Right-angled polytopes, hyperbolic manifolds and torus actions

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Polytopes and moment-angle manifolds

A convex polytope in $\mathbb{R}^n$ is a bounded intersection of m halfspaces:

$$P = \{ \boldsymbol{x} \in \mathbb{R}^n \colon \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle + b_i \geqslant 0 \quad \text{for } i = 1, \ldots, m \},$$

where $\boldsymbol{a}_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$.

Assume $\dim P = n$ and each $F_i = P \cap \{ \boldsymbol{x} \colon \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle + b_i = 0 \}$ is a facet.
$\mathcal{F} = \{F_1, \ldots, F_m\}$ the set of facets of P.

Define an affine map

$$i_P \colon \mathbb{R}^n \to \mathbb{R}^m, \quad i_P(\boldsymbol{x}) = \big(\langle \boldsymbol{a}_1, \boldsymbol{x} \rangle + b_1, \ldots, \langle \boldsymbol{a}_m, \boldsymbol{x} \rangle + b_m \big).$$

Then i_P is injective, and $i_P(P) \subset \mathbb{R}^m$ is the intersection of an n-dimensional plane with $\mathbb{R}^m_{\geqslant} = \{ \boldsymbol{y} = (y_1, \ldots, y_m) \colon y_i \geqslant 0 \}$.

Define the space $\mathcal{Z}_P$ from the diagram

$$\begin{array}{ccccc}
\mathcal{Z}_P & \xrightarrow{i_Z} & \mathbb{C}^m & & (z_1,\ldots,z_m) \\
\downarrow & & \downarrow \mu & & \downarrow \\
P & \xrightarrow{i_P} & \mathbb{R}^m_{\geqslant} & & (|z_1|^2,\ldots,|z_m|^2)
\end{array}$$

Explicitly, $\mathcal{Z}_P = \mu^{-1}(i_P(P))$. It has a T^m-action with the quotient $\mathcal{Z}_P/T^m = P$.

P is simple if there are $n = \dim P$ facets meeting at each vertex.

Proposition

If P is a simple polytope, then $\mathcal{Z}_P$ is a smooth $(m+n)$-dim manifold.

Proof.

Write $i_P(\mathbb{R}^n)$ by $(m-n)$ linear equations in $(y_1,\ldots,y_m) \in \mathbb{R}^m$. Replacing each y_k by $|z_k|^2$ we obtain a presentation of $\mathcal{Z}_P$ by Hermitian quadrics. $\square$

$\mathcal{Z}_P$ is the moment-angle manifold (corresponding to P).

Similarly, considering

$$\begin{array}{ccccc} \mathcal{R}_P & \longrightarrow & \mathbb{R}^m & & (u_1,\ldots,u_m) \\ \big\downarrow & & \big\downarrow \mu & & \big\downarrow \\ P & \xrightarrow{i_P} & \mathbb{R}^m_{\geqslant} & & (u_1^2,\ldots,u_m^2) \end{array}$$

we obtain a real moment-angle manifold $\mathcal{R}_P$.

Example

$P = \{(x_1,x_2) \in \mathbb{R}^2 \colon x_1 \geqslant 0,\ x_2 \geqslant 0,\ -\gamma_1 x_1 - \gamma_2 x_2 + 1 \geqslant 0\}$, $\gamma_1, \gamma_2 > 0$ (a 2-simplex). Then

$\mathcal{Z}_P = \{(z_1,z_2,z_3) \in \mathbb{C}^3 \colon \gamma_1|z_1|^2 + \gamma_2|z_2|^2 + |z_3|^2 = 1\}$ (a 5-sphere),

$\mathcal{R}_P = \{(u_1,u_2,u_3) \in \mathbb{R}^3 \colon \gamma_1|u_1|^2 + \gamma_2|u_2|^2 + |u_3|^2 = 1\}$ (a 2-sphere).

