A geometric view on SU-bordism

Taras Panov
joint with Zhi Lu, Ivan Limonchenko, Georgy Chernykh

Lomonosov Moscow State University

Algebraic Topology Seminar
Princeton University, 16 September 2020

1. Unitary bordism

Unitary bordism ring Ω^U consists of complex bordism classes of stably complex manifolds.

A stably complex manifold is a pair $(M, c_{\mathcal T})$ consisting of a smooth manifold M and a stably complex structure $c_{\mathcal T}$, determined by a choice of an isomorphism

$$c_{\mathcal T}\colon \mathcal T M \oplus \underline{\mathbb R}^N \xrightarrow{\cong} \xi$$

between the stable tangent bundle of M and a complex vector bundle ξ.

Theorem (Milnor–Novikov)

- *Two stably complex manifolds M and N represent the same bordism classes in Ω^U iff their sets of Chern characteristic numbers coincide.*
- *Ω^U is a polynomial ring on generators in every even degree:*

$$\Omega^U \cong \mathbb Z[a_1, a_2, \ldots, a_i, \ldots], \quad \deg a_i = 2i.$$

Polynomial generators of Ω^U can be detected using a special characteristic class s_n. It is the polynomial in the universal Chern classes $c_1, \ldots, c_n$ obtained by expressing the symmetric polynomial $x_1^n + \cdots + x_n^n$ via the elementary symmetric functions $\sigma_i(x_1, \ldots, x_n)$ and replacing each σ_i by c_i.

$s_n[M] = s_n(\mathcal{T}M)\langle M\rangle$: the corresponding characteristic number.

Theorem

The bordism class of a stably complex manifold M^{2i} may be taken to be the polynomial generator $a_i \in \Omega^U_{2i}$ iff

$$s_i[M^{2i}] = \begin{cases} \pm 1 & \text{if} \quad i+1 \neq p^s \quad \text{for any prime } p, \\ \pm p & \text{if} \quad i+1 = p^s \quad \text{for some prime } p \text{ and integer } s > 0. \end{cases}$$

Problem

Find nice geometric representatives in (unitary) bordism classes; e.g., smooth algebraic varieties and/or manifolds with large symmetry groups.

3. Special unitary bordism

A stably complex manifold $(M, c_{\mathcal{T}})$ is special unitary (an SU-manifold) if $c_1(M) = 0$. Bordism classes of SU-manifolds form the special unitary bordism ring Ω^{SU}.

The ring structure of Ω^{SU} is more subtle than that of Ω^{U}. Novikov described $\Omega^{SU} \otimes \mathbb{Z}[\frac{1}{2}]$ (it is a polynomial ring). The 2-torsion was described by Conner and Floyd. We shall need the following facts.

Theorem

- *The kernel of the forgetful map $\Omega^{SU} \to \Omega^{U}$ consists of torsion.*
- *Every torsion element in Ω^{SU} has order 2.*
- *$\Omega^{SU} \otimes \mathbb{Z}[\frac{1}{2}]$ is a polynomial algebra on generators in every even degree > 2:*

$$\Omega^{SU} \otimes \mathbb{Z}[\tfrac{1}{2}] \cong \mathbb{Z}[\tfrac{1}{2}][y_i \colon i > 1], \quad \deg y_i = 2i.$$

Let $\partial\colon \Omega^U_{2n} \to \Omega^U_{2n-2}$ be the homomorphism sending a bordism class $[M^{2n}]$ to the bordism class $[V^{2n-2}]$ of a submanifold $V^{2n-2} \subset M$ dual to $c_1(M)$.

