# Foliations arising from configurations of vectors, Gale duality, and moment-angle manifolds

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# Vector configurations and their associated foliations

$V \cong \mathbb{R}^k$ a $k$-dimensional real vector space
$\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ a configuration of $m$ vectors in the dual space $V^*$.
Allow repetitions, but assume that $\gamma_1, \ldots, \gamma_m$ span $V^*$.

Consider the action of $V$ on $\mathbb{R}^m$ given by

$$\begin{aligned} V \times \mathbb{R}^m &\longrightarrow \mathbb{R}^m \\ (\boldsymbol{v}, \boldsymbol{x}) &\mapsto \boldsymbol{v} \cdot \boldsymbol{x} = \big(e^{\langle \gamma_1, \boldsymbol{v} \rangle} x_1, \ldots, e^{\langle \gamma_m, \boldsymbol{v} \rangle} x_m\big). \end{aligned}$$

This is a very classical dynamical system taking its origin in the works of Poincaré. There is a well-known relationship between the linear properties of $\Gamma$ and the topology of the foliation of $\mathbb{R}^m$ by the orbits of the action. We attempt for systematising the existing knowledge on this relationship and proceed by analysing the topology of the quotient (the leaf space) using some recent constructions of toric topology.

# Motivation

The above action $V \times \mathbb{R}^m \to \mathbb{R}^m$ and its holomorphic analogue arise in several important constructions of algebraic geometry and topology:

- Topology of intersections of real and Hermitian quadrics (topology & holomorphic dynamics)
- The quotient construction of toric varieties (the Cox construction) (toric geometry)
- Smooth and complex-analytic structures on moment-angle manifolds (toric topology)

## Example

Consider two actions of $V = \mathbb{R}$ on $\mathbb{R}^2$ given by

$$(v,(x_1,x_2)) \mapsto (e^v x_1, e^v x_2), \tag{1}$$
$$(v,(x_1,x_2)) \mapsto (e^v x_1, e^{-v} x_2). \tag{2}$$

The only non-free orbit for both actions is $\mathbf{0} \in \mathbb{R}^2$, so both actions become free when restricted to $\mathbb{R}^2 \setminus \{\mathbf{0}\}$.

For (1), the quotient $(\mathbb{R}^2 \setminus \{\mathbf{0}\})/\mathbb{R}$ is a circle (a smooth manifold).

For (2), the quotient $(\mathbb{R}^2 \setminus \{\mathbf{0}\})/\mathbb{R}$ is a non-Hausdorff space.

The difference is that (1) is a proper action, while (2) is not.

# Nondegenerate leaves

We consider invariant subsets $U \subset \mathbb{R}^m$ with the property that the restriction of the action $V \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ to $U$ is free.

## Proposition

*The orbit $V\boldsymbol{x}$ of a point $\boldsymbol{x} = (x_1, \dots, x_m) \in \mathbb{R}^m$ under the action $V \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is free iff the subset $\{\gamma_i \colon x_i \neq 0\} \subseteq \Gamma$ spans the whole $V^*$.*

## Proof.

Suppose the orbit $V\boldsymbol{x}$ is not free, i. e. there exists $\boldsymbol{v} \neq 0$ such that

$$(x_1 e^{\langle \gamma_1, \boldsymbol{v} \rangle}, \dots, x_m e^{\langle \gamma_m, \boldsymbol{v} \rangle}) = (x_1, \dots, x_m).$$

Then $\langle \gamma_i, \boldsymbol{v} \rangle = 0$ for $x_i \neq 0$, which implies that the vectors $\gamma_i$ with $x_i \neq 0$ do not span $V^*$. The opposite statement is proved similarly. □

Denote $[m] = \{1, \dots, m\}$ and consider subsets $I = \{i_1, \dots, i_p\} \subseteq [m]$. For each $I$ we denote

$$\Gamma_I := \{\gamma_i : i \in I\} \subseteq \Gamma.$$

Let $\widehat{I} := [m] \setminus I$ denote the complementary subset. We set

$$\mathcal{K}(\Gamma) = \{I \subseteq [m] : \Gamma_{\widehat{I}} \text{ spans } V^*\}.$$

## Proposition

*$\mathcal{K}(\Gamma)$ is a pure simplicial complex of dimension $m - k - 1$.*

## Proof.

If $\Gamma_{\widehat{I}}$ spans $V^*$, then so does $\Gamma_{\widehat{J}} \supset \Gamma_{\widehat{I}}$ for any $J \subset I$. Hence, $\mathcal{K}(\Gamma)$ is a simplicial complex. Also, if $\Gamma_{\widehat{I}}$ spans $V^*$, then it contains a basis of $V^*$. Such a basis has the form $\Gamma_{\widehat{L}}$ for some $L$ with $I \subset L$ and $|L| = m - |\Gamma_{\widehat{L}}| = m - k$. It follows that each face $I \in \mathcal{K}$ is contained in a $(m-k-1)$-dimensional face, so $\mathcal{K}(\Gamma)$ is pure $(m-k-1)$-dimensional. □

Given a simplicial complex $\mathcal{K}$ on $[m]$, define the following open subset in $\mathbb{R}^m$ (the complement of an arrangement of coordinate subspaces):

$$U(\mathcal{K}) = \mathbb{R}^m \setminus \bigcup_{\{i_1,\ldots,i_p\}\notin\mathcal{K}} \{\boldsymbol{x}\colon x_{i_1} = \cdots = x_{i_p} = 0\}.$$

For example, if $\mathcal{K} = \{\varnothing\}$, then $U(\mathcal{K}) = (\mathbb{R}^\times)^m$, where $\mathbb{R}^\times = \mathbb{R}\setminus\{0\}$, and if $\mathcal{K}$ consists of all proper subsets of $[m]$, then $U(\mathcal{K}) = \mathbb{R}^m \setminus \{\mathbf{0}\}$.

