Polyhedral products, loop homology and right-angled Coxeter groups

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Workshop on Polyhedral Products in Geometric Group Theory
Fields Institute, Toronto, Canada, 25–29 May 2020

1. Preliminaries

Polyhedral product

$(\boldsymbol{X}, \boldsymbol{A}) = \{(X_1, A_1), \ldots, (X_m, A_m)\}$ a sequence of pairs of spaces, $A_i \subset X_i$.

$\mathcal{K}$ a simplicial complex on $[m] = \{1, 2, \ldots, m\}$, $\quad \varnothing \in \mathcal{K}$.

Given $I = \{i_1, \ldots, i_k\} \subset [m]$, set

$$(\boldsymbol{X}, \boldsymbol{A})^I = Y_1 \times \cdots \times Y_m \qquad \text{where } Y_i = \begin{cases} X_i & \text{if } i \in I, \\ A_i & \text{if } i \notin I. \end{cases}$$

The $\mathcal{K}$-polyhedral product of $(\boldsymbol{X}, \boldsymbol{A})$ is

$$(\boldsymbol{X}, \boldsymbol{A})^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (\boldsymbol{X}, \boldsymbol{A})^I = \bigcup_{I \in \mathcal{K}} \Bigl(\prod_{i \in I} X_i \times \prod_{j \notin I} A_j \Bigr) \subset \prod_{i=1}^{m} X_i.$$

Notation: $(X, A)^{\mathcal{K}} = (\boldsymbol{X}, \boldsymbol{A})^{\mathcal{K}}$ when all $(X_i, A_i) = (X, A)$;
$\boldsymbol{X}^{\mathcal{K}} = (\boldsymbol{X}, pt)^{\mathcal{K}}$, $X^{\mathcal{K}} = (X, pt)^{\mathcal{K}}$.

Categorical approach

Category of faces $\mathrm{CAT}(\mathcal{K})$.
Objects: simplices $I \in \mathcal{K}$. Morphisms: inclusions $I \subset J$.

TOP the category of topological spaces.
Define the $\mathrm{CAT}(\mathcal{K})$-diagram

$$\begin{aligned} \mathcal{D}_{\mathcal{K}}(\boldsymbol{X}, \boldsymbol{A})\colon \mathrm{CAT}(\mathcal{K}) &\longrightarrow \mathrm{TOP}, \\ I &\longmapsto (\boldsymbol{X}, \boldsymbol{A})^I, \end{aligned}$$

which maps the morphism $I \subset J$ of $\mathrm{CAT}(\mathcal{K})$ to the inclusion of spaces $(\boldsymbol{X}, \boldsymbol{A})^I \subset (\boldsymbol{X}, \boldsymbol{A})^J$.

Then we have

$$(\boldsymbol{X}, \boldsymbol{A})^{\mathcal{K}} = \operatorname{colim} \mathcal{D}_{\mathcal{K}}(\boldsymbol{X}, \boldsymbol{A}) = \operatorname*{colim}_{I \in \mathcal{K}} (\boldsymbol{X}, \boldsymbol{A})^I.$$

Example

Let $(X, A) = (S^1, pt)$, where S^1 is a circle. Then

$$(S^1)^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (S^1)^I \subset (S^1)^m.$$

When $\mathcal{K} = \{\varnothing, \{1\}, \ldots, \{m\}\}$ (m disjoint points), the polyhedral product $(S^1)^{\mathcal{K}}$ is the wedge $(S^1)^{\vee m}$ of m circles.

When $\mathcal{K}$ consists of all proper subsets of $[m]$ (the boundary $\partial\Delta^{m-1}$ of an $(m-1)$-dimensional simplex), $(S^1)^{\mathcal{K}}$ is the fat wedge of m circles; it is obtained by removing the top-dimensional cell from the m-torus $(S^1)^m$.

For a general $\mathcal{K}$ on m vertices, $(S^1)^{\vee m} \subset (S^1)^{\mathcal{K}} \subset (S^1)^m$.

