Complex geometry of manifolds with torus action

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Symplectic reduction and moment-angle manifolds

An m-torus T^m acts on $\mathbb{C}^m$ coordinatewise. This is a Hamiltonian torus action with respect to $\omega = i\sum_{k=1}^m dz_k \wedge d\bar z_k$, with the moment map

$$\mu\colon \mathbb{C}^m \to \mathbb{R}^m = \mathrm{Lie}(T^m)^*, \qquad (z_1,\ldots,z_m) \mapsto (|z_1|^2,\ldots,|z_m|^2).$$

A Hamiltonian toric manifold M^{2n} is the symplectic quotient $\mathbb{C}^m/\!/K$ by an $(m-n)$-dimensional subtorus $K \subset T^m$. It has a residual Hamiltonian action of $T^m/K \cong T^n$.

In more detail, the moment map for the K-action on $\mathbb{C}^m$ is the composite

$$\mu_K\colon \mathbb{C}^m \xrightarrow{\mu} \mathbb{R}^m \to \mathfrak{k}^*.$$

Take a regular value $\delta \in \mathfrak{k}^* \cong \mathbb{R}^{m-n}$. Then $M^{2n} = \mu_K^{-1}(\delta)/K$. It has a symplectic form ω' satisfying $p^*\omega' = i^*\omega$, where $p\colon \mu_K^{-1}(\delta) \to M^{2n}$ and $i\colon \mu_K^{-1}(\delta) \hookrightarrow \mathbb{C}^m$.

We refer to $\mathcal{Z} := \mu_K^{-1}(\delta)$ as a (polytopal) moment-angle manifold.

It can be written as an intersection of $(m-n)$ Hermitian quadrics in $\mathbb{C}^m$:

$$\mathcal{Z} = \Big\{(z_1,\dots,z_m)\in\mathbb{C}^m\colon\ \sum_{k=1}^m \gamma_{jk}|z_k|^2=\delta_j,\qquad j=1,\dots,m-n\Big\}.$$

The quotient $\mathcal{Z}/T^m = M^{2n}/T^n$ is a convex polytope in $\mathrm{Lie}(T^n)^*\subset\mathbb{R}^m$ (the moment polytope) given by

$$P = \Big\{(y_1,\dots,y_m)\in\mathbb{R}^m_{\geqslant}\colon\ \sum_{k=1}^m \gamma_{jk}y_k=\delta_j,\qquad j=1,\dots,m-n\Big\}.$$

Its facet normals $\mathbf{a}_1,\dots,\mathbf{a}_m$ form the Gale dual configuration to $\gamma_1,\dots,\gamma_m\in\mathfrak{k}^*$. They satisfy the Delzant condition: $\{\mathbf{a}_{i_1},\dots,\mathbf{a}_{i_n}\}$ is a lattice basis whenever the facets $F_{i_1},\dots,F_{i_n}$ intersect at a vertex.

Now consider an arbitrary (not necessarily rational) polytope

$$P = \{\mathbf{x} \in \mathbb{R}^n \colon \langle \mathbf{a}_i, \mathbf{x} \rangle + b_i \geqslant 0 \quad \text{for } i = 1, \ldots, m\}.$$

Define an affine map

$$i_P \colon \mathbb{R}^n \to \mathbb{R}^m, \quad i_P(\mathbf{x}) = \big(\langle \mathbf{a}_1, \mathbf{x} \rangle + b_1, \ldots, \langle \mathbf{a}_m, \mathbf{x} \rangle + b_m\big).$$

Then $i_P(P)$ is the intersection of an n-plane with $\mathbb{R}^m_{\geqslant} = \{\mathbf{y} = (y_1, \ldots, y_m) \colon y_i \geqslant 0\}$.

Define the space $\mathcal{Z}_P$ from the diagram

$$\begin{array}{ccccc} \mathcal{Z}_P & \xrightarrow{i_Z} & \mathbb{C}^m & & (z_1, \ldots, z_m) \\ \big\downarrow & & \big\downarrow \mu & & \big\downarrow \\ P & \xrightarrow{i_P} & \mathbb{R}^m_{\geqslant} & & (|z_1|^2, \ldots, |z_m|^2) \end{array}$$

$\mathcal{Z}_P$ has a T^m-action, $\mathcal{Z}_P / T^m = P$, and i_Z is a T^m-equivariant inclusion.

Proposition

If P is a simple polytope (more generally, if the presentation of P by inequalities is generic), then $\mathcal{Z}_P$ is a smooth manifold of dimension $m+n$.

