SU-bordism: geometric representatives, operations, multiplications and projections

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1. Unitary bordism

The unitary bordism ring Ω^U consists of complex bordism classes of stably complex manifolds.
A stably complex manifold is a pair $(M, c_\mathcal{T})$ consisting of a smooth manifold M and a stably complex structure $c_\mathcal{T}$, determined by a choice of an isomorphism

$$c_\mathcal{T}\colon \mathcal{T}M \oplus \underline{\mathbb{R}}^N \xrightarrow{\cong} \xi$$

between the stable tangent bundle of M and a complex vector bundle ξ.

Theorem (Milnor–Novikov)

- *Two stably complex manifolds M and N represent the same bordism classes in Ω^U iff their sets of Chern characteristic numbers coincide.*
- *Ω^U is a polynomial ring on generators in every even degree:*

$$\Omega^U \cong \mathbb{Z}[a_1, a_2, \ldots, a_i, \ldots], \quad \deg a_i = 2i.$$

Polynomial generators of Ω^U can be detected using a special characteristic class s_n. It is the polynomial in the universal Chern classes $c_1, \ldots, c_n$ obtained by expressing the symmetric polynomial $x_1^n + \cdots + x_n^n$ via the elementary symmetric functions $\sigma_i(x_1, \ldots, x_n)$ and replacing each σ_i by c_i.

$s_n[M] = s_n(\mathcal{T}M)\langle M\rangle$: the corresponding characteristic number.

Theorem

The bordism class of a stably complex manifold M^{2i} may be taken to be the polynomial generator $a_i \in \Omega^U_{2i}$ iff

$$s_i[M^{2i}] = \begin{cases} \pm 1 & \text{if} \quad i+1 \neq p^s \quad \text{for any prime } p, \\ \pm p & \text{if} \quad i+1 = p^s \quad \text{for some prime } p \text{ and integer } s > 0. \end{cases}$$

Problem

Find nice geometric representatives in (unitary) bordism classes; e.g., smooth algebraic varieties and/or manifolds with large symmetry groups.

2. Special unitary bordism

A stably complex manifold $(M, c_{\mathcal{T}})$ is special unitary (an SU-manifold) if $c_1(M) = 0$. Bordism classes of SU-manifolds form the special unitary bordism ring Ω^{SU}.

The ring structure of Ω^{SU} is more subtle than that of Ω^{U}. Novikov described $\Omega^{SU} \otimes \mathbb{Z}[\frac{1}{2}]$ (it is a polynomial ring). The 2-torsion was described by Conner and Floyd. We shall need the following facts.

Theorem

- *The kernel of the forgetful map $\Omega^{SU} \to \Omega^{U}$ consists of torsion.*
- *Every torsion element in Ω^{SU} has order 2.*
- *$\Omega^{SU} \otimes \mathbb{Z}[\frac{1}{2}]$ is a polynomial algebra on generators in every even degree > 2:*

$$\Omega^{SU} \otimes \mathbb{Z}[\tfrac{1}{2}] \cong \mathbb{Z}[\tfrac{1}{2}][y_i \colon i > 1], \quad \deg y_i = 2i.$$

3. U- and SU-theory

$\Omega^U = U_*(pt) = \pi_*(MU)$ is the coefficient ring of the complex bordism theory, defined by the Thom spectrum $MU = \{MU(n)\}$, where $MU(n)$ is the Thom space of the universal $U(n)$-bundle $EU(n) \to BU(n)$:

$$U_n(X, A) = \lim_{k\to\infty} \pi_{2k+n}\bigl((X/A) \wedge MU(k)\bigr),$$
$$U^n(X, A) = \lim_{k\to\infty} \bigl[\Sigma^{2k-n}(X/A), MU(k)\bigr]$$

for a CW-pair (X, A).