Let $\mathcal{W}_{2n}$ be the subgroup of Ω^U_{2n} consisting of bordism classes $[M^{2n}]$ such that every Chern number of M^{2n} of which c_1^2 is a factor vanishes.
The restriction of the boundary homomorphism $\partial\colon \mathcal{W}_{2n} \to \mathcal{W}_{2n-2}$ is defined. It satisfies

$$\partial(a \cdot b) = a \cdot \partial b + \partial a \cdot b - [\mathbb{C}P^1] \cdot \partial a \cdot \partial b.$$

The direct sum $\mathcal{W} = \bigoplus_{i \geqslant 0} \mathcal{W}_{2i}$ is *not* a subring of Ω^U: one has $[\mathbb{C}P^1] \in \mathcal{W}_2$, but $c_1^2[\mathbb{C}P^1 \times \mathbb{C}P^1] = 8 \neq 0$, so $[\mathbb{C}P^1] \times [\mathbb{C}P^1] \notin \mathcal{W}_4$.

$\mathcal{W}$ is a commutative ring with respect to the twisted product

$$a * b = a \cdot b + 2[V^4] \cdot \partial a \cdot \partial b,$$

where $\cdot$ denotes the product in Ω^U and V^4 is a stably complex manifold with $c_1^2[V^4] = -1$, e.g. $V^4 = \mathbb{C}P^1 \times \mathbb{C}P^1 - \mathbb{C}P^2$.

Set

$$m_i = \begin{cases} 1 & \text{if} \quad i+1 \neq p^s \quad \text{for any prime } p, \\ p & \text{if} \quad i+1 = p^s \quad \text{for some prime } p \text{ and integer } s>0, \end{cases}$$

so that $[M^{2i}] \in \Omega^U_{2i}$ represents a polynomial generator iff $s_i[M^{2i}] = \pm m_i$.

Theorem

$\mathcal{W}$ is a polynomial ring on generators in every even degree except 4:

$$\mathcal{W} \cong \mathbb{Z}[x_1, x_i \colon i>2], \quad x_1 = [\mathbb{C}P^1], \quad \deg x_i = 2i,$$

with $s_i[x_i] = m_i m_{i-1}$ and the boundary operator $\partial \colon \mathcal{W} \to \mathcal{W}$, $\partial^2 = 0$, given by $\partial x_1 = 2$, $\partial x_{2i} = x_{2i-1}$, and satisfying the identity $\partial(a * b) = a * \partial b + \partial a * b - x_1 * \partial a * \partial b$.

Theorem

There is an exact sequence of groups

$$0 \longrightarrow \Omega^{SU}_{2n-1} \xrightarrow{\theta} \Omega^{SU}_{2n} \xrightarrow{\alpha} \mathcal{W}_{2n} \xrightarrow{\beta} \Omega^{SU}_{2n-2} \xrightarrow{\theta} \Omega^{SU}_{2n-1} \longrightarrow 0,$$

where θ is the multiplication by the generator $\theta \in \Omega^{SU}_1 \cong \mathbb{Z}_2$, α is the forgetful homomorphism, and $\alpha\beta = -\partial$.

We have

$$\mathcal{W} \otimes \mathbb{Z}[\tfrac{1}{2}] \cong \mathbb{Z}[\tfrac{1}{2}][x_1, x_{2k-1}, 2x_{2k} - x_1 x_{2k-1} \colon k > 1],$$

where $x_1^2 = x_1 * x_1$ is a ∂-cycle, and each x_{2k-1}, $2x_{2k} - x_1 x_{2k-1}$ is a ∂-cycle.

Theorem

There exist elements $y_i \in \Omega^{SU}_{2i}$, $i > 1$, such that $s_2(y_2) = -48$ and

$$s_i(y_i) = \begin{cases} m_i m_{i-1} & \text{if } i \text{ is odd}, \\ 2m_i m_{i-1} & \text{if } i \text{ is even and } i > 2. \end{cases}$$

These elements are mapped as follows under the forgetful homomorphism $\alpha \colon \Omega^{SU} \to \mathcal{W}$:

$$y_2 \mapsto 2x_1^2, \quad y_{2k-1} \mapsto x_{2k-1}, \quad y_{2k} \mapsto 2x_{2k} - x_1 x_{2k-1}, \quad k > 1.$$

In particular, $\Omega^{SU} \otimes \mathbb{Z}[\tfrac{1}{2}]$ embeds into $\mathcal{W} \otimes \mathbb{Z}[\tfrac{1}{2}]$ as the polynomial subring generated by x_1^2, x_{2k-1} and $2x_{2k} - x_1 x_{2k-1}$.

