

Foliations arising from configurations of vectors,
Gale duality, and moment-angle manifolds
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Vector configurations and their associated foliations

$V \cong \mathbb{R}^k$ a k -dimensional real vector space

$\Gamma = \{\gamma_1, \dots, \gamma_m\}$ a **configuration** of m vectors in the dual space V^* .

Allow repetitions, but assume that $\gamma_1, \dots, \gamma_m$ span V^* .

Consider the action of V on \mathbb{R}^m given by

$$V \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$$

$$(\mathbf{v}, \mathbf{x}) \mapsto \mathbf{v} \cdot \mathbf{x} = (e^{\langle \gamma_1, \mathbf{v} \rangle} x_1, \dots, e^{\langle \gamma_m, \mathbf{v} \rangle} x_m).$$

This is a very classical dynamical system taking its origin in the works of Poincaré. There is a well-known relationship between the linear properties of Γ and the topology of the foliation of \mathbb{R}^m by the orbits of the action. We attempt for systematising the existing knowledge on this relationship and proceed by analysing the topology of the quotient (the **leaf space**) using some recent constructions of toric topology.

The above action $V \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ and its holomorphic analogue arise in several important constructions of algebraic geometry and topology:

- Topology of intersections of real and Hermitian quadrics
(topology & holomorphic dynamics)
- The quotient construction of toric varieties (the Cox construction)
(toric geometry)
- Smooth and complex-analytic structures on moment-angle manifolds
(toric topology)

Example

Consider two actions of $V = \mathbb{R}$ on \mathbb{R}^2 given by

$$(v, (x_1, x_2)) \mapsto (e^v x_1, e^v x_2), \quad (1)$$

$$(v, (x_1, x_2)) \mapsto (e^v x_1, e^{-v} x_2). \quad (2)$$

The only non-free orbit for both actions is $\mathbf{0} \in \mathbb{R}^2$, so both actions become free when restricted to $\mathbb{R}^2 \setminus \{\mathbf{0}\}$.

For (1), the quotient $(\mathbb{R}^2 \setminus \{\mathbf{0}\})/\mathbb{R}$ is a circle (a smooth manifold).

For (2), the quotient $(\mathbb{R}^2 \setminus \{\mathbf{0}\})/\mathbb{R}$ is a non-Hausdorff space.

The difference is that (1) is a **proper** action, while (2) is not.

Nondegenerate leaves

We consider invariant subsets $U \subset \mathbb{R}^m$ with the property that the restriction of the action $V \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ to U is free.

Proposition

The orbit Vx of a point $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ under the action $V \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is free iff the subset $\{\gamma_i : x_i \neq 0\} \subseteq \Gamma$ spans the whole V^ .*

Proof.

Suppose the orbit Vx is not free, i.e. there exists $\mathbf{v} \neq 0$ such that

$$(x_1 e^{\langle \gamma_1, \mathbf{v} \rangle}, \dots, x_m e^{\langle \gamma_m, \mathbf{v} \rangle}) = (x_1, \dots, x_m).$$

Then $\langle \gamma_i, \mathbf{v} \rangle = 0$ for $x_i \neq 0$, which implies that the vectors γ_i with $x_i \neq 0$ do not span V^* . The opposite statement is proved similarly. \square

Denote $[m] = \{1, \dots, m\}$ and consider subsets $I = \{i_1, \dots, i_p\} \subseteq [m]$. For each I we denote

$$\Gamma_I := \{\gamma_i : i \in I\} \subseteq \Gamma.$$

Let $\widehat{I} := [m] \setminus I$ denote the complementary subset. We set

$$\mathcal{K}(\Gamma) = \{I \subseteq [m] : \Gamma_{\widehat{I}} \text{ spans } V^*\}.$$

Proposition

$\mathcal{K}(\Gamma)$ is a pure simplicial complex of dimension $m - k - 1$.

Proof.

If $\Gamma_{\widehat{I}}$ spans V^* , then so does $\Gamma_{\widehat{J}} \supset \Gamma_{\widehat{I}}$ for any $J \subset I$. Hence, $\mathcal{K}(\Gamma)$ is a simplicial complex. Also, if $\Gamma_{\widehat{I}}$ spans V^* , then it contains a basis of V^* .

Such a basis has the form $\Gamma_{\widehat{L}}$ for some L with $I \subset L$ and

$|L| = m - |\widehat{L}| = m - k$. It follows that each face $I \in \mathcal{K}$ is contained in a $(m - k - 1)$ -dimensional face, so $\mathcal{K}(\Gamma)$ is pure $(m - k - 1)$ -dimensional. \square

Given a simplicial complex \mathcal{K} on $[m]$, define the following open subset in \mathbb{R}^m (the **complement of an arrangement of coordinate subspaces**):

$$U(\mathcal{K}) = \mathbb{R}^m \setminus \bigcup_{\{i_1, \dots, i_p\} \notin \mathcal{K}} \{\mathbf{x} : x_{i_1} = \dots = x_{i_p} = 0\}.$$

For example, if $\mathcal{K} = \{\emptyset\}$, then $U(\mathcal{K}) = (\mathbb{R}^\times)^m$, where $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$, and if \mathcal{K} consists of all proper subsets of $[m]$, then $U(\mathcal{K}) = \mathbb{R}^m \setminus \{\mathbf{0}\}$.