Proof.

Write $i_P(\mathbb{R}^n)$ by $m-n$ linear equations in $(y_1,\ldots,y_m)\in\mathbb{R}^m$. Replace y_k by $|z_k|^2$ to obtain a presentation of $\mathcal{Z}_P$ by quadrics. □

$\mathcal{Z}_P$: polytopal moment-angle manifold corresponding to P.

When P is a Delzant (in particular, rational) polytope, $\mathcal{Z}_P$ is the level set $\mu_K^{-1}(\delta)$ of the moment map for a subtorus $K\subset T^m$ given by

$$K=\mathrm{Ker}(q\colon T^m\to T^n),\quad q\colon\mathbb{R}^m\to\mathbb{R}^n,\quad \mathbf{e}_i\mapsto\mathbf{a}_i.$$

The moment-angle complex (as a polyhedral product)

$\mathcal{K}$ an abstract simplicial complex on the set $[m] = \{1, 2, \ldots, m\}$
$I = \{i_1, \ldots, i_k\} \in \mathcal{K}$ a simplex; always assume $\varnothing \in \mathcal{K}$.

Consider the unit m-dimensional polydisc:

$$\mathbb{D}^m = \{(z_1, ..., z_m) \in \mathbb{C}^m : |z_i|^2 \leqslant 1 \text{ for } i = 1, ..., m\}.$$

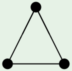
The moment-angle complex is

$$\mathcal{Z}_\mathcal{K} := \bigcup_{I \in \mathcal{K}} \Bigl(\prod_{i \in I} \mathbb{D} \times \prod_{i \notin I} \mathbb{S} \Bigr) \subset \mathbb{D}^m,$$

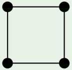
where $\mathbb{S}$ is the boundary of the unit disk $\mathbb{D}$.

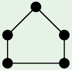
$\mathcal{Z}_\mathcal{K}$ has a natural action of the torus T^m.
When $\mathcal{K}$ is simplicial subdivision of a sphere (e.g., the boundary of a simplicial polytope), $\mathcal{Z}_\mathcal{K}$ is a topological manifold, called the moment-angle manifold.

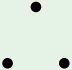
Example

1. Let $\mathcal{K} =$  (the boundary of a triangle). Then

$$\mathcal{Z}_{\mathcal{K}} = (\mathbb{D} \times \mathbb{D} \times \mathbb{S}) \cup (\mathbb{D} \times \mathbb{S} \times \mathbb{D}) \cup (\mathbb{S} \times \mathbb{D} \times \mathbb{D}) = \partial(\mathbb{D}^3) \cong S^5.$$

2. Let $\mathcal{K} =$  (the boundary of a square). Then $\mathcal{Z}_{\mathcal{K}} \cong S^3 \times S^3$.

3. Let $\mathcal{K} =$  Then $\mathcal{Z}_{\mathcal{K}} \cong (S^3 \times S^4) \# \cdots \# (S^3 \times S^4)$ (5 times).

4. Let $\mathcal{K} =$  (three disjoint points). Then

$$\mathcal{Z}_{\mathcal{K}} = (\mathbb{D} \times \mathbb{S} \times \mathbb{S}) \cup (\mathbb{S} \times \mathbb{D} \times \mathbb{S}) \cup (\mathbb{S} \times \mathbb{S} \times \mathbb{D}) \simeq S^3 \vee S^3 \vee S^3 \vee S^4 \vee S^4$$

(not a manifold).

We define an open submanifold $U(\mathcal{K}) \subset \mathbb{C}^m$ in a similar way:

$$U(\mathcal{K}) := \bigcup_{I \in \mathcal{K}} \Big(\prod_{i \in I} \mathbb{C} \times \prod_{i \notin I} \mathbb{C}^\times \Big), \qquad \mathbb{C}^\times = \mathbb{C} \setminus \{0\}.$$

$U(\mathcal{K})$ is a toric variety with the corresponding fan given by

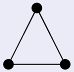
$$\Sigma_{\mathcal{K}} = \{\mathbb{R}_\geqslant \langle \mathbf{e}_i \colon i \in I \rangle \colon I \in \mathcal{K}\},$$

where $\mathbf{e}_i$ denotes the i-th standard basis vector of $\mathbb{R}^m$.