Example

Let $(X, A) = (\mathbb{R}, \mathbb{Z})$. Then

$$\mathcal{L}_{\mathcal{K}} := (\mathbb{R}, \mathbb{Z})^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (\mathbb{R}, \mathbb{Z})^{I} \subset \mathbb{R}^m.$$

When $\mathcal{K}$ consists of m disjoint points, $\mathcal{L}_{\mathcal{K}}$ is a grid in $\mathbb{R}^m$ consisting of all lines parallel to one of the coordinate axis and passing though integer points.

When $\mathcal{K} = \partial\Delta^{m-1}$, the complex $\mathcal{L}_{\mathcal{K}}$ is the union of all integer hyperplanes parallel to coordinate hyperplanes.

Example

Let $(X, A) = (\mathbb{R}P^\infty, pt)$, where $\mathbb{R}P^\infty = B\mathbb{Z}_2$. Then

$$(\mathbb{R}P^\infty)^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (\mathbb{R}P^\infty)^I \subset (\mathbb{R}P^\infty)^m.$$

Similarly,

$$(\mathbb{C}P^\infty)^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (\mathbb{C}P^\infty)^I \subset (\mathbb{C}P^\infty)^m.$$

Example

Let $(X,A)=(D^1,S^0)$, where $D^1=[-1,1]$ and $S^0=\{1,-1\}$. The real moment-angle complex is

$$\mathcal{R}_{\mathcal{K}} := (D^1,S^0)^{\mathcal{K}} = \bigcup_{I\in\mathcal{K}} (D^1,S^0)^I.$$

It is a cubic subcomplex in the m-cube $(D^1)^m=[-1,1]^m$.

When $\mathcal{K}$ consists of m disjoint points, $\mathcal{R}_{\mathcal{K}}$ is the 1-dimensional skeleton of the cube $[-1,1]^m$. When $\mathcal{K}=\partial\Delta^{m-1}$, $\mathcal{R}_{\mathcal{K}}$ is the boundary of the cube $[-1,1]^m$. Also, $\mathcal{R}_{\mathcal{K}}$ is a topological manifold when $|\mathcal{K}|$ is a sphere.

Let $(X,A)=(D^2,S^1)$. The moment-angle complex is

$$\mathcal{Z}_{\mathcal{K}} := (D^2,S^1)^{\mathcal{K}} = \bigcup_{I\in\mathcal{K}} (D^2,S^1)^I.$$

It is a topological $(m+n)$-manifold when $|\mathcal{K}|\cong S^{n-1}$ is a sphere.

Replacing spaces by groups in the construction of the polyhedral product $\boldsymbol{X}^{\mathcal{K}} = \operatorname{colim}_{I\in\mathcal{K}} \boldsymbol{X}^{I}$ we arrive at the following

Graph product

$\boldsymbol{G} = (G_1, \ldots, G_m)$ a sequence of m (topological) groups, $G_i \neq \{1\}$.

Given $I = \{i_1, \ldots, i_k\} \subset [m]$, set

$$\boldsymbol{G}^{I} = \Bigl\{(g_1, \ldots, g_m) \in \prod_{k=1}^{m} G_k \colon \ g_k = 1 \quad \text{for } k \notin I\Bigr\}.$$

Consider the following $\mathrm{CAT}(\mathcal{K})$-diagram of groups:

$$\mathcal{D}_{\mathcal{K}}(\boldsymbol{G}) \colon \mathrm{CAT}(\mathcal{K}) \longrightarrow \mathrm{GRP}, \qquad I \longmapsto \boldsymbol{G}^{I},$$

which maps a morphism $I \subset J$ to the canonical monomorphism $\boldsymbol{G}^{I} \to \boldsymbol{G}^{J}$.