4. (Quasi)toric manifolds

A toric variety is a normal complex algebraic variety V containing an algebraic torus $(\mathbb{C}^\times)^n$ as a Zariski open subset in such a way that the natural action of $(\mathbb{C}^\times)^n$ on itself extends to an action on V.

Toric varieties are classified by convex-geometrical objects called rational fans, and projective toric varieties correspond to convex lattice polytopes P.

A toric manifold is a complete (compact) nonsingular toric variety.

A quasitoric manifold is a smooth $2n$-dimensional closed manifold M with a locally standard action of a (compact) torus T^n whose quotient M/T^n is a simple polytope P. An omniorientation of a quasitoric manifold provides it with an intrinsic stably complex structure.

Theorem (Danilov–Jurkiewicz, Davis–Januszkiewicz)

Let V be a (quasi)toric manifold of real dimension $2n$. The cohomology ring $H^*(V;\mathbb{Z})$ is generated by the degree-two classes v_i dual to the torus-invariant codimension-two submanifolds V_i, and is given by

$$H^*(V;\mathbb{Z}) \cong \mathbb{Z}[v_1,\ldots,v_m]/\mathcal{I}, \qquad \deg v_i = 2,$$

where $\mathcal{I}$ is the ideal generated by elements of the following two types:

- $v_{i_1}\cdots v_{i_k}$ *such that the facets $i_1,\ldots,i_k$ do not intersect in P;*
- $\displaystyle\sum_{i=1}^m \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle v_i$, *for any vector $\boldsymbol{x} \in \mathrm{Hom}(T^n, S^1) \cong \mathbb{Z}^n$.*

Here $\boldsymbol{a}_i \in \mathrm{Hom}(S^1, T^n) \cong \mathbb{Z}^n$ is the primitive vector defining the one-parameter subgroup fixing V_i.

It is convenient to consider the integer $n \times m$ characteristic matrix

$$\Lambda = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$$

whose columns are the vectors $\boldsymbol{a}_i$ written in the standard basis of $\mathbb{Z}^n$.
Then the n linear forms $a_{j1}v_1 + \cdots + a_{jm}v_m$ corresponding to the rows of Λ vanish in $H^*(V;\mathbb{Z})$.

Theorem

There is the following isomorphism of complex vector bundles:

$$\mathcal{T}V \oplus \underline{\mathbb{C}}^{m-n} \cong \rho_1 \oplus \cdots \oplus \rho_m,$$

where $\mathcal{T}V$ is the tangent bundle, $\underline{\mathbb{C}}^{m-n}$ is the trivial $(m-n)$-plane bundle, and ρ_i is the line bundle corresponding to V_i, with $c_1(\rho_i) = v_i$.
In particular, the total Chern class of V is given by

$$c(V) = (1 + v_1)\cdots(1 + v_m).$$

5. Toric representatives in unitary bordism classes

The classical family of generators for Ω^U is formed by the Milnor hypersufaces $H(n_1, n_2)$.

$H(n_1, n_2)$ is a hyperplane section of the Segre embedding $\mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2} \to \mathbb{C}P^{(n_1+1)(n_2+1)-1}$, given by the equation

$$z_0 w_0 + \cdots + z_{n_1} w_{n_1} = 0$$

where $[z_0 : \cdots : z_{n_1}] \in \mathbb{C}P^{n_1}$, $[w_0 : \cdots : w_{n_2}] \in \mathbb{C}P^{n_2}$, $n_1 \leqslant n_2$.

Also, $H(n_1, n_2)$ can be identified with the projectivisation $\mathbb{C}P(\zeta)$ of a certain n_2-plane bundle over $\mathbb{C}P^{n_1}$. The bundle ζ is not a sum of line bundles when $n_1 > 1$, so $H(n_1, n_2)$ is *not* a toric manifold in this case.