Similarly, $\Omega^{SU} = SU_*(pt) = \pi_*(MSU)$ is the coefficient ring of the SU-theory, defined by the Thom spectrum $MSU = \{MSU(n)\}$:

$$SU_n(X, A) = \lim_{k\to\infty} \pi_{2k+n}\bigl((X/A) \wedge MSU(k)\bigr),$$
$$SU^n(X, A) = \lim_{k\to\infty} \bigl[\Sigma^{2k-n}(X/A), MSU(k)\bigr].$$

4. Operations in U-theory

A (stable) operation θ of degree n in complex cobordism is a family of additive maps

$$\theta\colon U^k(X,A) \to U^{k+n}(X,A),$$

which are functorial in (X,A) and commute with the suspension isomorphisms. The set of all operations is a Ω_U-algebra, denoted by A^U; it was described in the works of Landweber and Novikov in 1967.

There is an isomorphism of Ω_U-modules

$$A^U \cong [MU, MU] = U^*(MU) = \varprojlim U^{*+2N}(MU(N)).$$

There is also an isomorphism of left Ω_U-modules

$$A^U = U^*(MU) \cong \Omega_U \mathbin{\widehat{\otimes}} S,$$

where S is the Landweber–Novikov algebra, generated by the operations $S_\omega = \varphi^*(s_\omega^U)$ corresponding to universal characteristic classes $s_\omega^U \in U^*(BU)$ defined by symmetrising monomials $t_1^{i_1} \cdots t_k^{i_k}$, $\omega = (i_1, \ldots, i_k)$.

5. *SU*-linear operations

Lemma (Novikov)

The representations of A^U on $\Omega_U = U^*(pt)$ and $\Omega^U = U_*(pt)$ are faithful.

Remark

More generally, given spectra E, F of finite type, the natural homomorphism $F^*(E) \to \mathrm{Hom}^*(\pi_*(E), \pi_*(F))$ is injective when $\pi_*(F)$ and $H_*(E)$ do not have torsion; see Rudyak1998.

An operation $\theta \in A^U = [MU, MU]$ is *SU*-linear if it is an MSU-module map $MU \to MU$.

By the lemma above, it is equivalent to requiring that the induced map $\theta \colon \Omega^U \to \Omega^U$ is Ω^{SU}-linear, i. e. $\theta(ab) = a\theta(b)$ for $a \in \Omega^{SU}$, $b \in \Omega^U$.

Construction (Conner–Floyd and Novikov's geometric operations)

Let $\partial\colon \Omega^U_{2n} \to \Omega^U_{2n-2}$ be the homomorphism sending a bordism class $[M^{2n}]$ to the bordism class $[V^{2n-2}]$ of a submanifold $V^{2n-2} \subset M$ dual to $c_1(M) = c_1(\det \mathcal{T}M)$.

Similarly, given positive integers k_1, k_2, let

$$\Delta_{(k_1,k_2)}\colon \Omega^U_{2n} \to \Omega^U_{2n-2k_1-2k_2}$$

be the homorphism sending $[M]$ to the submanifold dual to $(\det \mathcal{T}M)^{\oplus k_1} \oplus (\overline{\det \mathcal{T}M})^{\oplus k_2}$.

We denote

$$\Delta = \Delta_{(1,1)}, \quad \partial_k = \Delta_{(k,0)}, \quad \partial = \partial_1, \quad \overline{\partial}_k = \Delta_{(0,k)}.$$

Each $\Delta_{(k_1,k_2)}$ extends to an operation in $U^{2k_1+2k_2}(MU) = [MU, MU]_{2k_1+2k_2}$, which is SU-linear by inspection.

Theorem (Chernykh-P.)

Any SU-linear operation $f \in [MU, MU]_{MSU,*}$ *can be written uniquely as a power series* $f = \sum_{i \geqslant 0} \mu_i \partial_i$, *where* $\mu_i \in \Omega_U^{-2i+*}$.

Proof (sketch).