Proposition

For any subcomplex

$$\mathcal{K} \subseteq \mathcal{K}(\Gamma) = \{I \subseteq [m] : \Gamma_{\hat{I}} \text{ spans } V^*\},$$

the restriction of the action $V \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ to $U(\mathcal{K})$ is free.

We restate this by saying that $U(\mathcal{K})$ consists of **nondegenerate leaves** of the foliation defined by $V \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ for any $\mathcal{K} \subseteq \mathcal{K}(\Gamma)$.

Linear Gale duality

Given $\Gamma = (\gamma_1, \dots, \gamma_m)$, define a linear map $\Gamma: \mathbb{R}^m \rightarrow V^*$, $\mathbf{e}_i \mapsto \gamma_i$. Let $W := \text{Ker } \Gamma$, so we have dual exact sequences

$$\begin{aligned} 0 \longrightarrow W \longrightarrow \mathbb{R}^m \xrightarrow{\Gamma} V^* \longrightarrow 0, \\ 0 \longrightarrow V \xrightarrow{\Gamma^*} \mathbb{R}^m \xrightarrow{A} W^* \longrightarrow 0, \end{aligned}$$

where Γ^* takes \mathbf{v} to $(\langle \gamma_1, \mathbf{v} \rangle, \dots, \langle \gamma_m, \mathbf{v} \rangle)$. Set $\mathbf{a}_i := A(\mathbf{e}_i)$.

The vector configuration $A = \{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ in W^* is called the **Gale dual** of $\Gamma = \{\gamma_1, \dots, \gamma_m\}$. The Gale dual of A is Γ .

If we choose bases in V and W , then Γ becomes a $k \times m$ -matrix with columns $\gamma_1, \dots, \gamma_m$ and A becomes an $(m - k) \times m$ -matrix with columns $\mathbf{a}_1, \dots, \mathbf{a}_m$. The identity $A\Gamma^* = 0$ implies that the rows of A form a basis in the space of linear relations between the vectors $\gamma_1, \dots, \gamma_m$.

Proposition

For any $I \subseteq [m]$, the vectors in Λ_I are linearly independent in W^* iff $\Gamma_{\hat{I}}$ spans V^* .

A **simplicial cone** σ in W^* consists of nonnegative linear combinations of a set of linearly independent vectors in W^* .

A **simplicial fan** is a finite collection $\Sigma = \{\sigma_1, \dots, \sigma_s\}$ of simplicial cones such that every face of a cone in Σ belongs to Σ and the intersection of any two cones in Σ is a face of each.

Let Σ be a simplicial fan in W^* , and let $\mathbf{a}_1, \dots, \mathbf{a}_m$ be generators of one-dimensional cones of Σ . The **underlying simplicial complex** $\mathcal{K} = \mathcal{K}_{\Sigma}$ is the collection of subsets $I \subseteq [m]$ such that $\{\mathbf{a}_i : i \in I\}$ spans a cone of Σ .

A simplicial fan Σ is therefore determined by two pieces of data:

- a simplicial complex \mathcal{K} on $[m]$;
- a configuration of vectors $A = \{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ in W^* such that for any simplex $I \in \mathcal{K}$ the subset $A_I = \{\mathbf{a}_i : i \in I\}$ is linearly independent.

Conversely, given a simplicial complex \mathcal{K} and a vector configuration A , we can define the simplicial cone $\sigma_I = \text{cone}(A_I)$ for each $I \in \mathcal{K}$.

The 'bunch of cones' $\{\sigma_I : I \in \mathcal{K}\}$ patches into a fan Σ whenever any two cones σ_I and σ_J intersect in a common face (which has to be $\sigma_{I \cap J}$). Under this condition, we say that the data $\{\mathcal{K}, A\}$ **define a fan** Σ .

We have the following criterion in terms of the vector configuration $\Gamma = \{\gamma_1, \dots, \gamma_m\}$ Gale dual to A .

Theorem

Let \mathcal{K} be a simplicial complex on $[m]$, let $A = \{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ be a vector configuration in W^* such that for any simplex $I \in \mathcal{K}$ the subset A_I is linearly independent, and let $\Gamma = \{\gamma_1, \dots, \gamma_m\}$ be the Gale dual vector configuration. The following conditions are equivalent:

- (a) $\{\mathcal{K}, A\}$ define a fan Σ ;
- (b) $\text{relint cone}(A_I) \cap \text{relint cone}(A_J) = \emptyset$ for any $I, J \in \mathcal{K}$, $I \neq J$;
- (c) $\text{relint cone}(\Gamma_{\hat{I}}) \cap \text{relint cone}(\Gamma_{\hat{J}}) \neq \emptyset$ for any $I, J \in \mathcal{K}$.