Right-angled polytopes and hyperbolic manifolds

Let P be a polytope in n-dimensional Lobachevsky space $\mathbb{L}^n$ with right angles between adjacent facets (a right-angled n-polytope).

Denote by $G(P)$ the group generated by reflections in the facets of P. It is a right-angled Coxeter group given by the presentation

$$G(P) = \langle g_1, \dots, g_m \mid g_i^2 = 1,\ g_i g_j = g_j g_i \text{ if } F_i \cap F_j \neq \varnothing \rangle,$$

where g_i denotes the reflection in the facet F_i.

The group $G(P)$ acts on $\mathbb{L}^n$ discretely with finite isotropy subgroups and with fundamental domain P.

Lemma (A. Vesnin, 1987)

Consider an epimorphism $\varphi\colon G(P) \to \mathbb{Z}_2^k$, $k \geqslant n$. The subgroup $\operatorname{Ker}\varphi \subset G(P)$ does not contain elements of finite order if and only if the images of the reflections in any n facets of P that have a common vertex are linearly independent in $\mathbb{Z}_2^k$.
In this case the group $\operatorname{Ker}\varphi$ acts freely on $\mathbb{L}^n$.

The quotient $N = \mathbb{L}^n / \operatorname{Ker}\varphi$ is a hyperbolic n-manifold. It is composed of $|\mathbb{Z}_2^k| = 2^k$ copies of P and has a Riemannian metric of constant negative curvature. Furthermore, the manifold N is aspherical (the Eilenberg–Mac Lane space $K(\operatorname{Ker}\varphi, 1)$), as its universal cover $\mathbb{L}^n$ is contractible.

Which combinatorial n-polytopes have right-angled realisations in $\mathbb{L}^n$?
In dim 3, there is a nice criterion going back to Pogorelov's work of 1967:

МАТЕМАТИЧЕСКИЕ ЗАМЕТКИ

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О ПРАВИЛЬНОМ РАЗБИЕНИИ ПРОСТРАНСТВА ЛОБАЧЕВСКОГО

А. В. Погорелов

Пусть P — замкнутый выпуклый многогранник с прямыми двугранными углами в пространстве Лобачевского. Отразим многогранник P зеркально в каждой из его граней. Полученные при этом многогранники отразим в гранях, не примыкающих к P, и т. д. Продолжая этот процесс неограниченно, мы заполним все пространство многогранниками, равными P. Построенное разбиение пространства Лобачевского на равные многогранники осуществляется системой плоскостей. Каждые две плоскости этой системы либо не пересекаются, либо пересекаются под прямым углом.

Рассмотрим пример. На плоскости Лобачевского можно построить правильный пятиугольник с любыми внутренними углами, меньшими $\frac{3\pi}{5}$. В частности, существует правильный пятиугольник с прямыми внутренними углами. Из таких плоских пятиугольников, очевидно, составляется правильный додекаэдр в пространстве Лобачевского. Все его двугранные углы прямые, и, следовательно, такими додекаэдрами покрывается все пространство Лобачевского указанным способом. Можно было бы привести и другие примеры многогранников, которыми заполняется пространство Лобачевского.

Целью настоящей заметки является установление необходимых и достаточных условий, при которых существует замкнутый выпуклый многогранник заданного строения с прямыми двугранными углами. При этом

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Alexei Pogorelov (1919–2002)

Theorem (Pogorelov, Andreev)

A combinatorial 3-polytope $P \neq \Delta^3$ can be realised as a right-angled polytope in $\mathbb{L}^3$ if and only if it is simple, and does not have 3- and 4-belts of facets. Furthermore, such a realisation is unique up to isometry.

We refer to the class of combinatorial 3-polytope satisfying the conditions above as a Pogorelov class $\mathcal{P}$.

A Pogorelov polytope does not have triangular or quadrangular facets.

Every fullerene (a simple 3-polytope with only pentagonal and hexagonal facets) is a Pogorelov polytope.