Buchstaber and Ray introduced a family $B(n_1, n_2)$ of *toric* generators of Ω^U. Each $B(n_1, n_2)$ is the projectivisation of a sum of n_2 line bundles over the bounded flag manifold BF_{n_1}. Then $B(n_1, n_2)$ is a toric manifold, because BF_{n_1} is toric and the projectivisation of a sum of line bundles over a toric manifold is toric.

We have $H(0, n_2) = B(0, n_2) = \mathbb{C}P^{n_2-1}$, so $s_{n_2-1}[H(0, n_2)] = s_{n_2-1}[B(0, n_2)] = n_2$. Furthermore,

$$s_{n_1+n_2-1}[H(n_1, n_2)] = s_{n_1+n_2-1}[B(n_1, n_2)] = -\binom{n_1+n_2}{n_1} \quad \text{for } n_1 > 1.$$

The fact that each of the families $\{[H(n_1, n_2)]\}$ and $\{[B(n_1, n_2)]\}$ generates the unitary bordism ring Ω^U follows from the well-known identity

$$\gcd\left\{\binom{n}{i},\ 0 < i < n\right\} = \begin{cases} 1 & \text{if} \quad n \neq p^s \quad \text{for any prime } p, \\ p & \text{if} \quad n = p^s \quad \text{for a prime } p \text{ and } s > 0. \end{cases}$$

We proceed to describing another family of toric generators for Ω^U.

Given two nonnegative integers n_1, n_2, define

$$L(n_1, n_2) = \mathbb{C}P(\eta \oplus \underline{\mathbb{C}}^{n_2}),$$

where η is the tautological line bundle over $\mathbb{C}P^{n_1}$.
It is a projective toric manifold with

$$\Lambda = \begin{pmatrix} \overbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{matrix}}^{n_1} & \begin{matrix} -1 \\ \vdots \\ -1 \end{matrix} & & \text{\Large 0} \\ & \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} & \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{matrix}}_{n_2} & \begin{matrix} -1 \\ \vdots \\ -1 \end{matrix} \\ \text{\Large 0} & & & \end{pmatrix}$$

The cohomology ring is given by

$$H^*\big(L(n_1,n_2)\big) \cong \mathbb{Z}[u,v]/\big(u^{n_1+1}, v^{n_2+1} - uv^{n_2}\big)$$

with $u^{n_1}v^{n_2}\langle L(n_1,n_2)\rangle = 1$.

There is an isomorphism of complex bundles

$$\mathcal{T}L(n_1,n_2) \oplus \underline{\mathbb{C}}^2 \cong \underbrace{p^*\bar\eta \oplus \cdots \oplus p^*\bar\eta}_{n_1+1} \oplus (\bar\gamma \otimes p^*\eta) \oplus \underbrace{\bar\gamma \oplus \cdots \oplus \bar\gamma}_{n_2},$$

where γ is the tautological line bundle over $L(n_1,n_2) = \mathbb{C}P(\eta \oplus \underline{\mathbb{C}}^{n_2})$.

The total Chern class is

$$c\big(L(n_1,n_2)\big) = (1+u)^{n_1+1}(1+v-u)(1+v)^{n_2}$$

with $u = c_1(p^*\bar\eta)$ and $v = c_1(\bar\gamma)$.

Lemma

For $n_2 > 0$, we have

$$s_{n_1+n_2}\big[L(n_1, n_2)\big] = \tbinom{n_1+n_2}{0} - \tbinom{n_1+n_2}{1} + \cdots + (-1)^{n_1}\tbinom{n_1+n_2}{n_1} + n_2.$$

Theorem (Lu-P.)

The bordism classes $[L(n_1, n_2)] \in \Omega^U_{2(n_1+n_2)}$ generate the ring Ω^U.