The graph product of the groups $G_1, \ldots, G_m$ is

$$\boldsymbol{G}^{\mathcal{K}} = \operatorname{colim}^{\mathrm{GRP}} \mathcal{D}_{\mathcal{K}}(\boldsymbol{G}) = \operatorname{colim}^{\mathrm{GRP}}_{I\in\mathcal{K}} \boldsymbol{G}^{I}.$$

The graph product $\boldsymbol{G}^{\mathcal{K}}$ depends only on the 1-skeleton (graph) of $\mathcal{K}$. Namely,

Proposition

The is an isomorphism of groups

$$\boldsymbol{G}^{\mathcal{K}} \cong \mathop{\bigstar}_{k=1}^{m} G_k \big/ (g_i g_j = g_j g_i \ \text{ for } g_i \in G_i,\ g_j \in G_j,\ \{i,j\} \in \mathcal{K}),$$

where $\bigstar_{k=1}^{m} G_k$ *denotes the free product of the groups* G_k*.*

Example

Let $G_i = \mathbb{Z}$. Then $\boldsymbol{G}^{\mathcal{K}}$ is the right-angled Artin group

$$RA_{\mathcal{K}} = F(g_1, \ldots, g_m)\big/(g_i g_j = g_j g_i \text{ for } \{i,j\} \in \mathcal{K}),$$

where $F(g_1, \ldots, g_m)$ is a free group with m generators.

When $\mathcal{K}$ is a full simplex, we have $RA_{\mathcal{K}} = \mathbb{Z}^m$. When $\mathcal{K}$ is m points, we obtain a free group of rank m.

Example

Let $G_i = \mathbb{Z}_2$. Then $\boldsymbol{G}^{\mathcal{K}}$ is the right-angled Coxeter group

$$RC_{\mathcal{K}} = F(g_1, \ldots, g_m)\big/(g_i^2 = 1,\ g_i g_j = g_j g_i \text{ for } \{i,j\} \in \mathcal{K}).$$

2. Classifying spaces

A natural question: when $B(\boldsymbol{G}^{\mathcal{K}}) \simeq (B\boldsymbol{G})^{\mathcal{K}}$?

Proposition

There is a homotopy fibration

$$(E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}} \longrightarrow (B\boldsymbol{G})^{\mathcal{K}} \longrightarrow \prod_{k=1}^{m} BG_k.$$

In particular, there are homotopy fibrations

$$(\mathbb{R}, \mathbb{Z})^{\mathcal{K}} = \mathcal{L}_{\mathcal{K}} \longrightarrow (S^1)^{\mathcal{K}} \longrightarrow (S^1)^m \qquad G = \mathbb{Z}$$

$$(D^1, S^0)^{\mathcal{K}} = \mathcal{R}_{\mathcal{K}} \longrightarrow (\mathbb{R}P^\infty)^{\mathcal{K}} \longrightarrow (\mathbb{R}P^\infty)^m \qquad G = \mathbb{Z}_2$$

$$(D^2, S^1)^{\mathcal{K}} = \mathcal{Z}_{\mathcal{K}} \longrightarrow (\mathbb{C}P^\infty)^{\mathcal{K}} \longrightarrow (\mathbb{C}P^\infty)^m \qquad G = S^1$$

A missing face (a minimal non-face) of $\mathcal{K}$ is a subset $I \subset [m]$ such that $I \notin \mathcal{K}$, but $J \in \mathcal{K}$ for each $J \subsetneq I$.

$\mathcal{K}$ a flag complex if each of its missing faces consists of two vertices. Equivalently, $\mathcal{K}$ is flag if any set of vertices of $\mathcal{K}$ which are pairwise connected by edges spans a simplex.

Every flag complex $\mathcal{K}$ is determined by its 1-skeleton $\mathcal{K}^1$, and is obtained from the graph $\mathcal{K}^1$ by filling in all complete subgraphs by simplices.

Theorem (P.–Ray–Vogt, 2002)

$B(\boldsymbol{G}^{\mathcal{K}}) \simeq (B\boldsymbol{G})^{\mathcal{K}}$ *if and only if* $\mathcal{K}$ *is flag.*

Higher Whitehead products in $\pi_*((B\boldsymbol{G})^{\mathcal{K}})$ are what obstructs the identity $B(\boldsymbol{G}^{\mathcal{K}}) \simeq (B\boldsymbol{G})^{\mathcal{K}}$ in the general case.
This can be fixed by replacing colim by $\mathrm{hocolim}$ in the definition of the graph product $\boldsymbol{G}^{\mathcal{K}} = \mathrm{colim}^{\mathrm{GRP}}_{I\in\mathcal{K}} \boldsymbol{G}^I$.