Use Conner and Floyd's equivalence of MSU-modules

$$MU \simeq MSU \wedge \Sigma^{-2}\mathbb{C}P^\infty.$$

It implies that the abelian group $[MU, MU]_{MSU,\,k}$ of SU-linear operations is isomorphic to $\widetilde{U}^{k+2}(\mathbb{C}P^\infty)$. More precisely, if $u \in \widetilde{U}^2(\mathbb{C}P^\infty)$ is the canonical orientation, then

$$[MU, MU]_* \to \widetilde{U}^{*+2}(\mathbb{C}P^\infty), \quad f \mapsto f(u),$$

becomes an isomorphism when restricted to $[MU, MU]_{MSU,*}$.

Under this isomorphism, a power series $\sum_{i \geqslant 0} \lambda_i u^{i+1} \in \widetilde{U}^{2k+2}(\mathbb{C}P^\infty)$ corresponds to the operation $\sum_{i \geqslant 0} \lambda_i \overline{\partial}_i$, because $\overline{\partial}_i(u) = u^{i+1}$. □

6. c_1-spherical bordism W

Consider closed manifolds M with a c_1-spherical structure, which consists of

- a stably complex structure on the tangent bundle $\mathcal{T}M$;
- a $\mathbb{C}P^1$-reduction of the determinant bundle, that is, a map $f\colon M \to \mathbb{C}P^1$ and an equivalence $f^*(\eta) \cong \det \mathcal{T}M$, where η is the tautological bundle over $\mathbb{C}P^1$.

This is a natural generalisation of an SU-structure, which can be thought of as a "$\mathbb{C}P^0$-reduction", that is, a trivialisation of the determinant bundle.

The corresponding bordism theory is called c_1-spherical bordism and is denoted W_*. It is instrumental in describing the SU-bordism ring and other calculations in the SU-theory.

As in the case of stable complex structures, a c_1-spherical complex structure on the stable tangent bundle is equivalent to such a structure on the stable normal bundle. There are forgetful transformations $MSU_* \to W_* \to MU_*$.

Homotopically, a c_1-spherical structure on a stable complex bundle $\xi\colon M \to BU$ is defined by a choice of lifting to a map $M \to X$, where X is the (homotopy) pullback:

$$\begin{array}{ccccc} & & X & \longrightarrow & \mathbb{C}P^1 \\ & \nearrow & \downarrow & & \downarrow{\scriptstyle i} \\ M & \xrightarrow{\ \xi\ } & BU & \xrightarrow{\ \det\ } & \mathbb{C}P^\infty \end{array}$$

The Thom spectrum corresponding to the map $X \to BU$ defines the bordism theory of manifolds with a $\mathbb{C}P^1$-reduction of the stable normal bundle, that is, the theory W_*. We denote this spectrum by W.

Proposition (Conner–Floyd)

There is an equivalence of MSU-modules

$$W \simeq MSU \wedge \Sigma^{-2}\mathbb{C}P^2.$$

Under this equivalence, the forgetful map $W \to MU$ is identified with the free MSU-module map $MSU \wedge \Sigma^{-2}\mathbb{C}P^2 \to MSU \wedge \Sigma^{-2}\mathbb{C}P^\infty$.

Theorem (Conner–Floyd, Stong)

(a) *The image of the forgetful homomorphism $\pi_*(W) \to \pi_*(MU)$ coincides with* $\ker \Delta$.

(b) *The spectrum W is the fibre of $MU \xrightarrow{\Delta} \Sigma^4 MU$.*

7. Multiplications and projections

$\Omega^W_{2n} = \pi_{2n}(W)$ can be identified with the subgroup of Ω^U_{2n} consisting of bordism classes $[M^{2n}]$ such that every Chern number of M^{2n} of which c_1^2 is a factor vanishes.

However, $\Omega^W = \bigoplus_{i \geqslant 0} \Omega^W_{2i}$ is *not* a subring of Ω^U: one has $[\mathbb{C}P^1] \in \Omega^W_2$, but $c_1^2[\mathbb{C}P^1 \times \mathbb{C}P^1] = 8 \neq 0$, so $[\mathbb{C}P^1] \times [\mathbb{C}P^1] \notin \Omega^W_4$.