Proof. $s_{n_1+n_2}\big[L(n_1, n_2) - 2L(n_1 - 1, n_2 + 1) + L(n_1 - 2, n_2 + 2)\big]$

$$= (-1)^{n_1-1}\tbinom{n_1+n_2}{n_1-1} + (-1)^{n_1}\tbinom{n_1+n_2}{n_1} - 2(-1)^{n_1-1}\tbinom{n_1+n_2}{n_1-1} = (-1)^{n_1}\tbinom{n_1+n_2+1}{n_1}$$

It follows that any unitary bordism class can be represented by a disjoint union of products of projective toric manifolds. Products of toric manifolds are toric, but disjoint unions are not, as toric manifolds are connected.

A disjoint union may be replaced by a connected sum, representing the same bordism class. However, connected sum is not an algebraic operation, and a connected sum of two algebraic varieties is rarely algebraic.

One can form equivariant connected sum of quasitoric manifolds, but the resulting invariant stably complex structure does not represent the cobordism sum of the two original manifolds. A more intricate connected sum construction is needed, as described in [Buchstaber, P. and Ray].

The conclusion, which can be derived from the above construction and any of the toric generating sets $\{B(n_1, n_2)\}$ or $\{L(n_1, n_2)\}$ for Ω^U, is as follows:

Theorem (Buchstaber-P.-Ray)

In dimensions > 2, every unitary bordism class contains a quasitoric manifold, necessarily connected, whose stably complex structure is induced by an omniorientation, and is therefore compatible with the torus action.

6. Toric generators of the SU-bordism ring

Proposition

An omnioriented quasitoric manifold M has $c_1(M) = 0$ if and only if there exists a linear function $\varphi\colon \mathbb{Z}^n \to \mathbb{Z}$ such that $\varphi(\boldsymbol{a}_i) = 1$ for $i = 1, \ldots, m$. Here the $\boldsymbol{a}_i$ are the columns of characteristic matrix.
In particular, if some n vectors of $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ form the standard basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$, then M is SU iff the column sums of Λ are all equal to 1.

Corollary

A toric manifold V cannot be SU.

Proof. If $\varphi(\boldsymbol{a}_i) = 1$ for all i, then the vectors $\boldsymbol{a}_i$ lie in the positive halfspace of φ, so they cannot span a complete fan.

Theorem (Buchstaber-P.-Ray)

A quasitoric SU-manifold M^{2n} represents 0 in Ω^U_{2n} whenever $n < 5$.

Example

Assume that $n_1 = 2k_1$ is positive even and $n_2 = 2k_2 + 1$ is positive odd, and consider the manifold $L(n_1, n_2) = \mathbb{C}P(\eta \oplus \underline{\mathbb{C}}^{n_2})$. We change its stably complex structure to the following:

$$\begin{aligned}&\mathcal{T}L(n_1, n_2) \oplus \underline{\mathbb{R}}^4 \\ \cong\ &\underbrace{p^*\bar\eta \oplus p^*\eta \oplus \cdots \oplus p^*\bar\eta \oplus p^*\eta}_{2k_1} \oplus p^*\bar\eta \oplus (\bar\gamma \otimes p^*\eta) \oplus \underbrace{\bar\gamma \oplus \gamma \oplus \cdots \oplus \bar\gamma \oplus \gamma}_{2k_2} \oplus \gamma\end{aligned}$$

and denote the resulting stably complex manifold by $\widetilde{L}(n_1, n_2)$. It has

$$c\big(\widetilde{L}(n_1, n_2)\big) = (1 - u^2)^{k_1}(1 + u)(1 + v - u)(1 - v^2)^{k_2}(1 - v),$$

so $\widetilde{L}(n_1, n_2)$ is an SU-manifold of dimension $2(n_1 + n_2) = 4(k_1 + k_2) + 2$.