In the case of discrete groups we obtain

Theorem

Let $\boldsymbol{G}^{\mathcal{K}}$ be a graph product of discrete groups.

1. $\pi_1((B\boldsymbol{G})^{\mathcal{K}}) \cong \boldsymbol{G}^{\mathcal{K}}$.
2. *Both spaces $(B\boldsymbol{G})^{\mathcal{K}}$ and $(E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}}$ are aspherical iff $\mathcal{K}$ is flag.*
3. $\pi_i((B\boldsymbol{G})^{\mathcal{K}}) \cong \pi_i((E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}})$ *for* $i \geqslant 2$.
4. $\pi_1((E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}})$ *is isomorphic to the kernel of the canonical projection* $\boldsymbol{G}^{\mathcal{K}} \to \prod_{k=1}^{m} G_k$ *(the Cartesian subgroup of* $\boldsymbol{G}^{\mathcal{K}}$*).*

Part of proof

Assume now that $\mathcal{K}$ is not flag. Choose a missing face $J = \{j_1, \ldots, j_k\} \subset [m]$ with $k \geqslant 3$ vertices. Let $\mathcal{K}_J = \{I \in \mathcal{K} \colon I \subset J\}$.

Then $(B\boldsymbol{G})^{\mathcal{K}_J}$ is the fat wedge of the spaces $\{BG_j, j \in J\}$, and it is a retract of $(B\boldsymbol{G})^{\mathcal{K}}$.

The homotopy fibre of the inclusion $(B\boldsymbol{G})^{\mathcal{K}_J} \to \prod_{j \in J} BG_j$ is $\Sigma^{k-1} G_{j_1} \wedge \cdots \wedge G_{j_k}$, a wedge of $(k-1)$-dimensional spheres.
Hence, $\pi_{k-1}((B\boldsymbol{G})^{\mathcal{K}_J}) \neq 0$ where $k \geqslant 3$.
Thus, $(B\boldsymbol{G})^{\mathcal{K}_J}$ and $(B\boldsymbol{G})^{\mathcal{K}}$ are non-aspherical.

The rest of the proof (the asphericity of $(E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}}$ and statements (3) and (4)) follow from the homotopy exact sequence of the fibration $(E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}} \to (B\boldsymbol{G})^{\mathcal{K}} \to \prod_{k=1}^{m} BG_k$.

Specialising to the cases $G_k = \mathbb{Z}$ and $G_k = \mathbb{Z}_2$ respectively we obtain:

Corollary

Let $RA_\mathcal{K}$ be a right-angled Artin group.

1. $\pi_1((S^1)^\mathcal{K}) \cong RA_\mathcal{K}$.
2. *Both* $(S^1)^\mathcal{K}$ *and* $\mathcal{L}_\mathcal{K} = (\mathbb{R}, \mathbb{Z})^\mathcal{K}$ *are aspherical iff* $\mathcal{K}$ *is flag.*
3. $\pi_i((S^1)^\mathcal{K}) \cong \pi_i(\mathcal{L}_\mathcal{K})$ *for* $i \geqslant 2$.
4. $\pi_1(\mathcal{L}_\mathcal{K})$ *is isomorphic to the commutator subgroup* $RA'_\mathcal{K}$.

Corollary

Let $RC_\mathcal{K}$ be a right-angled Coxeter group.

1. $\pi_1((\mathbb{R}P^\infty)^\mathcal{K}) \cong RC_\mathcal{K}$.
2. *Both* $(\mathbb{R}P^\infty)^\mathcal{K}$ *and* $\mathcal{R}_\mathcal{K} = (D^1, S^0)^\mathcal{K}$ *are aspherical iff* $\mathcal{K}$ *is flag.*
3. $\pi_i((\mathbb{R}P^\infty)^\mathcal{K}) \cong \pi_i(\mathcal{R}_\mathcal{K})$ *for* $i \geqslant 2$.
4. $\pi_1(\mathcal{R}_\mathcal{K})$ *is isomorphic to the commutator subgroup* $RC'_\mathcal{K}$.