Let $\pi\colon MU \to W$ be an SU-linear projection (an idempotent operation with image W). It defines an SU-bilinear multiplication on W by the formula

$$W \wedge W \to MU \wedge MU \xrightarrow{m_{MU}} MU \xrightarrow{\pi} W.$$

This multiplication has a unit, obtained from the unit of MSU by the forgetful morphism.

Example

1. Define a homomorphism $p_0\colon \Omega^U \to \Omega^W$ sending a bordism class $[M]$ to the class of the submanifold $N \subset \mathbb{C}P^1 \times M$ dual to $\overline{\eta} \otimes \det \mathcal{T}M$. We have $\det \mathcal{T}N \cong i^*\overline{\eta}$, where i is the embedding $N \hookrightarrow \mathbb{C}P^1 \times M$, so N has a natural c_1-spherical stably complex structure.

The homomorphism p_0 extends to an idempotent SU-linear operation $p_0\colon MU \to MU$, called the Stong projection.

2. Conner and Floyd defined geometrically a right inverse to the operation $\Delta\colon \Omega^U_* \to \Omega^U_{*-4}$. Novikov extended it to a cohomological operation $\Psi \in [\Sigma^4 MU, MU]$, $\Delta\Psi = 1$.

Then $1 - \Psi\Delta\colon MU \to MU$ is an idempotent SU-linear operation with image $\ker \Delta$, called the Conner–Floyd projection.

The two projections are different, although they define the same multiplication on W. This reflects the fact that both projections have the same coefficient of ∂_2 in their expansions $1 + \sum_{i\geqslant 2} \lambda_i \partial_i$.

Theorem (Chernykh-P)

Any SU-linear multiplication on W with the standard unit has the form

$$a * b = ab + (2[V] - w)\partial a \partial b,$$

where $[V] = [\mathbb{C}P^1]^2 - [\mathbb{C}P^2]$ and $w \in \Omega_4^W$. Any such multiplication is associative and commutative. Furthermore, the multiplications obtained from SU-linear projections are those with $w = 2\widetilde{w}$, $\widetilde{w} \in \Omega_4^W$.

In this way, W becomes a complex oriented multiplicative cohomology theory.

Let $m_i = \gcd\left\{\binom{i+1}{k}, 1 \leqslant k \leqslant i\right\}$

$$= \begin{cases} 1 & \text{if} \quad i+1 \neq p^\ell \quad \text{for any prime } p, \\ p & \text{if} \quad i+1 = p^\ell \quad \text{for some prime } p \text{ and integer } \ell > 0. \end{cases}$$

Then $[M^{2i}] \in \Omega^U_{2i}$ represents a polynomial generator iff $s_i[M^{2i}] = \pm m_i$.

Theorem (Stong)

Ω^W is a polynomial ring on generators in every even degree except 4:

$$\Omega^W \cong \mathbb{Z}[x_1, x_i \colon i \geqslant 3], \quad x_1 = [\mathbb{C}P^1], \quad x_i \in \pi_{2i}(W).$$

The polynomial generators x_i are specified by the condition $s_i(x_i) = \pm m_i m_{i-1}$ for $i \geqslant 3$. The boundary operator $\partial \colon \Omega^W \to \Omega^W$, $\partial^2 = 0$, is given by $\partial x_1 = 2$, $\partial x_{2i} = x_{2i-1}$, and satisfies the identity

$$\partial(a * b) = a * \partial b + \partial a * b - x_1 * \partial a * \partial b.$$

We have

$$\Omega^W \otimes \mathbb{Z}[\tfrac{1}{2}] \cong \mathbb{Z}[\tfrac{1}{2}][x_1, x_{2k-1}, 2x_{2k} - x_1x_{2k-1} \colon k > 1],$$

where $x_1^2 = x_1 * x_1$ is a ∂-cycle, and each x_{2k-1}, $2x_{2k} - x_1x_{2k-1}$ is a ∂-cycle.