Theorem

(a) $U(\mathcal{K}) = \mathbb{C}^m \setminus \bigcup_{\{i_1,\dots,i_k\} \notin \mathcal{K}} \{z_{i_1} = \dots = z_{i_k} = 0\}$ *(the complement to a coordinate subspace arrangement);*

(b) *There is a T^m-equivariant deformation retraction $U(\mathcal{K}) \xrightarrow{\simeq} \mathcal{Z}_{\mathcal{K}}$.*

E.g., $\mathcal{K} =$  Then $U(\mathcal{K}) = \mathbb{C}^3 \setminus \{z_1 = z_2 = z_3 = 0\} \xrightarrow{\simeq} S^5 = \mathcal{Z}_{\mathcal{K}}$.

Complex-analytic structrures on moment-angle manifolds

General approach: realise the deformation retraction $U(\mathcal{K}) \to \mathcal{Z}_\mathcal{K}$ as the orbit quotient map for a holomorphic, free and proper action of a complex-analytic subgroup $H \subset (\mathbb{C}^\times)^m$, i. e. $\mathcal{Z}_\mathcal{K} = U(\mathcal{K})/H$. This will make $\mathcal{Z}_\mathcal{K}$ into a compact complex manifold.

Let $\mathcal{K}$ be a sphere triangulation, i.e. $|\mathcal{K}| \cong S^{n-1}$.
A realisation $|\mathcal{K}| \subset \mathbb{R}^n$ is starshaped if there is a point $\mathbf{x} \notin |\mathcal{K}|$ such that any ray from $\mathbf{x}$ intersects $|\mathcal{K}|$ in exactly one point.

A convex triangulation $\mathcal{K}_P$ is starshaped, but not vice versa!

$\mathcal{K}$ has a starshaped realisation if and only if it is the underlying complex of a complete simplicial fan Σ.

$\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}^n$ the generators of the 1-dim cones of Σ. Define a map

$$q \colon \mathbb{R}^m \to \mathbb{R}^n, \quad \mathbf{e}_i \mapsto \mathbf{a}_i.$$

Set $\mathbb{R}^m_> = \{(y_1, \ldots, y_m) \in \mathbb{R}^m \colon y_i > 0\}$ and define

$$R := \exp(\operatorname{Ker} q) = \Bigl\{(y_1, \ldots, y_m) \in \mathbb{R}^m_> \colon \prod_{i=1}^m y_i^{\langle \mathbf{a}_i, \mathbf{u}\rangle} = 1 \text{ for all } \mathbf{u} \in \mathbb{R}^n\Bigr\},$$

$R \subset \mathbb{R}^m_>$ acts on $U(\mathcal{K}) \subset \mathbb{C}^m$ by coordinatewise multiplications.

Theorem

Let Σ be a complete simplicial fan in $\mathbb{R}^n$ with m one-dimensional cones, and let $\mathcal{K} = \mathcal{K}_\Sigma$ be its underlying simplicial complex. Then

(a) *the group $R \cong \mathbb{R}^{m-n}$ acts on $U(\mathcal{K})$ freely and properly, so the quotient $U(\mathcal{K})/R$ is a smooth $(m+n)$-dimensional manifold;*

(b) *$U(\mathcal{K})/R$ is T^m-equivariantly homeomorphic to $\mathcal{Z}_\mathcal{K}$.*

Therefore, $\mathcal{Z}_\mathcal{K}$ can be smoothed canonically.

Assume $m-n$ is even and set $\ell=\frac{m-n}{2}$.

Choose a linear map $\psi\colon\mathbb{C}^\ell\to\mathbb{C}^m$ satisfying the two conditions:

(a) $\mathrm{Re}\circ\psi\colon\mathbb{C}^\ell\to\mathbb{R}^m$ is a monomorphism;

(b) $q\circ\mathrm{Re}\circ\psi=0$.

$$\begin{array}{ccccccc}
\mathbb{C}^\ell & \xrightarrow{\ \psi\ } & \mathbb{C}^m & \xrightarrow{\ \mathrm{Re}\ } & \mathbb{R}^m & \xrightarrow{\ q\ } & \mathbb{R}^n\\
 & & \big\downarrow \exp & & \big\downarrow \exp & & \big\downarrow \exp\\
 & & (\mathbb{C}^\times)^m & \xrightarrow{\ |\cdot|\ } & \mathbb{R}^m_> & \xrightarrow{\ \exp q\ } & \mathbb{R}^n_>
\end{array}$$

here $|\cdot|$ denotes the map $(z_1,\ldots,z_m)\mapsto(|z_1|,\ldots,|z_m|)$. Now set

$$H=\exp\psi(\mathbb{C}^\ell)=\bigl\{\bigl(e^{\langle\psi_1,\mathbf{w}\rangle},\ldots,e^{\langle\psi_m,\mathbf{w}\rangle}\bigr)\in(\mathbb{C}^\times)^m\bigr\}$$

where $\mathbf{w}=(w_1,\ldots,w_\ell)\in\mathbb{C}^\ell$.

