Double cohomology of moment-angle complexes

Joint with Tony Bahri, Ivan Limonchenko, Jongbaek Song and Donald Stanley.

Taras Panov

Moscow State University

Advances in Homotopy Theory
University of Southampton (virtual), 15–17 September 2021

1. Preliminaries

$\mathcal{K}$ a simplicial complex on $[m] = \{1, 2, \dots, m\}$, $\varnothing \in \mathcal{K}$.
$I = \{i_1, \dots, i_k\} \in \mathcal{K}$ a face (or a simplex).
Assume $\varnothing \in \mathcal{K}$ and $\{i\} \in \mathcal{K}$ for each $i = 1, \dots, m$ (no ghost vertices).

$\mathrm{CAT}(\mathcal{K})$ the face category of $\mathcal{K}$, with objects $I \in \mathcal{K}$ and morphisms $I \subset J$.
For $I \in \mathcal{K}$, consider

$$(D^2, S^1)^I \colon \{(z_1, \dots, z_m) \in (D^2)^m \colon |z_j| = 1 \text{ if } j \notin I\} \subset (D^2)^m.$$

Note that $(D^2, S^1)^I \subset (D^2, S^1)^J$ whenever $I \subset J$. Have a diagram

$$\mathscr{D}_{\mathcal{K}} \colon \mathrm{CAT}(\mathcal{K}) \to \mathrm{TOP}$$

mapping $I \in \mathcal{K}$ to $(D^2, S^1)^I$.

The moment-angle complex corresponding to $\mathcal{K}$ is

$$\mathcal{Z}_{\mathcal{K}} := \operatorname{colim} \mathscr{D}_K = \bigcup_{I \in \mathcal{K}} (D^2, S^1)^I \subset (D^2)^m.$$

The face ring of $\mathcal{K}$ is

$$\mathbb{Z}[\mathcal{K}] := \mathbb{Z}[v_1, \ldots, v_m]/\mathcal{I}_{\mathcal{K}},$$

where $\mathcal{I}_{\mathcal{K}}$ is generated by $\prod_{i\in I} v_i$ for which $I \subset [m]$ is not a simplex of $\mathcal{K}$.

Theorem

There are isomorphisms of bigraded commutative algebras

$$\begin{aligned} H^*(\mathcal{Z}_{\mathcal{K}}) &\cong \mathrm{Tor}_{\mathbb{Z}[v_1,\ldots,v_m]}(\mathbb{Z}[\mathcal{K}],\mathbb{Z}) \\ &\cong H\big(\Lambda[u_1,\ldots,u_m]\otimes\mathbb{Z}[\mathcal{K}], d\big) \\ &\cong \bigoplus_{I\subset[m]} \widetilde{H}^*(\mathcal{K}_I). \end{aligned}$$

Here $\big(\Lambda[u_1,\ldots,u_m]\otimes\mathbb{Z}[\mathcal{K}], d\big)$ *is the Koszul complex with* $\mathrm{bideg}\, u_i = (-1,2)$, $\mathrm{bideg}\, v_i = (0,2)$ *and* $du_i = v_i$, $dv_i = 0$.
$\widetilde{H}^*(\mathcal{K}_I)$ *denotes the reduced simplicial cohomology of the full subcomplex* $\mathcal{K}_I \subset \mathcal{K}$ *(the restriction of* $\mathcal{K}$ *to* $I \subset [m]$*).*

The bigraded components of the cohomology of $\mathcal{Z}_\mathcal{K}$ are given by

$$H^{-k,2\ell}(\mathcal{Z}_\mathcal{K}) \cong \bigoplus_{I\subset[m]\colon\, |I|=\ell} \widetilde{H}^{\ell-k-1}(\mathcal{K}_I), \quad H^p(\mathcal{Z}_\mathcal{K}) = \bigoplus_{-k+2\ell=p} H^{-k,2\ell}(\mathcal{Z}_\mathcal{K}).$$

Consider the following quotient of the Koszul ring $\Lambda[u_1,\ldots,u_m]\otimes\mathbb{Z}[\mathcal{K}]$:

$$R^*(\mathcal{K}) = \Lambda[u_1,\ldots,u_m]\otimes\mathbb{Z}[\mathcal{K}]\big/(v_i^2 = u_iv_i = 0,\ 1\leqslant i\leqslant m).$$

Then $R^*(\mathcal{K})$ has finite rank as an abelian group, with a basis of monomials u_Jv_I where $J\subset[m]$, $I\in\mathcal{K}$ and $J\cap I=\varnothing$.