The conditions specifying Pogorelov polytopes also feature as the no-$\triangle$ and no-$\square$ condition in Gromov's theory of hyperbolic groups.

Classification of right-angled polytopes in $\mathbb{L}^4$ is wide open.

There are no right-angled polytopes in $\mathbb{L}^n$ for $n \geqslant 5$, [Nikulin, Vinberg].

Given a right-angled polytope P, how to find an epimorphism $\varphi\colon G(P) \to \mathbb{Z}_2^k$ with $\operatorname{Ker}\varphi$ acting freely on $\mathbb{L}^n$?

One can consider the abelianisation: $G(P) \xrightarrow{\mathrm{ab}} \mathbb{Z}_2^m$, with $\operatorname{Ker}\mathrm{ab} = G'(P)$, the commutator subgroup.

The corresponding n-manifold $\mathbb{L}^n/G'(P)$ is the real moment-angle manifold $\mathcal{R}_P$, described as an intersection of quadrics in the beginning of this talk.

Corollary

If P is a right-angled polytope in $\mathbb{L}^n$, then the real moment-angle manifold $\mathcal{R}_P$ admits a hyperbolic structure as $\mathbb{L}^n/G'(P)$, where $G'(P)$ is the commutator subgroup of the corresponding right-angled Coxeter group. The manifold $\mathcal{R}_P$ is composed of 2^m copies of P.

A more econimical way to obtain a hyperbolic manifold is to consider $\varphi\colon G(P) \to \mathbb{Z}_2^n$. Such an epimorphism factors as $G(P) \xrightarrow{\mathrm{ab}} \mathbb{Z}_2^m \xrightarrow{\Lambda} \mathbb{Z}_2^n$, where Λ is a linear map.

The subgroup $\operatorname{Ker}\varphi$ acts freely on $\mathbb{L}^n$ if and only the Λ-images of any n facets of P that meet at a vertex form a basis of $\mathbb{Z}_2^n$.
Such Λ is called a $\mathbb{Z}_2$-characteristic function (Davis, Januszkiewicz'1991).

Proposition

Any simple 3-polytope admits a characteristic function.

Proof.

Given a 4-colouring of the facets of P, we assign to a facet of ith colour the ith basis vector $\boldsymbol{e}_i \in \mathbb{Z}^3$ for $i = 1, 2, 3$ and the vector $\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3$ for $i = 4$. The resulting map $\Lambda\colon \mathbb{Z}_2^m \to \mathbb{Z}_2^3$ satisfies the required condition, as any three of the four vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3$ form a basis of $\mathbb{Z}^3$. □

Definition (A. Vesnin, 1987)

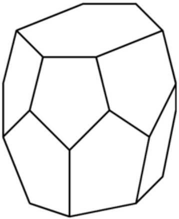
$N(P,\Lambda) = \mathbb{L}^3/\operatorname{Ker}\varphi$ a hyperbolic 3-manifold of Löbell type.
It is composed of $|\mathbb{Z}_2^3| = 8$ copies of a right-angled 3-polytope $P \in \mathcal{P}$ glued along their facets;
the gluing is prescribed by the characteristic function $\Lambda\colon \mathbb{Z}_2^m \to \mathbb{Z}_2^3$.

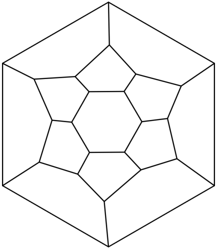
In particular, one obtains a hyperbolic 3-manifold $N(P,\chi)$ from any regular 4-colouring $\chi\colon \mathcal{F} \to \{1,2,3,4\}$ of a right-angled 3-polytope P.

Löbell (1931) was first to consider hyperbolic 3-manifolds coming from 4-colourings of a family of right-angled polytopes starting from the dodecahedron.