Then $H\cong\mathbb{C}^\ell$ is a complex-analytic (but not algebraic) subgroup of $(\mathbb{C}^\times)^m$. It acts on $U(\mathcal{K})$ by holomorphic transformations.

Example (holomorphic tori)

Let $\mathcal{K}$ be empty on 2 elements (that is, $\mathcal{K}$ has two ghost vertices). We therefore have $n = 0$, $m = 2$, $\ell = 1$, and $q\colon \mathbb{R}^2 \to 0$ is a zero map.

Let $\psi\colon \mathbb{C} \to \mathbb{C}^2$ be given by $z \mapsto (z, \alpha z)$ for some $\alpha \in \mathbb{C}$, so that

$$H = \{(e^z, e^{\alpha z})\} \subset (\mathbb{C}^\times)^2.$$

Condition (b) above is void, while (a) is equivalent to $\alpha \notin \mathbb{R}$. Then $\exp\psi\colon H \to (\mathbb{C}^\times)^2$ is an embedding, and the quotient $(\mathbb{C}^\times)^2/H$ is a complex torus $T^2_{\mathbb{C}}$ with parameter $\alpha \in \mathbb{C}$:

$$(\mathbb{C}^\times)^2/H \cong \mathbb{C}/(\mathbb{Z} \oplus \alpha\mathbb{Z}) = T^2_{\mathbb{C}}(\alpha).$$

Similarly, if $\mathcal{K}$ is empty on 2ℓ elements (so that $n = 0$, $m = 2\ell$), we can obtain any complex torus $T^{2\ell}_{\mathbb{C}}$ as the quotient $(\mathbb{C}^\times)^{2\ell}/H$.

Theorem (P.-Ustinovsky)

Let Σ be a complete simplicial fan in $\mathbb{R}^n$ with m one-dimensional cones, and let $\mathcal{K} = \mathcal{K}_\Sigma$ be its underlying simplicial complex. Assume that $m - n = 2\ell$. Then

(a) *the holomorphic action of the group $H \cong \mathbb{C}^\ell$ on $U(\mathcal{K})$ is free and proper, so the quotient $U(\mathcal{K})/H$ is a compact complex $(m-\ell)$-manifold;*

(b) *there is a T^m-equivariant diffeomorphism $U(\mathcal{K})/H \cong \mathcal{Z}_\mathcal{K}$ defining a complex structure on $\mathcal{Z}_\mathcal{K}$ in which T^m acts by holomorphic transformations.*

Conversely, assume $\mathcal{Z}_{\mathcal{K}}$ admits a T^m-invariant complex structure. Then the T^m-action extends to a holomorphic action of $(\mathbb{C}^\times)^m$ on $\mathcal{Z}_{\mathcal{K}}$. Have a complex-analytic subgroup of global stabilisers

$$H = \{g \in (\mathbb{C}^\times)^m \colon g \cdot x = x \text{ for all } x \in \mathcal{Z}_{\mathcal{K}}\}.$$

$\mathfrak{h} = \mathrm{Lie}(H)$ is a complex subalgebra of $\mathrm{Lie}(\mathbb{C}^\times)^m = \mathbb{C}^m$ and satisfies

(a) the composite $\mathfrak{h} \hookrightarrow \mathbb{C}^m \xrightarrow{\mathrm{Re}} \mathbb{R}^m$ is injective;

(b) the quotient map $q \colon \mathbb{R}^m \to \mathbb{R}^m / \mathrm{Re}(\mathfrak{h})$ sends the fan $\Sigma_{\mathcal{K}}$ to a complete fan $q(\Sigma_{\mathcal{K}})$ in $\mathbb{R}^m / \mathrm{Re}(\mathfrak{h})$.

Theorem (Ishida)

Every complex moment-angle manifold $\mathcal{Z}_{\mathcal{K}}$ is T^m-equivariantly biholomorphic to the quotient manifold $U(\mathcal{K})/H$.

