eta_{\ell}] = H_{\bar{\partial}}^{*,*}(K)$ ,  $\xi_j \in H_{\bar{\partial}}^{1,0}(K)$ ,  $\eta_j \in H_{\bar{\partial}}^{0,1}(K)$ ,  
 $dv_j = d\eta_j = 0$ ,  $d\xi_j = c(\xi_j)$ ,

$c: H_{\bar{\partial}}^{1,0}(K) \rightarrow H_{\bar{\partial}}^{1,1}(V_{\Sigma})$  is the first Chern class map.

## Corollary

- (a) The Borel spectral sequence of the holomorphic fibration  $\mathcal{Z}_{\mathcal{K}} \xrightarrow{K} V_{\Sigma}$  (converging to Dolbeault cohomology of  $\mathcal{Z}_{\mathcal{K}}$ ) collapses at the  $E_3$  page;
- (b) The Frölicher spectral sequence (with  $E_1 = H_{\bar{\partial}}^{*,*}(\mathcal{Z}_{\mathcal{K}})$ , converging to  $H^*(\mathcal{Z}_{\mathcal{K}})$ ) collapses at  $E_2$ .

# Transverse Kähler form and analytic subsets

The complex structure on  $\mathcal{Z}_{\mathcal{K}}$  is determined by two pieces of data:

- a complete simplicial fan  $\Sigma$  with generators  $\mathbf{a}_1, \dots, \mathbf{a}_m$ ;
- an  $\ell$ -dimensional holomorphic subgroup  $H \subset (\mathbb{C}^\times)^m$ .

If this data is *generic* (in particular, the fan  $\Sigma$  is not rational), then there is no holomorphic principal torus fibration  $\mathcal{Z}_{\mathcal{K}} \rightarrow V_{\Sigma}$  over a toric variety  $V_{\Sigma}$ .

Instead, there is a holomorphic  $\ell$ -dimensional *foliation*  $\mathcal{F}$ , which sometimes admits a **transverse Kähler form**  $\omega_{\mathcal{F}}$ . This form can be used to describe submanifolds and analytic subsets in  $\mathcal{Z}_{\mathcal{K}}$ .

A  $(1, 1)$ -form  $\omega_{\mathcal{F}}$  on the complex manifold  $\mathcal{Z}_{\mathcal{K}}$  is **transverse Kähler** with respect to the foliation  $\mathcal{F}$  if

- (a)  $\omega_{\mathcal{F}}$  is closed, i. e.  $d\omega_{\mathcal{F}} = 0$ ;
- (b)  $\omega_{\mathcal{F}}$  is nonnegative and the zero space of  $\omega_{\mathcal{F}}$  is the tangent space of  $\mathcal{F}$ .

A complete simplicial fan  $\Sigma$  in  $\mathbb{R}^n$  is **weakly normal** if there exists a (not necessarily simple)  $n$ -dimensional polytope  $P$  such that  $\Sigma$  is a simplicial subdivision of the normal fan  $\Sigma_P$ .

### Theorem (P.–Ustinovsky–Verbitsky)

*Assume that  $\Sigma$  is a weakly normal fan. Then there exists an exact  $(1, 1)$ -form  $\omega_{\mathcal{F}}$  on  $\mathcal{Z}_{\mathcal{K}} = U(\mathcal{K})/H$  which is transverse Kähler for the foliation  $\mathcal{F}$  on the dense open subset  $(\mathbb{C}^{\times})^m/H \subset U(\mathcal{K})/H$ .*

If there is a transverse Kähler form defined *on the whole* of  $\mathcal{Z}_{\mathcal{K}}$ , then  $\Sigma$  is a normal fan of a simple polytope [Ishida], and  $\mathcal{Z}_{\mathcal{K}}$  can be written as an intersection of Hermitian quadrics as in the beginning of the talk.

For each  $J \subset [m]$ , the **coordinate submanifold** of  $\mathcal{Z}_{\mathcal{K}}$  is

$$\mathcal{Z}_{\mathcal{K}_J} = \{(z_1, \dots, z_m) \in \mathcal{Z}_{\mathcal{K}} : z_i = 0 \text{ for } i \notin J\}.$$

The closure of any  $(\mathbb{C}^\times)^m$ -orbit of  $U(\mathcal{K})$  has the form  $U(\mathcal{K}_J)$  for some  $J \subset [m]$  (in particular, the dense orbit corresponds to  $J = [m]$ ). Similarly, the closure of any  $(\mathbb{C}^\times)^m/C$ -orbit of  $\mathcal{Z}_{\mathcal{K}} \cong U(\mathcal{K})/C$  has the form  $\mathcal{Z}_{\mathcal{K}_J}$ .