Proper actions

A continuous action $G \times X \rightarrow X$, $(g, x) \mapsto g \cdot x$ of a topological group G on a topological space X is **proper** if the map $h: G \times X \rightarrow X \times X$, $(g, x) \mapsto (g \cdot x, x)$ is proper, that is, $h^{-1}(C)$ is compact for any compact $C \subseteq X \times X$.

Properness is a key property for noncompact Lie group actions:

- the quotient M/G of a proper action of a Lie group action G on a manifold M is Hausdorff;
- the quotient M/G of a smooth, free and proper action of a Lie group G on a smooth manifold M is a smooth manifold.

For our action $V \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ we have the following result:

Theorem

Let $\Gamma = \{\gamma_1, \dots, \gamma_m\}$ be a vector configuration in V^* defining the action $V \times \mathbb{R}^m \rightarrow \mathbb{R}^m$, and let $A = \{a_1, \dots, a_m\}$ be the Gale dual configuration. Let \mathcal{K} be a simplicial complex on $[m]$ such that for any $I \in \mathcal{K}$ the subset Γ_I spans V^* (equivalently, the subset A_I is linearly independent). Then

- (1) the restricted action $V \times U(\mathcal{K}) \rightarrow U(\mathcal{K})$ is free;
- (2) the action $V \times U(\mathcal{K}) \rightarrow U(\mathcal{K})$ is proper iff $\{\mathcal{K}, A\}$ define a fan.

If $\{\mathcal{K}, A\}$ define a **complete** fan in W^* (i. e. the union of all cones is the whole W^*), then the quotient $U(\mathcal{K})/V$ is a compact smooth manifold. It is known in toric topology as the **real moment-angle manifold** corresponding to \mathcal{K} .

Polytopal fans and intersections of quadrics

The **normal fan** Σ_P of a simple convex polytope P in W is an important example of a complete simplicial fan. In this case, the vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$ are the inward-pointing normals to the facets of P , and a subset A_I spans a cone iff the intersection of facets with normals \mathbf{a}_i , $i \in I$, is nonempty.

Not every complete simplicial fan is a normal fan! In fact, we have

Theorem

Let $A = \{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ and $\Gamma = \{\gamma_1, \dots, \gamma_m\}$ be a pair of Gale dual vector configurations. Assume that $\Sigma = \{\text{cone } A_I : I \in \mathcal{K}\}$ is a fan with convex support (respectively, complete fan). The following conditions are equivalent:

- (a) Σ is a normal fan of polyhedron (respectively, polytope);
- (b) $\bigcap_{I \in \mathcal{K}} \text{relint cone}(\Gamma_{\hat{I}}) \neq \emptyset$.

Therefore, the data $\{\mathcal{K}, A\}$ define a fan Σ iff the relative interiors of Gale dual cones $\text{cone } \Gamma_{\hat{\gamma}}$ have pairwise nonempty intersections, and Σ is the normal fan of a polytope iff all the cones $\text{cone } \Gamma_{\hat{\gamma}}$ have a common relative interior point.

In the polytopal case, the leaf space $U(\mathcal{K})/V$ can be described as an intersection of quadrics:

Theorem

For any $\mathbf{c} \in \bigcap_{I \in \mathcal{K}} \text{relint } \text{cone}(\Gamma_{\hat{\gamma}})$, the quotient $U(\mathcal{K})/V$ is diffeomorphic to

$$\{\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m : \gamma_1 x_1^2 + \dots + \gamma_m x_m^2 = \mathbf{c}\}.$$

Idea of proof.

The function $f: \mathbb{R}^m \rightarrow \mathbb{R}$, $f(\mathbf{x}) = \|\gamma_1 x_1^2 + \dots + \gamma_m x_m^2 - \mathbf{c}\|^2$ has a unique minimum at each orbit $V\mathbf{x}$, and the set of these minima is the intersection of quadrics above. □

Holomorphic actions

$V \cong \mathbb{C}^\ell$ a complex space (think of endowing $V \cong \mathbb{R}^k$ with a complex structure, provided that $k = 2\ell$ is even).

$\Gamma = \{\gamma_1, \dots, \gamma_m\}$ a configuration of vectors in V^* .

Consider the action of V on \mathbb{C}^m given by

$$\begin{aligned} V \times \mathbb{C}^m &\longrightarrow \mathbb{C}^m \\ (\mathbf{v}, \mathbf{z}) &\mapsto \mathbf{v} \cdot \mathbf{z} = (e^{\langle \gamma_1, \mathbf{v} \rangle} z_1, \dots, e^{\langle \gamma_m, \mathbf{v} \rangle} z_m). \end{aligned}$$

Provided that the holomorphic action $V \times U(\mathcal{K}) \rightarrow U(\mathcal{K})$ is free and proper (the *fan condition*), the quotient $\mathcal{Z}_K = U(\mathcal{K})/V$ is a complex-analytic manifold (the **complex moment-angle manifold**).

This construction leads to a new family on *non-Kähler* complex manifolds, which includes the classical series of **Hopf** and **Calabi–Eckmann manifolds**.

References (to some earlier works)

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