Thus, $\mathcal{Z}_{\mathcal{K}}$ admits a complex structure if and only if $\mathcal{K}$ is the underlying complex of a complete simplicial fan (i. e., a star-shaped sphere).

Example (Hopf manifold)

Let Σ be a complete fan in $\mathbb{R}^n$ whose cones are generated by all proper subsets of $n+1$ vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n, -\mathbf{e}_1 - \ldots - \mathbf{e}_n$.

Add one 'empty' 1-cone to make $m-n$ even: $m = n+2$, $\ell = 1$.
Then $q\colon \mathbb{R}^{n+2} \to \mathbb{R}^n$ is given by the matrix $(\mathbf{0}\ I\ -\mathbf{1})$, where I is the unit $n \times n$ matrix, and $\mathbf{0}$, $\mathbf{1}$ are the n-columns of zeros and units respectively.

The underlying complex $\mathcal{K} = \partial\Delta^n$ with $n+1$ vertices and 1 ghost vertex, $\mathcal{Z}_\mathcal{K} \cong S^1 \times S^{2n+1}$, and $U(\mathcal{K}) = \mathbb{C}^\times \times (\mathbb{C}^{n+1} \setminus \{0\})$.

Take $\psi\colon \mathbb{C} \to \mathbb{C}^{n+2}$, $z \mapsto (z, \alpha z, \ldots, \alpha z)$ for some $\alpha \in \mathbb{C}$, $\alpha \notin \mathbb{R}$. Then

$$H = \bigl\{(e^z, e^{\alpha z}, \ldots, e^{\alpha z})\colon z \in \mathbb{C}\bigr\} \subset (\mathbb{C}^\times)^{n+2},$$

and $\mathcal{Z}_\mathcal{K}$ acquires a complex structure as the quotient $U(\mathcal{K})/H$:

$$\mathbb{C}^\times \times \bigl(\mathbb{C}^{n+1} \setminus \{0\}\bigr) / \ \{(t, \mathbf{w}) \sim (e^z t, e^{\alpha z}\mathbf{w})\} \cong \bigl(\mathbb{C}^{n+1} \setminus \{0\}\bigr) / \ \{\mathbf{w} \sim e^{2\pi i\alpha}\mathbf{w}\},$$

where $t \in \mathbb{C}^\times$, $\mathbf{w} \in \mathbb{C}^{n+1} \setminus \{0\}$. The Hopf manifold.

A holomorphic foliation on $\mathcal{Z}_\mathcal{K}$

Define the Lie subalgebra and the corresponding Lie group

$$\mathfrak{k} = \mathrm{Re}(\mathfrak{h}) \subset \mathbb{R}^m = \mathrm{Lie}(T^m), \qquad K = \exp(\mathfrak{k}) \subset T^m.$$

The restriction of the T^m-action on $U(\mathcal{K})/H$ to $K \subset T^m$ is almost free. We obtain a *holomorphic* foliation $\mathcal{F}$ on $\mathcal{Z}_\mathcal{K}$ by the orbits of K.

If the subspace $\mathfrak{k} \subset \mathbb{R}^m$ is rational (i. e., generated by integer vectors), then K is a subtorus of T^m and the complete simplicial fan $\Sigma := q(\Sigma_\mathcal{K})$ is rational. The rational fan Σ defines a toric variety

$$V_\Sigma = \mathcal{Z}_\mathcal{K}/K = U(\mathcal{K})/K_\mathbb{C}.$$

The holomorphic foliation of $\mathcal{Z}_\mathcal{K}$ by the orbits of K becomes a holomorphic Seifert fibration over the toric orbifold V_Σ with fibres compact complex tori $K_\mathbb{C}/H \cong T^{m-n}$.

The rational case:

$$
\begin{array}{ccccc}
\mathbb{C}^m \supset & & U(\mathcal{K}) & & \\
& K_{\mathbb{C}} \cong (\mathbb{C}^\times)^{m-n} & \downarrow & \searrow^{H \cong \mathbb{C}^\ell} & \\
& & & & \mathcal{Z}_{\mathcal{K}} \quad \text{moment-angle manifold} \\
& & \downarrow & \swarrow_{K_{\mathbb{C}}/H \cong T^{m-n}} & \\
\text{toric variety} & & V_\Sigma & &
\end{array}
$$

The non-rational case:

Have $U(\mathcal{K}) \xrightarrow{H} \mathcal{Z}_{\mathcal{K}}$,

and a holomorphic foliation $\mathcal{F}$ of $\mathcal{Z}_{\mathcal{K}}$ by the orbits of $K \subset T^m$.