Theorem

There exist elements $y_i \in \Omega^{SU}_{2i}$, $i > 1$, such that $s_2(y_2) = -48$ and

$$s_i(y_i) = \begin{cases} m_i m_{i-1} & \text{if } i \text{ is odd}, \\ 2m_i m_{i-1} & \text{if } i \text{ is even and } i > 2. \end{cases}$$

These elements are mapped as follows under the forgetful homomorphism $\Omega^{SU} \to \Omega^W$:

$$y_2 \mapsto 2x_1^2, \quad y_{2k-1} \mapsto x_{2k-1}, \quad y_{2k} \mapsto 2x_{2k} - x_1x_{2k-1}, \quad k > 1.$$

In particular, $\Omega^{SU} \otimes \mathbb{Z}[\frac{1}{2}]$ embeds in $\Omega^W \otimes \mathbb{Z}[\frac{1}{2}]$ as the polynomial subring generated by x_1^2, x_{2k-1} and $2x_{2k} - x_1x_{2k-1}$.

8. (Quasi)toric representatives in U-bordism classes

The classical family of generators for Ω^U is formed by the Milnor hypersufaces $H(n_1, n_2)$.

$H(n_1, n_2)$ is a hyperplane section of the Segre embedding $\mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2} \to \mathbb{C}P^{(n_1+1)(n_2+1)-1}$, given by the equation

$$z_0 w_0 + \cdots + z_{n_1} w_{n_1} = 0$$

where $[z_0 : \cdots : z_{n_1}] \in \mathbb{C}P^{n_1}$, $[w_0 : \cdots : w_{n_2}] \in \mathbb{C}P^{n_2}$, $n_1 \leqslant n_2$.

Also, $H(n_1, n_2)$ can be identified with the projectivisation $\mathbb{C}P(\zeta)$ of a certain n_2-plane bundle over $\mathbb{C}P^{n_1}$. The bundle ζ is not a sum of line bundles when $n_1 > 1$, so $H(n_1, n_2)$ is *not* a toric manifold in this case.

Buchstaber and Ray introduced a family $B(n_1, n_2)$ of *toric* generators of Ω^U. Each $B(n_1, n_2)$ is the projectivisation of a sum of n_2 line bundles over the bounded flag manifold BF_{n_1}. Then $B(n_1, n_2)$ is a toric manifold, because BF_{n_1} is toric and the projectivisation of a sum of line bundles over a toric manifold is toric.

We have $H(0, n_2) = B(0, n_2) = \mathbb{C}P^{n_2-1}$, so $s_{n_2-1}[H(0, n_2)] = s_{n_2-1}[B(0, n_2)] = n_2$. Furthermore,

$$s_{n_1+n_2-1}[H(n_1, n_2)] = s_{n_1+n_2-1}[B(n_1, n_2)] = -\binom{n_1+n_2}{n_1} \quad \text{for } n_1 > 1.$$

It follows that each of the families $\{[H(n_1, n_2)]\}$ and $\{[B(n_1, n_2)]\}$ generates the unitary bordism ring Ω^U.

We proceed to describing another family of toric generators for Ω^U.

Given two nonnegative integers n_1, n_2, define

$$L(n_1, n_2) = \mathbb{C}P(\eta \oplus \underline{\mathbb{C}}^{n_2}),$$

where η is the tautological line bundle over $\mathbb{C}P^{n_1}$.
It is a projective toric manifold with

$$\Lambda = \begin{pmatrix} \overbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{matrix}}^{n_1} & \begin{matrix} -1 \\ \vdots \\ -1 \end{matrix} & & \Large 0 & \\ & \Large 0 & \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} & \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{matrix}}_{n_2} & \begin{matrix} -1 \\ \vdots \\ -1 \end{matrix} \end{pmatrix}$$

The cohomology ring is given by

$$H^*\big(L(n_1,n_2)\big) \cong \mathbb{Z}[u,v]/\big(u^{n_1+1}, v^{n_2+1} - uv^{n_2}\big)$$

with $u^{n_1}v^{n_2}\langle L(n_1,n_2)\rangle = 1$.