Example (continued)

$\widetilde{L}(n_1, n_2)$ is an omnioriented quasitoric manifold over $\Delta^{n_1} \times \Delta^{n_2}$ corresponding to the matrix

$$\Lambda = \left(\begin{array}{ccccccccccc}
\multicolumn{5}{c}{\overbrace{\hphantom{1 \quad 0 \quad 0 \quad \cdots \quad 0}}^{n_1=2k_1}} & & & & & & \\
1 & 0 & 0 & \cdots & 0 & 1 & & & & & \\
0 & 1 & 0 & \cdots & 0 & -1 & & & & & \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & & \text{\Large 0} & & \\
0 & 0 & 0 & 1 & 0 & 1 & & & & & \\
0 & 0 & 0 & 0 & 1 & -1 & & & & & \\
 & & & & & 1 & 1 & 0 & \cdots & 0 & 1 \\
 & & & & & & 0 & 1 & \cdots & 0 & -1 \\
 & & \text{\Large 0} & & & & \vdots & \ddots & \ddots & 0 & \vdots \\
 & & & & & & 0 & 0 & 0 & 1 & 1 \\
 & & & & & & \multicolumn{4}{c}{\underbrace{\hphantom{0 \quad 0 \quad 0 \quad 1}}_{n_2=2k_2+1}} &
\end{array}\right)$$

The columns sum of this matrix are 1 by inspection.

Lemma

- *For $k > 1$, there is a linear combination y_{2k+1} of SU-bordism classes $[\widetilde{L}(n_1, n_2)]$ with $n_1 + n_2 = 2k + 1$ such that $s_{2k+1}(y_{2k+1}) = m_{2k+1} m_{2k}$.*
- *For $k > 2$, there is a linear combination y_{2k} of SU-bordism classes $[\widetilde{N}(n_1, n_2)]$ with $n_1 + n_2 + 1 = 2k$ such that $s_{2k}(y_{2k}) = 2m_{2k} m_{2k-1}$.*

Theorem (Lu-P.)

There exist quasitoric SU-manifolds M^{2i}, $i \geqslant 5$, with $s_i(M^{2i}) = m_i m_{i-1}$ if i is odd and $s_i(M^{2i}) = 2m_i m_{i-1}$ if i is even. These quasitoric manifolds represent polynomial generators of $\Omega^{SU} \otimes \mathbb{Z}[\frac{1}{2}]$.

7. Calabi–Yau hypersurfaces and SU-bordism

A Calabi–Yau manifold is a compact Kähler manifold M with $c_1(M) = 0$. By definition, a Calabi–Yau manifold is an SU-manifold.

A toric manifold V is Fano if its anticanonical class $V_1 + \cdots + V_m$ (representing $c_1(V)$) is ample. In geometric terms, the projective embedding $V \hookrightarrow \mathbb{C}P^s$ corresponding to $V_1 + \cdots + V_m$ comes from a lattice polytope P in which the lattice distance from 0 to each hyperplane containing a facet is 1. Such a lattice polytope P is called reflexive; its polar polytope P^* is also a lattice polytope.

The submanifold N dual to $c_1(V)$ is given by the hyperplane section of the embedding $V \hookrightarrow \mathbb{C}P^s$ defined by $V_1 + \cdots + V_m$. Therefore, $N \subset V$ is a smooth algebraic hypersurface in V, so N is a Calabi–Yau manifold of complex dimension $n - 1$.

Lemma

The s-number of the Calabi–Yau manifold N is given by

$$s_{n-1}(N) = \big\langle (v_1^{n-1} + \cdots + v_m^{n-1})(v_1 + \cdots + v_m) - (v_1 + \cdots + v_m)^n, [V] \big\rangle.$$

Example

Consider the Calabi–Yau hypersurface N_3 in $V = \mathbb{C}P^3$.

We have $c_1(\mathcal{T}\mathbb{C}P^3) = 4u$, where $u \in H^2(\mathbb{C}P^3; \mathbb{Z})$ is the canonical generator dual to a hyperplane section.

Therefore, N_3 can be given by a generic quartic equation in homogeneous coordinates on $\mathbb{C}P^3$.

The standard example is the quartic given by $z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0$, which is a $K3$-surface. Lemma above gives

$$s_3(N_3) = \langle 4u^2 \cdot 4u - (4u)^3, [\mathbb{C}P^3] \rangle = -48,$$

so N_3 represents the generator $-y_2 \in \Omega_4^{SU}$.