Example: Löbell polytopes Q_k ("barrel" fullerenes)

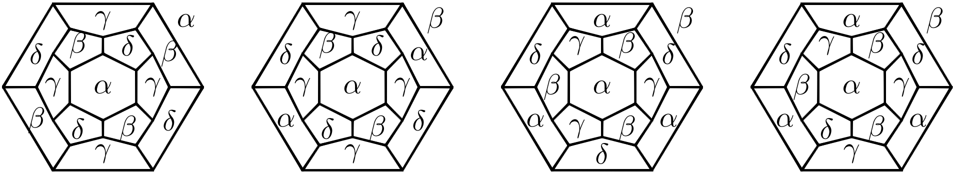
For $k \geqslant 5$, let Q_k be a simple 3-polytope with two "top" and "bottom" k-gonal facets and $2k$ pentagonal facets forming two k-belts around the top and bottom, so that Q_k has $2k+2$ facets in total.
Note that Q_5 is a combinatorial dodecahedron, while Q_6 is a fullerene.





It is easy to see that $Q_k \in \mathcal{P}$, so it admits a right-angled realisation in $\mathbb{L}^3$.

Consider hyperbolic 3-manifolds $N(Q_k, \chi)$ corresponding to 4-colourings χ of Q_k. For example, a dodecahedron Q_5 has a unique 4-colouring up to equivalence, while Q_6 has four non-equivalent regular 4-colourings (4-colourings are equivalent if they differ by a permutation of colours).



Conjecture (Vesnin, 1991)

Hyperbolic 3-manifolds $N(Q_k, \chi_1)$ and $N(Q_k, \chi_2)$ are diffeomorphic (isometric) if and only if the 4-colourings χ_1 and χ_2 are equivalent.

By 2009 the conjecture was verified for all k except $6, 8$ using deep results on arithmetic groups (Margulis commensurator theorem).
However, it remained open for Q_6 and Q_8.

Pairs (P,Λ) and (P',Λ') are equivalent if P and P' are combinatorially equivalent, and $\Lambda,\Lambda'\colon \mathbb{Z}_2^m \to \mathbb{Z}_2^n$ differ by an automorphism of $\mathbb{Z}_2^n$.

Theorem (Buchstaber–Erokhovets–Masuda–P–Park)

Let $N = N(P,\Lambda)$ and $N' = N(P',\Lambda')$ be two hyperbolic 3-manifolds of Löbell type corresponding to right-angled 3-polytopes P and P'. Then the following conditions are equivalent:

(a) *there is a cohomology ring isomorphism* $\varphi\colon H^*(N;\mathbb{Z}_2) \xrightarrow{\cong} H^*(N';\mathbb{Z}_2)$*;*

(b) *there is a diffeomorphism $N \cong N'$;*

(c) *there is an equivalence of $\mathbb{Z}_2$-characteristic pairs $(P,\Lambda) \sim (P',\Lambda')$.*

The difficult implication is (a)⇒(c). Its proof builds upon the wealth of cohomological techniques of toric topology.

Specifying to $\mathbb{Z}_2$-characteristic functions Λ coming from colourings χ we obtain:

Theorem (Buchstaber–P)

Hyperbolic 3-manifolds $N(P, \chi_1)$ and $N(P', \chi_2)$ corresponding to right-angled polytopes P and P' are diffeomorphic (isometric) if and only if the 4-colourings χ_1 and χ_2 are equivalent.

In particular, Vesnin's conjecture holds for *all* Löbell polytopes Q_k.

Cohomology of moment-angle manifolds

The face ring (the Stanley–Reisner ring) of a simple polytope P

$$\mathbb{Z}[P] := \mathbb{Z}[v_1, \ldots, v_m] \big/ \big(v_{i_1} \cdots v_{i_k} = 0 \quad \text{for } F_{i_1} \cap \cdots \cap F i_k = \varnothing \big)$$

where $\deg v_i = 2$.