Furthermore, $R^*(\mathcal{K})$ can be identified with the cellular cochains $C^*(\mathcal{Z}_\mathcal{K})$ of $\mathcal{Z}_\mathcal{K}$ with the standard cell decomposition, the quotient ideal $(v_i^2 = u_iv_i = 0,\ 1\leqslant i\leqslant m)$ is d-invariant and acyclic, and there is a ring isomorphism

$$H^*(\mathcal{Z}_\mathcal{K}) \cong H\bigl(R^*(\mathcal{K}), d\bigr).$$

The algebras above have the following functorial properties.

Proposition

Let $\mathcal{K}$ be a simplicial complex on m vertices, and let $\mathcal{L} \subset \mathcal{K}$ be its subcomplex on ℓ vertices. The inclusion $\mathcal{L} \subset \mathcal{K}$ induces an inclusion $\mathcal{Z}_{\mathcal{L}} \to \mathcal{Z}_{\mathcal{K}}$ and homomorphisms of (differential) graded algebras

(a) $\mathbb{Z}[\mathcal{K}] \to \mathbb{Z}[\mathcal{L}]$,

(b) $\big(\Lambda[u_1, \ldots, u_m] \otimes \mathbb{Z}[\mathcal{K}], d\big) \to \big(\Lambda[u_1, \ldots, u_\ell] \otimes \mathbb{Z}[\mathcal{L}], d\big)$,

(c) $\big(R^*(\mathcal{K}), d\big) \to \big(R^*(\mathcal{L}), d\big)$,

(d) $H^*(\mathcal{Z}_{\mathcal{K}}) \to H^*(\mathcal{Z}_{\mathcal{L}})$,

defined by sending u_i, v_i to 0 for $i \notin [\ell]$.

Furthermore, if $\mathcal{K}_I$ is a full subcomplex for some $I \subset [m]$, then we have a retraction $\mathcal{Z}_{\mathcal{K}} \to \mathcal{Z}_{\mathcal{K}_I}$ and homomorphisms

(e) $\mathbb{Z}[\mathcal{K}_I] \to \mathbb{Z}[\mathcal{K}]$,

(f) $H^*(\mathcal{Z}_{\mathcal{K}_I}) \to H^*(\mathcal{Z}_{\mathcal{K}})$.

There are also homology versions of these homomorphisms.

2. Double (co)homology

We have

$$H_p(\mathcal{Z}_\mathcal{K}) \cong \bigoplus_{I\subset[m]} \widetilde{H}_{p-|I|-1}(\mathcal{K}_I),$$

Given $j \in [m] \setminus I$, consider the homomorphism

$$\phi_{p;I,j}\colon \widetilde{H}_p(\mathcal{K}_I) \to \widetilde{H}_p(\mathcal{K}_{I\cup\{j\}})$$

induced by the inclusion $\mathcal{K}_I \hookrightarrow \mathcal{K}_{I\cup\{j\}}$. Then, we define

$$\partial'_p = (-1)^{p+1} \bigoplus_{I\subset[m],\, j\in[m]\setminus I} \varepsilon(j, I)\, \phi_{p;I,j},$$

where

$$\varepsilon(j, I) = (-1)^{\#\{i\in I\colon\, i<j\}}.$$

Lemma

$\partial'_p\colon \bigoplus_{I\subset[m]} \widetilde{H}_p(\mathcal{K}_I) \to \bigoplus_{I\subset[m]} \widetilde{H}_p(\mathcal{K}_I)$ *satisfies* $(\partial'_p)^2 = 0$.