$\sigma = (\sigma_1, \ldots, \sigma_k)$ an unordered partition of n, $\quad \sigma_1 + \cdots + \sigma_k = n$
$\varDelta^{\sigma_i}$ the standard reflexive simplex of dimension σ_i.
$P_\sigma = \varDelta^{\sigma_1} \times \cdots \times \varDelta^{\sigma_k}$ is a reflexive polytope with the corresponding toric Fano manifold $V_\sigma = \mathbb{C}P^{\sigma_1} \times \cdots \times \mathbb{C}P^{\sigma_k}$.
N_σ the canonical Calabi–Yau hypersurface in V_σ.

Theorem (Limonchenko-Lu-P.)

The SU-bordism classes of the canonical Calabi–Yau hypersurfaces N_σ in $\mathbb{C}P^{\sigma_1} \times \cdots \times \mathbb{C}P^{\sigma_k}$ multiplicatively generate the SU-bordism ring $\varOmega^{SU}[\frac{1}{2}]$.

Idea of proof.

Denote by $\widehat{P}(n)$ the set of all partitions σ with parts of size at most $n-2$:

$$\widehat{P}(n) := \{\sigma = (\sigma_1, \ldots, \sigma_k)\colon \sigma_1 + \cdots + \sigma_k = n, \quad \sigma \neq (n), (1, n-1).\}$$

For each σ we have the multinomial coefficient $\binom{n}{\sigma} = \frac{n!}{\sigma_1!\cdots\sigma_k!}$ and define

$$\alpha(\sigma) := \binom{n}{\sigma}(\sigma_1 + 1)^{\sigma_1} \cdots (\sigma_k + 1)^{\sigma_k}.$$

Then for for any $\sigma \in \widehat{P}(n)$ we have

$$s_{n-1}(N_\sigma) = -\alpha(\sigma).$$

Then we prove that

$$\gcd_{\sigma \in \widehat{P}(n)} \alpha(\sigma) = \begin{cases} 2m_{n-1}m_{n-2} & \text{if } n > 3 \text{ is odd;} \\ m_{n-1}m_{n-2} & \text{if } n > 3 \text{ is even;} \\ 48 & \text{if } n = 3. \end{cases}$$

Therefore, there is a linear combination of the bordism classes $[N_\sigma] \in \Omega^{SU}_{2n-2}$ whose s-number satisfies the condition for a polynomial generator y_{n-1} of $\Omega^{SU}[\frac{1}{2}]$. $\square$

Question

Which bordism classes in Ω^{SU} can be represented by Calabi–Yau manifolds?

This question is an SU-analogue of the following well-known problem of Hirzebruch: which bordism classes in Ω^{U} contain connected (i. e., irreducible) non-singular algebraic varieties?

If one drops the connectedness assumption, then any U-bordism class of positive dimension can be represented by an algebraic variety. Since a product and a positive integral linear combination of algebraic classes is an algebraic class (possibly, disconnected), one only needs to find in each dimension i algebraic varieties M and N with $s_i(M) = m_i$ and $s_i(N) = -m_i$. This was done by Milnor in 1960.

For SU-bordism, the situation is different: if a class $a \in \Omega^{SU}$ can be represented by a Calabi–Yau manifold, then $-a$ does not necessarily have this property. Therefore, the first step towards the answering the question above is whether y_i and $-y_i$ can be simultaneously represented by Calabi–Yau manifolds.

References

- Victor Buchstaber, Taras Panov and Nigel Ray. *Toric genera.* Internat. Math. Research Notices 16 (2010), 3207–3262.
- Victor Buchstaber and Taras Panov. *Toric Topology.* Mathematical Surveys and Monographs, vol. 204, American Mathematical Society, Providence, RI, 2015, 516 pages.
- Zhi Lü and Taras Panov. *On toric generators in the unitary and special unitary bordism rings.* Algebraic & Geometric Topology 16 (2016), no. 5, 2865–2893.
- Georgy Chernykh, Ivan Limonchenko and Taras Panov. *SU-bordism: structure results and geometric representatives.* Russian Math. Surveys 74 (2019), no. 3; arXiv:1903.07178.