Example

Let $\mathcal{K}$ be an m-cycle (the boundary of an m-gon).
A simple argument with Euler characteristic shows that $\mathcal{R}_{\mathcal{K}} = (D^1, S^0)^{\mathcal{K}}$ is homeomorphic to a closed orientable surface of genus $(m-4)2^{m-3} + 1$.
(This observation goes back to a 1938 work of Coxeter.)
Therefore, the commutator subgroup of the corresponding right-angled Coxeter group $RC_{\mathcal{K}}$ is a surface group.

Similarly, when $|\mathcal{K}| \cong S^2$ (which is equivalent to $\mathcal{K}$ being the boundary of a 3-dimensional simplicial polytope), $\mathcal{R}_{\mathcal{K}}$ is a 3-dimensional manifold.
Therefore, the commutator subgroup of the corresponding $RC_{\mathcal{K}}$ is a 3-manifold group.

3. Commutator subgroups and subalgebras

First consider the case $G_i = S^1$. The homotopy fibration

$$(D^2, S^1)^{\mathcal{K}} = \mathcal{Z}_{\mathcal{K}} \longrightarrow (\mathbb{C}P^\infty)^{\mathcal{K}} \longrightarrow (\mathbb{C}P^\infty)^m$$

splits after looping:

$$\Omega(\mathbb{C}P^\infty)^{\mathcal{K}} \simeq \Omega\mathcal{Z}_{\mathcal{K}} \times T^m$$

Warning: this is not an H-space splitting!

Proposition

There exists an exact sequence of Hopf algebras (over a base ring $\mathbf{k}$*)*

$$\mathbf{k} \longrightarrow H_*(\Omega\mathcal{Z}_{\mathcal{K}}) \longrightarrow H_*(\Omega(\mathbb{C}P^\infty)^{\mathcal{K}}) \xrightarrow{\mathrm{Ab}} \Lambda[u_1, \ldots, u_m] \longrightarrow 0$$

where $\Lambda[u_1, \ldots, u_m]$ *denotes the exterior algebra and* $\deg u_i = 1$.

Here, $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ is the commutator subalgebra of a largely non-commutative algebra $H_*(\Omega(\mathbb{C}P^\infty)^{\mathcal{K}})$.

Consider the graph product Lie algebra

$$L_{\mathcal{K}} = FL\langle u_1, \ldots, u_m \rangle \big/ \big([u_i, u_i] = 0,\ [u_i, u_j] = 0 \text{ for } \{i,j\} \in \mathcal{K} \big),$$

where $FL\langle u_1, \ldots, u_m \rangle$ is the free graded Lie algebra, $\deg u_i = 1$, and $[a, b] = -(-1)^{|a||b|}[b, a]$ denotes the graded Lie bracket.

We can write $L_{\mathcal{K}} = \operatorname{colim}^{\mathrm{GLA}}_{I \in \mathcal{K}} CL\langle u_i \colon i \in I \rangle$, where CL denotes the trivial graded Lie algebra and the colimit taken in the category of graded Lie algebras. (Similar to $RC_{\mathcal{K}} = \operatorname{colim}^{\mathrm{GRP}}_{I \in \mathcal{K}} (\mathbb{Z}_2)^I$.)

The universal enveloping algebra of $L_{\mathcal{K}}$ is the quotient of the free associative algebra $T\langle \lambda_1, \ldots, \lambda_m \rangle$ by the same relations.

Theorem

There is an injective homomorphism of Hopf algebras

$$T\langle u_1, \ldots, u_m \rangle \big/ \big(u_i^2 = 0,\ u_i u_j + u_j u_i = 0 \ \textit{for}\ \{i,j\} \in \mathcal{K} \big) \hookrightarrow H_*\big(\Omega(\mathbb{C}P^\infty)^{\mathcal{K}}\big)$$

which is an isomorphism if and only if $\mathcal{K}$ is flag.