There is an isomorphism of complex bundles

$$\mathcal{T}L(n_1,n_2) \oplus \underline{\mathbb{C}}^2 \cong \underbrace{p^*\bar\eta \oplus \cdots \oplus p^*\bar\eta}_{n_1+1} \oplus (\bar\gamma \otimes p^*\eta) \oplus \underbrace{\bar\gamma \oplus \cdots \oplus \bar\gamma}_{n_2},$$

where γ is the tautological line bundle over $L(n_1,n_2) = \mathbb{C}P(\eta \oplus \underline{\mathbb{C}}^{n_2})$.

The total Chern class is

$$c\big(L(n_1,n_2)\big) = (1+u)^{n_1+1}(1+v-u)(1+v)^{n_2}$$

with $u = c_1(p^*\bar\eta)$ and $v = c_1(\bar\gamma)$.

Lemma

For $n_2 > 0$, we have

$$s_{n_1+n_2}\big[L(n_1, n_2)\big] = \binom{n_1+n_2}{0} - \binom{n_1+n_2}{1} + \cdots + (-1)^{n_1}\binom{n_1+n_2}{n_1} + n_2.$$

Theorem (Lu-P.)

The bordism classes $[L(n_1, n_2)] \in \Omega^U_{2(n_1+n_2)}$ generate the ring Ω^U.

Proof. $s_{n_1+n_2}\big[L(n_1, n_2) - 2L(n_1 - 1, n_2 + 1) + L(n_1 - 2, n_2 + 2)\big]$
$= (-1)^{n_1-1}\binom{n_1+n_2}{n_1-1} + (-1)^{n_1}\binom{n_1+n_2}{n_1} - 2(-1)^{n_1-1}\binom{n_1+n_2}{n_1-1} = (-1)^{n_1}\binom{n_1+n_2+1}{n_1}$

It follows that any unitary bordism class can be represented by a disjoint union of products of projective toric manifolds. Products of toric manifolds are toric, but disjoint unions are not, as toric manifolds are connected.

A disjoint union may be replaced by a connected sum, representing the same bordism class. However, connected sum is not an algebraic operation, and a connected sum of two algebraic varieties is rarely algebraic.

One can form equivariant connected sum of quasitoric manifolds, but the resulting invariant stably complex structure does not represent the cobordism sum of the two original manifolds. A more intricate connected sum construction is needed, as described in [Buchstaber, P. and Ray].

The conclusion, which can be derived from the above construction and any of the toric generating sets $\{B(n_1, n_2)\}$ or $\{L(n_1, n_2)\}$ for Ω^U, is as follows:

Theorem (Buchstaber-P.-Ray)

In dimensions > 2, every unitary bordism class contains a quasitoric manifold, necessarily connected, whose stably complex structure is induced by an omniorientation, and is therefore compatible with the torus action.

9. Toric generators of the SU-bordism ring

Proposition

An omnioriented quasitoric manifold M has $c_1(M) = 0$ if and only if there exists a linear function $\varphi\colon \mathbb{Z}^n \to \mathbb{Z}$ such that $\varphi(\boldsymbol{a}_i) = 1$ for $i = 1, \ldots, m$. Here the $\boldsymbol{a}_i$ are the columns of characteristic matrix.
In particular, if some n vectors of $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ form the standard basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$, then M is SU iff the column sums of Λ are all equal to 1.

Corollary

A toric manifold V cannot be SU.

Proof. If $\varphi(\boldsymbol{a}_i) = 1$ for all i, then the vectors $\boldsymbol{a}_i$ lie in the positive halfspace of φ, so they cannot span a complete fan.