We therefore have a chain complex

$$CH_*(\mathcal{Z}_\mathcal{K}) := (H_*(\mathcal{Z}_\mathcal{K}), \partial'),$$

where

$$\partial' \colon \widetilde{H}_{-k,2\ell}(\mathcal{Z}_\mathcal{K}) \to \widetilde{H}_{-k-1,2\ell+2}(\mathcal{Z}_\mathcal{K})$$

with respect to the following bigraded decomposition of $H_p(\mathcal{Z}_\mathcal{K})$

$$H_p(\mathcal{Z}_\mathcal{K}) = \bigoplus_{-k+2\ell=p} H_{-k,2\ell}(\mathcal{Z}_\mathcal{K}), \quad H_{-k,2\ell}(\mathcal{Z}_\mathcal{K}) \cong \bigoplus_{I\subset[m]\colon |I|=\ell} \widetilde{H}_{\ell-k-1}(\mathcal{K}_I).$$

We define the bigraded double homology of $\mathcal{Z}_\mathcal{K}$ by

$$HH_*(\mathcal{Z}_\mathcal{K}) = H(H_*(\mathcal{Z}_\mathcal{K}), \partial').$$

For the cohomological version, given $i \in I$, consider the homomorphism

$$\psi_{p;i,I}\colon \widetilde{H}^p(\mathcal{K}_I) \to \widetilde{H}^p(\mathcal{K}_{I\setminus\{i\}})$$

induced by the inclusion $\mathcal{K}_{I\setminus\{i\}} \hookrightarrow \mathcal{K}_I$, and

$$d'_p = (-1)^{p+1} \sum_{i\in I} \varepsilon(i,I)\psi_{p;i,I}.$$

We define $d'\colon H^*(\mathcal{Z}_\mathcal{K}) \to H^*(\mathcal{Z}_\mathcal{K})$ using $H^*(\mathcal{Z}_\mathcal{K}) = \bigoplus_{I\subset[m]} \widetilde{H}^*(\mathcal{K}_I)$:

$$d'\colon H^{-k,2\ell}(\mathcal{Z}_\mathcal{K}) \to H^{-k+1,2\ell-2}(\mathcal{Z}_\mathcal{K}).$$

Similarly, have $(d')^2 = 0$, which turns $H^*(\mathcal{Z}_\mathcal{K})$ into a cochain complex

$$CH^*(\mathcal{Z}_\mathcal{K}) := (H^*(\mathcal{Z}_\mathcal{K}), d').$$

We define the bigraded double cohomology of $\mathcal{Z}_\mathcal{K}$ by

$$HH^*(\mathcal{Z}_\mathcal{K}) = H\big(H^*(\mathcal{Z}_\mathcal{K}), d'\big).$$

3. The bicomplexes

Given $I \subset [m]$, let $C^p(\mathcal{K}_I)$ be the pth simplicial cochain group of $\mathcal{K}_I$.

Denote by $\alpha_{L,I} \in C^{q-1}(\mathcal{K}_I)$ the basis cochain corresponding to an oriented simplex $L = (\ell_1, \ldots, \ell_q) \in \mathcal{K}_I$; it takes value 1 on L and vanishes on all other simplices.

The simplicial coboundary map (differential) $d\colon C^p(\mathcal{K}_I) \to C^{p+1}(\mathcal{K}_I)$ is

$$d\alpha_{L,I} = \sum_{j \in I \setminus L,\, L \cup \{j\} \in \mathcal{K}} \varepsilon(j, L)\alpha_{L \cup \{j\}, I}.$$

Consider $\psi_{p;i,I}\colon C^p(\mathcal{K}_I) \to C^p(\mathcal{K}_{I \setminus \{i\}})$ induced by the inclusion $\mathcal{K}_{I \setminus \{i\}} \hookrightarrow \mathcal{K}_I$, and define

$$d'_p = (-1)^{p+1} \sum_{i \in I} \varepsilon(i, I)\psi_{p;i,I}.$$

Recall that the differential d on the Koszul complex $\Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[\mathcal{K}]$ has bidegree $(1, 0)$ and satisfies

$$du_j = v_j, \quad dv_j = 0, \qquad \text{for } j = 1, \dots, m.$$

We introduce the second differential d' of bidegree $(1, -2)$ on $\Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[\mathcal{K}]$ by setting

$$d' u_j = 1, \quad d' v_j = 0, \qquad \text{for } j = 1, \dots, m,$$

and extending by the Leibniz rule. Explicitly, the differential d' is defined on square-free monomials $u_J v_I$ by

$$d'(u_J v_I) = \sum_{j \in J} \varepsilon(j, J) u_{J \setminus \{j\}} v_I, \quad d'(v_I) = 0.$$

The differential d' is also defined by the same formula on the submodule $R^*(\mathcal{K}) \subset \Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[\mathcal{K}]$ generated by the monomials $u_J v_I$ with $J \cap I = \varnothing$. However, the ideal $\left(v_i^2 = u_i v_i = 0,\ 1 \leqslant i \leqslant m\right)$ is not d'-invariant, so $(R^*(\mathcal{K}), d')$ is not a differential graded algebra.