Theorem

There are ring isomorphisms

$$\begin{aligned} H^*(\mathcal{Z}_P) &\cong \operatorname{Tor}_{\mathbb{Z}[v_1,\ldots,v_m]}(\mathbb{Z}[P], \mathbb{Z}) \\ &\cong H\big(\Lambda[u_1, \ldots, u_m] \otimes \mathbb{Z}[P], d\big) \qquad du_i = v_i, dv_i = 0 \\ &\cong \bigoplus_{I \subset \{1,\ldots,m\}} \widetilde{H}^{*-|I|-1}(P_I) \qquad P_I = \bigcup_{i \in I} F_i \end{aligned}$$

(Quasi)toric manifolds and small covers

P a simple n-polytope, $\mathcal{F} = \{F_1, \ldots, F_m\}$ the set of facets.

A characteristic function is map $\Lambda\colon \mathcal{F} \to \mathbb{Z}^n$ such that $\Lambda(F_{i_1}), \ldots, \Lambda(F_{i_n})$ is a basis of $\mathbb{Z}^n$ whenever $F_{i_1}, \ldots, F_{i_n}$ intersect in a vertex.
A characteristic function defines a linear map $\Lambda\colon \mathbb{Z}^{\mathcal{F}} = \mathbb{Z}^m \to \mathbb{Z}^n$ and a homomorphism of tori $\Lambda\colon T^m \to T^n$.

Proposition

The subgroup $\operatorname{Ker} \Lambda \cong T^{m-n}$ *acts freely on* $\mathcal{Z}_P$.

$M(P, \Lambda) = \mathcal{Z}_P / \operatorname{Ker} \Lambda$ a quasitoric manifold (Davis, Januszkiewicz'1991).
It is a smooth $2n$-dimensional manifold with an action of the n-torus $T^m / \operatorname{Ker} \Lambda \cong T^n$ and the quotient P.

By considering $\mathbb{Z}_2$-characteristic functions $\Lambda\colon \mathbb{Z}_2^m \to \mathbb{Z}_2^n$, we obtain small covers of P as the quotients $\mathcal{R}_P/\operatorname{Ker}\Lambda$.

A small cover $N(P,\Lambda)$ is a smooth n-dimensional manifold with an action of $\mathbb{Z}_2^n$ and the quotient P.

Proposition

In dimension 3, a real moment-angle manifold $\mathcal{R}_P$ and a small cover $N(P,\Lambda)$ admit a hyperbolic structure if and only if P is a Pogorelov polytope.

Proof.

P is a Pogorelov polytope $\quad\Leftrightarrow\quad$ dual $\mathcal{K}$ has no $\triangle$ and $\square$

If P is a Pogorelov polytope, then $\mathcal{R}_P$ and $N(P,\Lambda)$ are hyperbolic.

If $\mathcal{K}$ has a $\triangle$, then $R_\triangle \cong S^2$ retracts off $\mathcal{R}_P$, so $\mathcal{R}_P$ cannot be hyperbolic (as it has $\pi_2 \neq 0$).

If $\mathcal{K}$ has a $\square$, then $R_\square \cong T^2$ retracts off $\mathcal{R}_P$, which is impossible for a hyperbolic manifold. $\square$

Theorem (Davis–Januszkiewicz'1991)

Let $M = M(P, \Lambda)$ be a quasitoric manifold over a simple n-polytope P. The cohomology ring $H^*(M; \mathbb{Z})$ is generated by the 2-dimensional classes $[v_i]$ dual to the characteristic submanifolds M_i, $i = 1, \dots, m$, and is given by

$$H^*(M; \mathbb{Z}) \cong \mathbb{Z}[v_1, \dots, v_m]/\mathcal{I}, \qquad \deg v_i = 2,$$

where $\mathcal{I}$ is the ideal generated by elements of two kinds:

(a) $v_{i_1} \cdots v_{i_k}$, *where* $F_{i_1} \cap \cdots \cap F_{i_k} = \varnothing$ *in* P;

(b) $\sum_{i=1}^{m} \langle \Lambda(F_i), \boldsymbol{x} \rangle v_i$ *for any* $\boldsymbol{x} \in \mathbb{Z}^n$.