The holomorphic foliated manifold $(\mathcal{Z}_{\mathcal{K}}, \mathcal{F})$ is a model for 'non-commutative' toric varieties in the sense of [Katzarkov, Lupercio, Meersseman, Verjovsky] (arXiv:1308.2774) and [Ratiu, Zung] (arXiv:1705.11110).

De Rham and Dolbeault cohomology

The face ring (the Stanley–Reisner ring) of $\mathcal{K}$ is

$$\mathbb{C}[\mathcal{K}] := \mathbb{C}[v_1, ..., v_m]/I_{\mathcal{K}} = \mathbb{C}[v_1, ..., v_m]/(v_{i_1} \cdots v_{i_k} : \{i_1, \ldots, i_k\} \notin \mathcal{K}),$$

where $\mathbb{C}[v_1, ..., v_m]$ is the polynomial algebra, $\deg v_i = 2$, and $I_{\mathcal{K}}$ is the Stanley–Reisner ideal.

Proposition

The T^m-equivariant cohomology is given by

$$H^*_{T^m}(\mathcal{Z}_{\mathcal{K}}) = H^*_{T^m}(U(\mathcal{K})) \cong \mathbb{C}[\mathcal{K}].$$

The toric variety V_Σ is Kähler (equivalently, projective) if and only if Σ is the normal fan of lattice (Delzant) polytope P.

Theorem (Danilov)

The Dolbeault cohomology of V_Σ is given by

$$H^{*,*}_{\bar\partial}(V_\Sigma) \cong \mathbb{C}[v_1, ..., v_m]/(I_\mathcal{K} + J_\Sigma),$$

where $v_i \in H^{1,1}_{\bar\partial}(V_\Sigma)$, $I_\mathcal{K}$ is the Stanley–Reisner ideal,
J_Σ is the ideal generated by the linear forms $\sum_{k=1}^m \langle \mathbf{a}_k, \mathbf{u}\rangle v_k$,
$\mathbf{a}_k = q(\mathbf{e}_k)$ are the generators of 1-dim cones of Σ, $\mathbf{u} \in (\mathbb{R}^m/\mathfrak{k})^*$.

The nonzero Hodge numbers are given by $h^{p,p}(V_\Sigma) = h_p$,
where $h(\Sigma) = (h_0, h_1, \ldots, h_n)$ is the *h*-vector of Σ.

Theorem (Buchstaber-P.)

The de Rham cohomology ring of $\mathcal{Z}_{\mathcal{K}}$ is given by

$$\begin{aligned} H^*(\mathcal{Z}_{\mathcal{K}}) &\cong \mathrm{Tor}_{\mathbb{C}[v_1,\ldots,v_m]}(\mathbb{C}[\mathcal{K}],\mathbb{C}) \\ &\cong H\bigl(\Lambda[u_1,\ldots,u_m]\otimes\mathbb{C}[\mathcal{K}],d\bigr) \qquad du_i=v_i,\ dv_i=0 \\ &\cong H\bigl(\Lambda[t_1,\ldots,t_{m-n}]\otimes H^*(V_\Sigma),d\bigr) \qquad \Lambda[t_1,\ldots,t_{m-n}]=H^*(K) \\ &\cong \bigoplus_{I\subset[m]} \widetilde{H}^{*-|I|-1}(\mathcal{K}_I). \end{aligned}$$

Theorem (P.-Ustinovsky)

Let Σ be a rational fan, $\mathcal{Z}_\mathcal{K} \xrightarrow{K} V_\Sigma$ a holomorphic torus fibration. Then the Dolbeault cohomology ring of $\mathcal{Z}_\mathcal{K}$ is given by

$$H^{*,*}_{\bar\partial}(\mathcal{Z}_\mathcal{K}) \cong H\bigl(\Lambda[\xi_1,\dots,\xi_\ell,\eta_1,\dots,\eta_\ell] \otimes H^{*,*}_{\bar\partial}(V_\Sigma), d\bigr),$$

where $\Lambda[\xi_1,\dots,\xi_\ell,\eta_1,\dots,\eta_\ell] = H^{*,*}_{\bar\partial}(K)$, $\xi_j \in H^{1,0}_{\bar\partial}(K)$, $\eta_j \in H^{0,1}_{\bar\partial}(K)$,
$dv_j = d\eta_j = 0$, $d\xi_j = c(\xi_j)$,
$c\colon H^{1,0}_{\bar\partial}(K) \to H^{1,1}_{\bar\partial}(V_\Sigma)$ is the first Chern class map.