Lemma

With d and d' defined above, $\big(\bigoplus_{I\subset[m]} C^*(\mathcal{K}_I), d, d'\big)$, $\big(\Lambda[u_1,\ldots,u_m]\otimes\mathbb{Z}[\mathcal{K}], d, d'\big)$ and $\big(R^*(\mathcal{K}), d, d'\big)$ are bicomplexes, that is, d and d' satisfy $dd' = -d'd$.

By construction, $HH^*(\mathcal{Z}_{\mathcal{K}})$ is the first double cohomology of the bicomplex $\big(\bigoplus_{I\subset[m]} C^*(\mathcal{K}_I), d, d'\big)$:

$$HH^*(\mathcal{Z}_{\mathcal{K}}) = H\big(H\big(\bigoplus_{I\subset[m]} C^*(\mathcal{K}_I), d\big), d'\big).$$

Theorem

The bicomplexes $\left(\bigoplus_{I\subset[m]} C^*(\mathcal{K}_I), d, d'\right)$ *and* $\left(R^*(\mathcal{K}), d, d'\right)$ *are isomorphic. Therefore,* $HH^*(\mathcal{Z}_\mathcal{K})$ *is isomorphic to the first double cohomology of the bicomplex* $\left(\Lambda[u_1,\ldots,u_m]\otimes\mathbb{Z}[\mathcal{K}], d, d'\right)$*:*

$$HH^*(\mathcal{Z}_\mathcal{K}) \cong H\big(H\big(\Lambda[u_1,\ldots,u_m]\otimes\mathbb{Z}[\mathcal{K}], d\big), d'\big).$$

Proof (sketch).

Define a homomorphism

$$\begin{aligned} f\colon C^{q-1}(\mathcal{K}_I) &\longrightarrow R^{q-|I|,2|I|}(\mathcal{K}),\\ \alpha_{L,I} &\longmapsto \varepsilon(L,I)\, u_{I\setminus L} v_L, \end{aligned}$$

where $\varepsilon(L,I)=\prod_{i\in L}\varepsilon(i,I)=(-1)^{\sum_{\ell\in L}\#\{i\in I\colon\, i<\ell\}}$.
Then f is an isomorphism of free abelian groups commuting with d and d'.
That is, have an isomorphism of bicomplexes

$$f\colon \Big(\bigoplus_{I\subset[m]} C^*(\mathcal{K}_I), d, d'\Big) \longrightarrow \big(R^*(\mathcal{K}), d, d'\big).$$

□

Corollary

The double cohomology $HH^*(\mathcal{Z}_\mathcal{K})$ is a graded commutative algebra, with the product induced from the cohomology product on $H^*(\mathcal{Z}_\mathcal{K})$.

Proposition

(a) *For any $\mathcal{K}$, the d'-cohomology of $\Lambda[u_1,\ldots,u_m]\otimes\mathbb{Z}[\mathcal{K}]$ is zero:*

$$H\bigl(\Lambda[u_1,\ldots,u_m]\otimes\mathbb{Z}[\mathcal{K}],d'\bigr)=0.$$

(b) *If $\mathcal{K}\ne\Delta^{m-1}$ (the full simplex on $[m]$), then the d'-cohomology of the bicomplexes $\bigoplus_{I\subset[m]}C^*(\mathcal{K}_I)$ and $R^*(\mathcal{K})$ is zero:*

$$H\Bigl(\bigoplus_{I\subset[m]}C^*(\mathcal{K}_I),d'\Bigr)=H\bigl(R^*(\mathcal{K}),d'\bigr)=0.$$

Therefore, the second double cohomology and the total cohomology of the bicomplexes $\bigl(\bigoplus_{I\subset[m]}C^*(\mathcal{K}_I),d,d'\bigr)$ and $\bigl(R^*(\mathcal{K}),d,d'\bigr)$ is zero unless $\mathcal{K}=\Delta^{m-1}$.