Theorem (Buchstaber-P.-Ray)

A quasitoric SU-manifold M^{2n} represents 0 in Ω^U_{2n} whenever $n < 5$.

Example

Assume that $n_1 = 2k_1$ is positive even and $n_2 = 2k_2 + 1$ is positive odd, and consider the manifold $L(n_1, n_2) = \mathbb{C}P(\eta \oplus \underline{\mathbb{C}}^{n_2})$. We change its stably complex structure to the following:

$$\mathcal{T} L(n_1, n_2) \oplus \underline{\mathbb{R}}^4$$
$$\cong \underbrace{p^*\bar\eta \oplus p^*\eta \oplus \cdots \oplus p^*\bar\eta \oplus p^*\eta}_{2k_1} \oplus p^*\bar\eta \oplus (\bar\gamma \otimes p^*\eta) \oplus \underbrace{\bar\gamma \oplus \gamma \oplus \cdots \oplus \bar\gamma \oplus \gamma}_{2k_2} \oplus \gamma$$

and denote the resulting stably complex manifold by $\widetilde{L}(n_1, n_2)$. It has

$$c(\widetilde{L}(n_1, n_2)) = (1 - u^2)^{k_1}(1 + u)(1 + v - u)(1 - v^2)^{k_2}(1 - v),$$

so $\widetilde{L}(n_1, n_2)$ is an SU-manifold of dimension $2(n_1 + n_2) = 4(k_1 + k_2) + 2$.

Example (continued)

$\widetilde{L}(n_1, n_2)$ is an omnioriented quasitoric manifold over $\Delta^{n_1} \times \Delta^{n_2}$ corresponding to the matrix

$$
\Lambda = \left(\begin{array}{ccccccccccc}
\multicolumn{5}{c}{\overbrace{\hphantom{1 \quad 0 \quad 0 \quad \cdots \quad 0}}^{n_1=2k_1}} & & & & & & \\
1 & 0 & 0 & \cdots & 0 & 1 & & & & & \\
0 & 1 & 0 & \cdots & 0 & -1 & & & & & \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & & \text{\Large 0} & & \\
0 & 0 & 0 & 1 & 0 & 1 & & & & & \\
0 & 0 & 0 & 0 & 1 & -1 & & & & & \\
 & & & & & 1 & 1 & 0 & \cdots & 0 & 1 \\
 & & & & & & 0 & 1 & \cdots & 0 & -1 \\
 & & \text{\Large 0} & & & & \vdots & \ddots & \ddots & 0 & \vdots \\
 & & & & & & 0 & 0 & 0 & 1 & 1 \\
 & & & & & & \multicolumn{4}{c}{\underbrace{\hphantom{0 \quad 0 \quad 0 \quad 1}}_{n_2=2k_2+1}} &
\end{array}\right)
$$

The columns sum of this matrix are 1 by inspection.

Lemma

- *For* $k > 1$, *there is a linear combination* y_{2k+1} *of* SU*-bordism classes* $[\widetilde{L}(n_1, n_2)]$ *with* $n_1 + n_2 = 2k + 1$ *such that* $s_{2k+1}(y_{2k+1}) = m_{2k+1} m_{2k}$.
- *For* $k > 2$, *there is a linear combination* y_{2k} *of* SU*-bordism classes* $[\widetilde{N}(n_1, n_2)]$ *with* $n_1 + n_2 + 1 = 2k$ *such that* $s_{2k}(y_{2k}) = 2m_{2k} m_{2k-1}$.

Theorem (Lu-P.)

There exist quasitoric SU*-manifolds* M^{2i}, $i \geqslant 5$, *with* $s_i(M^{2i}) = m_i m_{i-1}$ *if* i *is odd and* $s_i(M^{2i}) = 2m_i m_{i-1}$ *if* i *is even. These quasitoric manifolds represent polynomial generators of* $\Omega^{SU} \otimes \mathbb{Z}[\frac{1}{2}]$.

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