Corollary

(a) *The Borel spectral sequence of the holomorphic fibration $\mathcal{Z}_\mathcal{K} \xrightarrow{K} V_\Sigma$ (converging to Dolbeault cohomology of $\mathcal{Z}_\mathcal{K}$) collapses at the E_3 page;*

(b) *The Frölicher spectral sequence (with $E_1 = H^{*,*}_{\bar\partial}(\mathcal{Z}_\mathcal{K})$, converging to $H^*(\mathcal{Z}_\mathcal{K})$) collapses at E_2.*

Transverse Kähler form and analytic subsets

The complex structure on $\mathcal{Z}_\mathcal{K}$ is determined by two pieces of data:

- a complete simplicial fan Σ with generators $\mathbf{a}_1, \ldots, \mathbf{a}_m$;
- an ℓ-dimensional holomorphic subgroup $H \subset (\mathbb{C}^\times)^m$.

If this data is *generic* (in particular, the fan Σ is not rational), then there is no holomorphic principal torus fibration $\mathcal{Z}_\mathcal{K} \to V_\Sigma$ over a toric variety V_Σ.

Instead, there is a holomorphic ℓ-dimensional *foliation* $\mathcal{F}$, which sometimes admits a transverse Kähler form $\omega_\mathcal{F}$. This form can be used to describe submanifolds and analytic subsets in $\mathcal{Z}_\mathcal{K}$.

A $(1,1)$-form $\omega_{\mathcal{F}}$ on the complex manifold $\mathcal{Z}_{\mathcal{K}}$ is transverse Kähler with respect to the foliation $\mathcal{F}$ if

(a) $\omega_{\mathcal{F}}$ is closed, i. e. $d\omega_{\mathcal{F}} = 0$;

(b) $\omega_{\mathcal{F}}$ is nonnegative and the zero space of $\omega_{\mathcal{F}}$ is the tangent space of $\mathcal{F}$.

A complete simplicial fan Σ in $\mathbb{R}^n$ is weakly normal if there exists a (not necessarily simple) n-dimensional polytope P such that Σ is a simplicial subdivision of the normal fan Σ_P.

Theorem (P.–Ustinovsky–Verbitsky)

Assume that Σ is a weakly normal fan. Then there exists an exact $(1,1)$-form $\omega_{\mathcal{F}}$ on $\mathcal{Z}_{\mathcal{K}} = U(\mathcal{K})/H$ which is transverse Kähler for the foliation $\mathcal{F}$ on the dense open subset $(\mathbb{C}^\times)^m/H \subset U(\mathcal{K})/H$.

If there is a transverse Kähler form defined *on the whole of $\mathcal{Z}_{\mathcal{K}}$*, then Σ is a normal fan of a simple polytope [Ishida], and $\mathcal{Z}_{\mathcal{K}}$ can be written as an intersection of Hermitian quadrics as in the beginning of the talk.

For each $J \subset [m]$, the coordinate submanifold of $\mathcal{Z}_\mathcal{K}$ is

$$\mathcal{Z}_{\mathcal{K}_J} = \{(z_1, \ldots, z_m) \in \mathcal{Z}_\mathcal{K} \colon z_i = 0 \quad \text{for } i \notin J\}.$$

The closure of any $(\mathbb{C}^\times)^m$-orbit of $U(\mathcal{K})$ has the form $U(\mathcal{K}_J)$ for some $J \subset [m]$ (in particular, the dense orbit corresponds to $J = [m]$). Similarly, the closure of any $(\mathbb{C}^\times)^m/C$-orbit of $\mathcal{Z}_\mathcal{K} \cong U(\mathcal{K})/C$ has the form $\mathcal{Z}_{\mathcal{K}_J}$.

Theorem (P.–Ustinovsky–Verbitsky)

Assume that the data defining a complex structure on $\mathcal{Z}_\mathcal{K} = U(\mathcal{K})/C$ is generic. Then any divisor of $\mathcal{Z}_\mathcal{K}$ is a union of coordinate divisors.

Furthermore, if Σ is a weakly normal fan, then any compact irreducible analytic subset $Y \subset \mathcal{Z}_\mathcal{K}$ of positive dimension is a coordinate submanifold.