(c) *If $\mathcal{K}=\Delta^{m-1}$, then the only nonzero d'-cohomology group of $\bigoplus_{I\subset[m]}C^*(\mathcal{K}_I)$ and $R^*(\mathcal{K})$ is $H^{2m}\cong\mathbb{Z}$, represented by $\alpha_{[m],[m]}$ and $v_1\cdots v_m$, respectively.*

Proposition

Let $\mathcal{L} \subset \mathcal{K}$ be simplicial complexes on the same vertex set $[m]$. There are homomorphisms

(a) $(\Lambda[u_1,\ldots,u_m] \otimes \mathbb{Z}[\mathcal{K}], d, d') \to (\Lambda[u_1,\ldots,u_\ell] \otimes \mathbb{Z}[\mathcal{L}], d, d')$,

(b) $(R^*(\mathcal{K}), d, d') \to (R^*(\mathcal{L}), d, d')$,

(c) $CH^*(\mathcal{Z}_\mathcal{K}) \to CH^*(\mathcal{Z}_\mathcal{L})$,

(d) $HH^*(\mathcal{Z}_\mathcal{K}) \to HH^*(\mathcal{Z}_\mathcal{L})$.

Furthermore, if $\mathcal{K}_I$ is a full subcomplex for some $I \subset [m]$, then we have homomorphisms

(e) $(\Lambda[u_i \colon i \in I] \otimes \mathbb{Z}[\mathcal{K}_I], d, d') \to (\Lambda[u_1,\ldots,u_m] \otimes \mathbb{Z}[\mathcal{K}], d, d')$,

(f) $(R^*(\mathcal{K}_I), d, d') \to (R^*(\mathcal{K}), d, d')$,

(g) $CH^*(\mathcal{Z}_{\mathcal{K}_I}) \to CH^*(\mathcal{Z}_\mathcal{K})$,

(h) $HH^*(\mathcal{Z}_{\mathcal{K}_I}) \to HH^*(\mathcal{Z}_\mathcal{K})$.

There are also homology versions of the homomorphisms, which map between H_*, CH_* and HH_* in the opposite direction.

4. Relation to the torus action

Given a circle action $S^1 \times X \to X$ on a space X, the induced map in cohomology has the form

$$H^*(X) \to H^*(S^1 \times X) = \Lambda[u] \otimes H^*(X), \quad \alpha \mapsto 1 \otimes \alpha + u \otimes \iota(\alpha),$$

where $u \in H^1(S^1)$ is a generator and $\iota\colon H^*(X) \to H^{*-1}(X)$ is a derivation.

Proposition

The derivation corresponding to the i^{th} coordinate circle action $S^1_i \times \mathcal{Z}_\mathcal{K} \to \mathcal{Z}_\mathcal{K}$ is induced by the derivation ι_i of the Koszul complex $(\Lambda[u_1, \ldots, u_m] \otimes \mathbb{Z}[\mathcal{K}], d)$ given on the generators by

$$\iota_i(u_j) = \delta_{ij}, \quad \iota_i(v_j) = 0, \qquad \text{for } j = 1, \ldots, m,$$

where δ_{ij} is the Kronecker delta.

The derivation corresponding to the diagonal circle action $S^1_d \times \mathcal{Z}_\mathcal{K} \to \mathcal{Z}_\mathcal{K}$ coincides with the differential d'.

Proof.

The i^{th} coordinate circle action $S^1_i \times \mathcal{Z}_\mathcal{K} \to \mathcal{Z}_\mathcal{K}$ is the map of moment-angle complexes $\varphi_\mathcal{Z} \colon \mathcal{Z}_{(\varnothing,\{i'\})\sqcup\mathcal{K}} \to \mathcal{Z}_\mathcal{K}$ induced by the simplicial map $\varphi\colon (\varnothing, \{i'\}) \sqcup \mathcal{K} \to \mathcal{K}$ sending the ghost vertex i' to i. The corresponding map of Koszul complexes is given by

$$\Lambda[u_1, \ldots, u_m] \otimes \mathbb{Z}[\mathcal{K}] \to \Lambda[u_i'] \otimes \Lambda[u_1, \ldots, u_m] \otimes \mathbb{Z}[\mathcal{K}],$$

where $u_i \mapsto u_i' + u_i = 1 \otimes u_i + u_i' \otimes 1$, $u_j \mapsto 1 \otimes u_j$ for $j \neq i$ and $v_j \mapsto 1 \otimes v_j$ for any j. Here u_i' represents the generator of $H^1(S^1_i)$. This proves the first assertion.