### Theorem (P.–Ustinovsky–Verbitsky)

*Assume that the data defining a complex structure on  $\mathcal{Z}_{\mathcal{K}} = U(\mathcal{K})/C$  is generic. Then any divisor of  $\mathcal{Z}_{\mathcal{K}}$  is a union of coordinate divisors.*

*Furthermore, if  $\Sigma$  is a weakly normal fan, then any compact irreducible analytic subset  $Y \subset \mathcal{Z}_{\mathcal{K}}$  of positive dimension is a coordinate submanifold.*

### Corollary

*Under generic assumptions, there are no non-constant meromorphic functions on  $\mathcal{Z}_{\mathcal{K}}$  (i. e. the algebraic dimension of  $\mathcal{Z}_{\mathcal{K}}$  is zero).*



# Basic cohomology

$M$  a manifold with an action of a connected Lie group  $G$ ,  $\mathfrak{g} = \text{Lie } G$ .

$$\Omega(M)_{\text{bas}, G} = \{\omega \in \Omega(M) : \iota_{\xi}\omega = L_{\xi}\omega = 0 \text{ for any } \xi \in \mathfrak{g}\},$$

$H_{\text{bas}, G}^*(M) = H(\Omega(M)_{\text{bas}, G}, d)$  the **basic cohomology** of  $M$ .

$S(\mathfrak{g}^*)$  the symmetric algebra on  $\mathfrak{g}^*$  with generators of degree 2.

The **Cartan model** is

$$\mathcal{C}_{\mathfrak{g}}(\Omega(M)) = ((S(\mathfrak{g}^*) \otimes \Omega(M))^{\mathfrak{g}}, d_{\mathfrak{g}}),$$

where  $(S(\mathfrak{g}^*) \otimes \Omega(M))^{\mathfrak{g}}$  denotes the  $\mathfrak{g}$ -invariant subalgebra.

An element  $\omega \in \mathcal{C}_{\mathfrak{g}}(\Omega(M))$  is a “ $\mathfrak{g}$ -equivariant polynomial map from  $\mathfrak{g}$  to  $\Omega(M)$ ”. The differential  $d_{\mathfrak{g}}$  is given by

$$d_{\mathfrak{g}}(\omega)(\xi) = d(\omega(\xi)) - \iota_{\xi}(\omega(\xi)).$$

## Theorem

$$H_{\text{bas}, G}^*(M) \cong H(\mathcal{C}_{\mathfrak{g}}(\Omega(M)), d_{\mathfrak{g}}).$$

If in addition  $G$  is a compact, then

$$H_{\text{bas}, G}^*(M) \cong H_G^*(M) = H^*(EG \times_G M) \quad \text{the equivariant cohomology.}$$

Now consider  $\mathcal{Z}_{\mathcal{K}}$  with the action of  $K$  (a holomorphic foliation  $\mathcal{F}$ ).

### Theorem (Ishida–Krutowski–P.)

*There is an isomorphism of algebras:*

$$H_{\text{bas}, K}^*(\mathcal{Z}_{\mathcal{K}}) \cong \mathbb{C}[v_1, \dots, v_m]/(I_{\mathcal{K}} + J_{\Sigma}),$$

where  $I_{\mathcal{K}}$  is the Stanley–Reisner ideal of  $\mathcal{K}$ , generated by the monomials

$$v_{i_1} \cdots v_{i_k} \quad \text{with } \{i_1, \dots, i_k\} \notin \mathcal{K},$$

and  $J_{\Sigma}$  is the ideal generated by the linear forms

$$\sum_{i=1}^m \langle \mathbf{a}_i, \mathbf{u} \rangle v_i \quad \text{with } \mathbf{u} \in (\mathbb{R}^m / \mathfrak{t})^*.$$

This settles a conjecture by [\[Battaglia and Zaffran\]](#) (arXiv:1108.1637).

If  $K$  is a compact torus (the fan  $\Sigma$  is rational), then we get

$$H_{\text{bas}, K}^*(\mathcal{Z}_{\mathcal{K}}) = H^*(\mathcal{Z}_{\mathcal{K}}/K) = H^*(V_{\Sigma})$$

and the result above turns into the well-known description of the cohomology of toric manifolds, due to [\[Danilov and Jurkiewicz\]](#).

### Idea of proof of the theorem.

Let  $\mathfrak{t} = \text{Lie}(T^m) \cong \mathbb{R}^m$  and consider the Cartan model

$$\mathcal{C}_{\mathfrak{t}}(\Omega(\mathcal{Z}_{\mathcal{K}})) = ((S(\mathfrak{t}^*) \otimes \Omega(\mathcal{Z}_{\mathcal{K}}))^{T^m}, d_{\mathfrak{t}}).$$

Then

$$H(\mathcal{C}_{\mathfrak{t}}(\Omega(\mathcal{Z}_{\mathcal{K}}))) = H_{T^m}^*(\mathcal{Z}_{\mathcal{K}}) = \mathbb{C}[v_1, \dots, v_m]/I_{\mathcal{K}}.$$

**Key lemma:** the dga  $\mathcal{C}_{\mathfrak{t}}(\Omega(\mathcal{Z}_{\mathcal{K}}))$  is formal (quasi-isomorphic to its cohomology). □

- [1] Taras Panov and Yuri Ustinovsky. *Complex-analytic structures on moment-angle manifolds*. Moscow Math. J. 12 (2012), no. 1, 149–172.
- [2] Taras Panov, Yuri Ustinovsky and Misha Verbitsky. *Complex geometry of moment-angle manifolds*. Math. Zeitschrift 284 (2016), no. 1, 309–333.
- [3] Hiroaki Ishida, Roman Krutowski and Taras Panov. *Basic cohomology of canonical holomorphic foliations on complex moment-angle manifolds*. Internat. Math. Research Notices, to appear, 2020.
- [4] Roman Krutowski and Taras Panov. *Dolbeault cohomology of complex manifolds with torus action*. arXiv:1908.06356.