Now consider the case of discrete G_i (e. g., $G_i = \mathbb{Z}_2$). The homotopy fibration

$$(E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}} \longrightarrow (B\boldsymbol{G})^{\mathcal{K}} \longrightarrow \prod_{k=1}^{m} BG_k.$$

gives rise to a short exact sequence of groups

$$1 \longrightarrow \pi_1((E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}}) \longrightarrow \boldsymbol{G}^{\mathcal{K}} \longrightarrow \prod_{k=1}^{m} G_k \longrightarrow 1$$

so

$$\mathrm{Ker}\Big(\boldsymbol{G}^{\mathcal{K}} \to \prod_{k=1}^{m} G_k\Big) = \pi_1((E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}}).$$

In the case of right-angled Artin or Coxeter groups (or when each G_i is abelian), the group above is the commutator subgroup $(\boldsymbol{G}^{\mathcal{K}})'$.

Theorem (Grbić–P–Theriault–Wu, 2012)

Assume that $\mathcal{K}$ is flag. The commutator subalgebra $H_*(\Omega\mathcal{Z}_\mathcal{K})$ is generated by $\sum_{I\subset[m]}\dim\widetilde{H}^0(\mathcal{K}_I)$ iterated commutators of the form

$$[u_j,u_i],\quad [u_{k_1},[u_j,u_i]],\quad \ldots,\quad [u_{k_1},[u_{k_2},\cdots[u_{k_{m-2}},[u_j,u_i]]\cdots]]$$

where $k_1<k_2<\cdots<k_p<j>i$, $k_s\ne i$ for any s, and i is the smallest vertex in a connected component not containing j of the subcomplex $\mathcal{K}_{\{k_1,\ldots,k_p,j,i\}}$. Furthermore, this multiplicative generating set is minimal, that is, the commutators above form a basis in the submodule of indecomposables in $H_*(\Omega\mathcal{Z}_\mathcal{K})$.

Theorem (P–Veryovkin, 2016)

The commutator subgroup $RC'_\mathcal{K}=\pi_1(\mathcal{R}_\mathcal{K})=H_*(\Omega\mathcal{R}_\mathcal{K})$ has a minimal generator set consisting of $\sum_{J\subset[m]}\operatorname{rank}\widetilde{H}_0(\mathcal{K}_J)$ iterated commutators

$$(g_j,g_i),\quad (g_{k_1},(g_j,g_i)),\quad \ldots,\quad (g_{k_1},(g_{k_2},\cdots(g_{k_{m-2}},(g_j,g_i))\cdots)),$$

with the same condition on the indices as in the previous theorem.

4. When the commutator subgroup is free?

A graph Γ is called chordal (in other terminology, triangulated) if each of its cycles with $\geqslant 4$ vertices has a chord.

By a result of Fulkerson–Gross, a graph is chordal if and only if its vertices can be ordered in such a way that, for each vertex i, the lesser neighbours of i form a complete subgraph. (A perfect elimination order.)

Theorem (Grbić–P–Theriault–Wu, 2012)

Let $\mathcal{K}$ be a flag complex and $\mathbf{k}$ a field. The following conditions are equivalent:

1. $H_*(\Omega\mathcal{Z}_\mathcal{K};\mathbf{k})$ *is a free associative algebra;*
2. $\mathcal{Z}_\mathcal{K}$ *has homotopy type of a wedge of spheres;*
3. $\mathcal{K}^1$ *is a chordal graph.*

Theorem (P–Veryovkin, 2016)

Let $\mathcal{K}$ be a flag complex. The following conditions are equivalent:

1. $\mathrm{Ker}(\boldsymbol{G}^{\mathcal{K}} \to \prod_{k=1}^{m} G_k)$ *is a free group;*
2. $(E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}}$ *is homotopy equivalent to a wedge of circles;*
3. $\mathcal{K}^1$ *is a chordal graph.*

Proof

(2)⇒(1) Because $\mathrm{Ker}\Big(\boldsymbol{G}^{\mathcal{K}} \to \prod_{k=1}^{m} G_k\Big) = \pi_1((E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}})$.