The second assertion follows from the fact that the derivation corresponding to the diagonal circle action is the sum of the derivations corresponding to the coordinate circle actions. □

The derivations ι_i were studied in the work of Amelotte and Briggs under the name primary cohomology operations for $\mathcal{Z}_\mathcal{K}$. We expect that their higher cohomology operations are related to the differentials in spectral sequence of the bicomplex $(\Lambda[u_1, \ldots, u_m] \otimes \mathbb{Z}[\mathcal{K}], d, d')$.

5. Techniques for computing $HH^*(\mathcal{Z}_\mathcal{K})$

There are several methods for the practical computation of $HH^*(\mathcal{Z}_\mathcal{K})$. We show that $HH^*(\mathcal{Z}_\mathcal{K})$ has rank one if and only if $\mathcal{K}$ is a full simplex, and identify several classes of simplicial complexes $\mathcal{K}$ for which $HH^*(\mathcal{Z}_\mathcal{K})$ has rank two. We also show that the rank of $HH^*(\mathcal{Z}_\mathcal{K})$ can be arbitrarily large.

Proposition

For a simplicial complex $\mathcal{K}$ on the vertex set $[m]$, the following conditions are equivalent:

(a) *all full subcomplexes of $\mathcal{K}$ are acyclic;*

(b) $\mathcal{K} = \Delta^{m-1}$ *and* $\mathcal{Z}_\mathcal{K} = (D^2)^m$;

(c) $\mathcal{Z}_\mathcal{K}$ *is acyclic;*

(d) $HH^*(\mathcal{Z}_\mathcal{K}) = HH^{0,0}(\mathcal{Z}_\mathcal{K}) = \mathbb{Z}$.

Proposition

Let $\mathcal{K} = \partial\Delta^{m-1}$, the boundary of an $(m-1)$-simplex. Then,

$$HH^{-k,2\ell}(\mathcal{Z}_{\mathcal{K}}) = \begin{cases} \mathbb{Z} & \text{for } (-k, 2\ell) = (0,0),\ (-1, 2m); \\ 0 & \text{otherwise.} \end{cases}$$

Theorem

For two simplicial complexes $\mathcal{K}$ and $\mathcal{L}$, if either $H^*(\mathcal{Z}_{\mathcal{K}})$ or $H^*(\mathcal{Z}_{\mathcal{L}})$ is free, then there is an isomorphism of chain complexes

$$CH^*(\mathcal{Z}_{\mathcal{K}*\mathcal{L}}) \cong CH^*(\mathcal{Z}_{\mathcal{K}}) \otimes CH^*(\mathcal{Z}_{\mathcal{L}}).$$

In particular, we have $HH^*(\mathcal{Z}_{\mathcal{K}*\mathcal{L}}; k) \cong HH^*(\mathcal{Z}_{\mathcal{K}}; k) \otimes HH^*(\mathcal{Z}_{\mathcal{L}}; k)$ with field coefficients.

In the previous examples $HH^*(\mathcal{Z}_\mathcal{K})$ behaved like $H^*(\mathcal{Z}_\mathcal{K})$. Here is an example of a major difference.

Theorem

Let $\mathcal{K} = \mathcal{K}' \sqcup pt$ be the disjoint union of a nonempty simplicial complex $\mathcal{K}'$ and a point. Then,

$$HH^{-k,2\ell}(\mathcal{Z}_\mathcal{K}) = \begin{cases} \mathbb{Z} & \text{for } (-k, 2\ell) = (0,0),\ (-1,4); \\ 0 & \text{otherwise.} \end{cases}$$

More generally,

Theorem

Let $\mathcal{K} = \mathcal{K}' \cup_\sigma \Delta^n$ be a simplicial complex obtained from a nonempty simplicial complex $\mathcal{K}'$ by gluing an n-simplex along a proper, possibly empty, face $\sigma \in \mathcal{K}$. Then either $\mathcal{K}$ is a simplex, or

$$HH^{-k,2\ell}(\mathcal{Z}_\mathcal{K}) = \begin{cases} \mathbb{Z} & \text{for } (-k, 2\ell) = (0,0),\ (-1,4); \\ 0 & \text{otherwise.} \end{cases}$$