The $\mathbb{Z}_2$-cohomology ring $H^*(N; \mathbb{Z}_2)$ of a small cover $N = N(P, \Lambda)$ has the same description, with generators v_i of degree 1.

Theorem (Buchstaber-Erokhovets-Masuda-P-Park)

Let $M = M(P, \Lambda)$ and $M' = M(P', \Lambda')$ be quasitoric 6-manifolds, where P is a Pogorelov 3-polytope. The following conditions are equivalent:

(a) *There is a ring isomorphism $\varphi\colon H^*(M;\mathbb{Z}) \xrightarrow{\cong} H^*(M';\mathbb{Z})$;*

(b) *There is a diffeomorphism $M \cong M'$;*

(c) *There is an equivalence of characteristic pairs $(P, \Lambda) \sim (P', \Lambda')$.*

Idea of proof (for both theorems).

We need to prove that a ring iso $\varphi\colon H^*(N;\mathbb{Z}_2)\xrightarrow{\cong}H^*(N';\mathbb{Z}_2)$ implies an equivalence $(P,\Lambda)\sim(P',\Lambda')$.

An iso $\varphi\colon H^*(N;\mathbb{Z}_2)\xrightarrow{\cong}H^*(N';\mathbb{Z}_2)$ implies an iso $\varphi\colon H^*(M;\mathbb{Z}_2)\xrightarrow{\cong}H^*(M';\mathbb{Z}_2)$ (as every $\mathbb{Z}_2$-characteristic function of a 3-polytope lifts to a $\mathbb{Z}$-characteristic function!)

An iso $\varphi\colon H^*(M;\mathbb{Z}_2)=\mathbb{Z}_2[P]/\mathcal{J}_\Lambda\xrightarrow{\cong}\mathbb{Z}_2[P']/\mathcal{J}_{\Lambda'}=H^*(M';\mathbb{Z}_2)$ implies an iso

$$\psi\colon H^*(\mathcal{Z}_P;\mathbb{Z}_2)=\operatorname{Tor}_{\mathbb{Z}_2[v_1,\ldots,v_m]/\mathcal{J}_\Lambda}\big(\mathbb{Z}_2[P]/\mathcal{J}_\Lambda,\mathbb{Z}_2\big)$$
$$\xrightarrow{\cong}\operatorname{Tor}_{\mathbb{Z}_2[v_1,\ldots,v_m]/\mathcal{J}_{\Lambda'}}\big(\mathbb{Z}_2[P']/\mathcal{J}_{\Lambda'},\mathbb{Z}_2\big)=H^*(\mathcal{Z}_{P'};\mathbb{Z}_2)$$

Pogorelov's conditions imply that ψ maps the set of canonical generators $\{[u_iv_j]\in H^3(\mathcal{Z}_P)\colon F_i\cap F_j=\varnothing\}$ bijectively to the corresponding set for $\mathcal{Z}_{P'}$.
This implies that φ maps the set $\{[v_i]\in H^2(M)\}$ bijectively to the corresponding set for M', giving an equivalence $(P,\Lambda)\sim(P',\Lambda')$. □

Cohomological rigidity

Problem

Let $M = M(P, \Lambda)$ and $M' = M(P', \Lambda')$ be quasitoric manifolds with isomorphic integer cohomology rings. Are they homeomorphic?

Our result gives a positive answer in the case of quasitoric 6-manifolds over Pogorelov polytopes.

Proposition

6-dimensional quasitoric manifolds M and M' are diffeomorphic if there is an isomorphism $\varphi\colon H^*(M;\mathbb{Z}) \xrightarrow{\cong} H^*(M';\mathbb{Z})$ preserving the first Pontryagin class, i. e. $\varphi(p_1(M)) = p_1(M')$.

We have $p_1(M) = v_1^2 + \cdots + v_m^2 \in H^4(M)$. However, we were not able to establish the invariance of $p_1(M)$ directly...

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