Corollary

Under generic assumptions, there are no non-constant meromorphic functions on $\mathcal{Z}_\mathcal{K}$ (i. e. the algebraic dimension of $\mathcal{Z}_\mathcal{K}$ is zero).

Basic cohomology

M a manifold with an action of a connected Lie group G, $\mathfrak{g} = \mathrm{Lie}\, G$.

$$\Omega(M)_{\mathrm{bas},\, G} = \{\omega \in \Omega(M)\colon \iota_\xi \omega = L_\xi \omega = 0 \text{ for any } \xi \in \mathfrak{g}\},$$

$H^*_{\mathrm{bas},\, G}(M) = H(\Omega(M)_{\mathrm{bas},\, G}, d)$ the basic cohomology of M.

$S(\mathfrak{g}^*)$ the symmetric algebra on $\mathfrak{g}^*$ with generators of degree 2.
The Cartan model is

$$\mathcal{C}_{\mathfrak{g}}(\Omega(M)) = ((S(\mathfrak{g}^*) \otimes \Omega(M))^{\mathfrak{g}}, d_{\mathfrak{g}}),$$

where $(S(\mathfrak{g}^*) \otimes \Omega(M))^{\mathfrak{g}}$ denotes the $\mathfrak{g}$-invariant subalgebra.
An element $\omega \in \mathcal{C}_{\mathfrak{g}}(\Omega(M))$ is a "$\mathfrak{g}$-equivariant polynomial map from $\mathfrak{g}$ to $\Omega(M)$". The differential $d_{\mathfrak{g}}$ is given by

$$d_{\mathfrak{g}}(\omega)(\xi) = d(\omega(\xi)) - \iota_\xi(\omega(\xi)).$$

Theorem

$$H^*_{\mathrm{bas},\,G}(M) \cong H\big(\mathcal{C}_{\mathfrak{g}}(\Omega(M)), d_{\mathfrak{g}}\big).$$

If in addition G is a compact, then

$$H^*_{\mathrm{bas},\,G}(M) \cong H^*_G(M) = H^*(EG \times_G M) \quad \text{the equivariant cohomology.}$$

Now consider $\mathcal{Z}_\mathcal{K}$ with the action of K (a holomorphic foliation $\mathcal{F}$).

Theorem (Ishida–Krutowski–P.)

There is an isomorphism of algebras:

$$H^*_{\mathrm{bas},\,K}(\mathcal{Z}_\mathcal{K}) \cong \mathbb{C}[v_1, \ldots, v_m]/(I_\mathcal{K} + J_\Sigma),$$

where $I_\mathcal{K}$ is the Stanley–Reisner ideal of $\mathcal{K}$, generated by the monomials

$$v_{i_1} \cdots v_{i_k} \quad \text{with } \{i_1, \ldots, i_k\} \notin \mathcal{K},$$

and J_Σ is the ideal generated by the linear forms

$$\sum_{i=1}^{m} \langle \mathbf{a}_i, \mathbf{u} \rangle v_i \quad \text{with } \mathbf{u} \in (\mathbb{R}^m/\mathfrak{k})^*.$$

This settles a conjecture by [Battaglia and Zaffran] (arXiv:1108.1637).

If K is a compact torus (the fan Σ is rational), then we get

$$H^*_{\mathrm{bas},\,K}(\mathcal{Z}_\mathcal{K}) = H^*(\mathcal{Z}_\mathcal{K}/K) = H^*(V_\Sigma)$$

and the result above turns into the well-known description of the cohomology of toric manifolds, due to [Danilov and Jurkiewicz].

Idea of proof of the theorem.

Let $\mathfrak{t} = \mathrm{Lie}(T^m) \cong \mathbb{R}^m$ and consider the Cartan model

$$\mathcal{C}_\mathfrak{t}(\Omega(\mathcal{Z}_\mathcal{K})) = \bigl((S(\mathfrak{t}^*) \otimes \Omega(\mathcal{Z}_\mathcal{K}))^{T^m}, d_\mathfrak{t}\bigr).$$

Then

$$H(\mathcal{C}_\mathfrak{t}(\Omega(\mathcal{Z}_\mathcal{K}))) = H^*_{T^m}(\mathcal{Z}_\mathcal{K}) = \mathbb{C}[v_1, \ldots, v_m]/I_\mathcal{K}.$$

Key lemma: the dga $\mathcal{C}_\mathfrak{t}(\Omega(\mathcal{Z}_\mathcal{K}))$ is formal (quasi-isomorphic to its cohomology). □

References

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