Theorem

For a simplicial complex $\mathcal{K}$, the following conditions are equivalent:

(a) *all full subcomplexes of $\mathcal{K}$ are homotopy discrete sets of points;*

(b) *$\mathcal{K}$ is flag and its 1-skeleton $\mathrm{sk}^1(\mathcal{K})$ is a chordal graph;*

(c) *$\mathcal{K}$ can be obtained by iterating the procedure of attaching a simplex along a (possibly empty) face, starting from a simplex.*

Each of the conditions above implies that $\mathcal{K}$ is either a simplex, or

$$HH^{-k,2\ell}(\mathcal{Z}_{\mathcal{K}}) = \begin{cases} \mathbb{Z} & \text{for } (-k,2\ell)=(0,0),\ (-1,4); \\ 0 & \text{otherwise.} \end{cases}$$

6. The case of an m-cycle

Let $\mathcal{Z}_{\mathcal{L}}$ be the moment-angle complex corresponding to an m-cycle $\mathcal{L}$. By a result of McGavran, $\mathcal{Z}_{\mathcal{L}}$ is homeomorphic to connected sum of sphere products:

$$\mathcal{Z}_{\mathcal{L}} \cong \mathop{\#}_{k=3}^{m-1} \left(S^k \times S^{m+2-k}\right)^{\#(k-2)\binom{m-2}{k-1}}.$$

Theorem

Let $\mathcal{L}$ be an m-cycle for $m \geq 5$. Then $HH^{-k,2\ell}(\mathcal{Z}_{\mathcal{L}})$ is $\mathbb{Z}$ in bidegrees $(-k, 2\ell) = (0,0)$, $(-1,4)$, $(-m+3, 2(m-2))$, $(-m+2, 2m)$, and is 0 otherwise.

7. Further observations, examples and questions

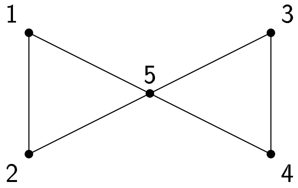
A simplicial complex $\mathcal{K}$ is wedge decomposable if it can be written as a nontrivial union $\mathcal{L} \cup_{\Delta^t} \mathcal{M}$ of two simplicial complexes $\mathcal{L}$ and $\mathcal{M}$ along a nonempty simplex Δ^t that is not the whole $\mathcal{K}$.

Question

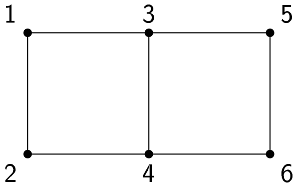
We have shown that if $\mathcal{L}$ or $\mathcal{M}$ is a simplex, then $HH^*(\mathcal{Z}_\mathcal{K}) \cong \mathbb{Z} \oplus \mathbb{Z}$ in bidegrees $(0,0)$ and $(-1,4)$. Is it true for all wedge decomposable $\mathcal{K}$? Do other $\mathcal{K}$ also have this property?

We know that $HH^*(\mathcal{Z}_\mathcal{K}) \cong \mathbb{Z} \oplus \mathbb{Z}$ when $\mathcal{K}$ is the boundary of a simplex, but the bidegrees are different.

Here are two examples of simplicial complexes $\mathcal{K}$ with $HH^*(\mathcal{Z}_\mathcal{K}) \cong \mathbb{Z} \oplus \mathbb{Z}$ in bidegrees $(0,0)$ and $(-1,4)$. They are not covered by the previous results, but are wedge decomposable.



(a) Non-flag and chordal.



(b) Flag and non-chordal

Question

Suppose $\mathcal{K}$ is a triangulated sphere or a Gorenstein complex of dimension $n-1$. Does the double cohomology $HH^*(\mathcal{Z}_\mathcal{K})$ satisfy bigraded Poincaré duality, like $H^*(\mathcal{Z}_\mathcal{K})$? In particular, do the ranks of the bigraded double cohomology groups satisfy*

$$\operatorname{rank} HH^{-k,2\ell}(\mathcal{Z}_\mathcal{K}) = \operatorname{rank} HH^{-(m-n)+k,2(m-\ell)}(\mathcal{Z}_\mathcal{K})?$$

This is the case when $\mathcal{K}$ is an ℓ-cycle.