(3)⇒(2) Use induction and perfect elimination order.

(1)⇒(3) Assume that $\mathcal{K}^1$ is not chordal. Then, for each chordless cycle of length $\geqslant 4$, one can find a subgroup in $\mathrm{Ker}(\boldsymbol{G}^{\mathcal{K}} \to \prod_{k=1}^{m} G_k)$ which is a surface group. Hence, $\mathrm{Ker}(\boldsymbol{G}^{\mathcal{K}} \to \prod_{k=1}^{m} G_k)$ is not a free group.

Corollary

Let $RA_{\mathcal{K}}$ and $RC_{\mathcal{K}}$ be the right-angled Artin and Coxeter groups corresponding to a simplicial complex $\mathcal{K}$.

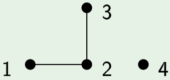
(a) *The commutator subgroup $RA'_{\mathcal{K}}$ is free iff $\mathcal{K}^1$ is a chordal graph.*

(b) *The commutator subgroup $RC'_{\mathcal{K}}$ is free iff $\mathcal{K}^1$ is a chordal graph.*

Part (a) is the result of Servatius, Droms and Servatius.

The difference between (a) and (b) is that the commutator subgroup $RA'_{\mathcal{K}}$ is infinitely generated, unless $RA_{\mathcal{K}} = \mathbb{Z}^m$, while the commutator subgroup $RC'_{\mathcal{K}}$ is finitely generated.

Example

Let $\mathcal{K} =$ 

Then the commutator subgroup $RC'_{\mathcal{K}}$ is free with the following basis:

$$
\begin{gathered}
(g_3, g_1),\ (g_4, g_1),\ (g_4, g_2),\ (g_4, g_3),\\
(g_2, (g_4, g_1)),\ (g_3, (g_4, g_1)),\ (g_1, (g_4, g_3)),\ (g_3, (g_4, g_2)),\\
(g_2, (g_3, (g_4, g_1))).
\end{gathered}
$$

Example

Let $\mathcal{K}$ be an m-cycle with $m \geqslant 4$ vertices.
Then $\mathcal{K}^1$ is not a chordal graph, so the group $RC'_{\mathcal{K}}$ is not free.

In fact, $\mathcal{R}_{\mathcal{K}}$ is an orientable surface of genus $(m-4)2^{m-3} + 1$, so $RC'_{\mathcal{K}} \cong \pi_1(\mathcal{R}_{\mathcal{K}})$ is a one-relator group.

5. One-relator groups

Theorem (Grbić–Ilyasova–P–Simmons, 2020)

Let $\mathcal{K}$ be a flag complex. The following conditions are equivalent:

1. $\pi_1(\mathcal{R}_\mathcal{K}) = RC'_\mathcal{K}$ *is a one-relator group;*
2. $H_2(\mathcal{R}_\mathcal{K}) = \mathbb{Z}$*;*
3. $\mathcal{K} = C_p$ *or* $\mathcal{K} = C_p * \Delta^q$ *for* $p \geqslant 4$ *and* $q \geqslant 0$*, where* C_p *is a* p*-cycle,* Δ^q *is a* q*-simplex, and* $*$ *denotes the join of simplicial complexes.*

Theorem (Grbić–Ilyasova–P–Simmons, 2020)

Let $\mathcal{K}$ be a flag complex. The following conditions are equivalent:

1. $H_*(\Omega\mathcal{Z}_\mathcal{K})$ *is a one-relator algebra;*
2. $H_{2-j,2j}(\mathcal{Z}_\mathcal{K}) = \begin{cases} \mathbb{Z} & \text{if } j = p \text{ for some } p,\ 4 \leqslant p \leqslant m \\ 0 & \text{otherwise;} \end{cases}$
3. $\mathcal{K} = C_p$ *or* $\mathcal{K} = C_p * \Delta^q$ *for* $p \geqslant 4$ *and* $q \geqslant 0$*.*

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