### Proposition

*For any subcomplex*

$$\mathcal{K} \subseteq \mathcal{K}(\Gamma) = \{I \subseteq [m]\colon \Gamma_{\widehat{I}} \text{ spans } V^*\},$$

*the restriction of the action $V \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ to $U(\mathcal{K})$ is free.*

We restate this by saying that $U(\mathcal{K})$ consists of nondegenerate leaves of the foliation defined by $V \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ for any $\mathcal{K} \subseteq \mathcal{K}(\Gamma)$.

# Linear Gale duality

Given $\Gamma = (\gamma_1, \ldots, \gamma_m)$, define a linear map $\varGamma\colon \mathbb{R}^m \to V^*$, $\boldsymbol{e}_i \mapsto \gamma_i$. Let $W := \operatorname{Ker} \varGamma$, so we have dual exact sequences

$$0 \longrightarrow W \longrightarrow \mathbb{R}^m \xrightarrow{\varGamma} V^* \longrightarrow 0,$$

$$0 \longrightarrow V \xrightarrow{\varGamma^*} \mathbb{R}^m \xrightarrow{A} W^* \longrightarrow 0,$$

where $\varGamma^*$ takes $\boldsymbol{v}$ to $(\langle\gamma_1, \boldsymbol{v}\rangle, \ldots, \langle\gamma_m, \boldsymbol{v}\rangle)$. Set $\boldsymbol{a}_i := A(\boldsymbol{e}_i)$.

The vector configuration $\mathrm{A} = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ in $W^*$ is called the Gale dual of $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$. The Gale dual of $\mathrm{A}$ is $\Gamma$.

If we choose bases in $V$ and $W$, then $\varGamma$ becomes a $k \times m$-matrix with columns $\gamma_1, \ldots, \gamma_m$ and $A$ becomes an $(m - k) \times m$-matrix with columns $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$. The identity $A\varGamma^* = 0$ implies that the rows of $A$ form a basis in the space of linear relations between the vectors $\gamma_1, \ldots, \gamma_m$.

# Simplicial cones and fans

## Proposition

*For any $I \subseteq [m]$, the vectors in $\mathrm{A}_I$ are linearly independent in $W^*$ iff $\Gamma_{\widehat{I}}$ spans $V^*$.*

A simplicial cone $\sigma$ in $W^*$ consists of nonnegative linear combinations of a set of linearly independent vectors in $W^*$.

A simplicial fan is a finite collection $\Sigma = \{\sigma_1, \ldots, \sigma_s\}$ of simplicial cones such that every face of a cone in $\Sigma$ belongs to $\Sigma$ and the intersection of any two cones in $\Sigma$ is a face of each.

Let $\Sigma$ be a simplicial fan in $W^*$, and let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ be generators of one-dimensional cones of $\Sigma$. The underlying simplicial complex $\mathcal{K} = \mathcal{K}_\Sigma$ is the collection of subsets $I \subseteq [m]$ such that $\{\boldsymbol{a}_i \colon i \in I\}$ spans a cone of $\Sigma$.

A simplicial fan $\Sigma$ is therefore determined by two pieces of data:

- a simplicial complex $\mathcal{K}$ on $[m]$;
- a configuration of vectors $\mathrm{A} = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ in $W^*$ such that for any simplex $I \in \mathcal{K}$ the subset $\mathrm{A}_I = \{\boldsymbol{a}_i \colon i \in I\}$ is linearly independent.

Conversely, given a simplicial complex $\mathcal{K}$ and a vector configuration $\mathrm{A}$, we can define the simplicial cone $\sigma_I = \mathrm{cone}(\mathrm{A}_I)$ for each $I \in \mathcal{K}$.

The 'bunch of cones' $\{\sigma_I \colon I \in \mathcal{K}\}$ patches into a fan $\Sigma$ whenever any two cones $\sigma_I$ and $\sigma_J$ intersect in a common face (which has to be $\sigma_{I \cap J}$). Under this condition, we say that the data $\{\mathcal{K}, \mathrm{A}\}$ define a fan $\Sigma$.

We have the following criterion in terms of the vector configuration $\Gamma = \{\gamma_1, \dots, \gamma_m\}$ Gale dual to $\mathrm{A}$.

## Theorem

*Let $\mathcal{K}$ be a simplicial complex on $[m]$, let $\mathrm{A} = \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_m\}$ be a vector configuration in $W^*$ such that for any simplex $I \in \mathcal{K}$ the subset $\mathrm{A}_I$ is linearly independent, and let $\Gamma = \{\gamma_1, \dots, \gamma_m\}$ be the Gale dual vector configuration. The following conditions are equivalent:*

(a) $\{\mathcal{K}, \mathrm{A}\}$ *define a fan* $\Sigma$;

(b) $\mathrm{relint}\,\mathrm{cone}(\mathrm{A}_I) \cap \mathrm{relint}\,\mathrm{cone}(\mathrm{A}_J) = \varnothing$ *for any* $I, J \in \mathcal{K}$, $I \neq J$;

(c) $\mathrm{relint}\,\mathrm{cone}(\Gamma_{\widehat{I}}) \cap \mathrm{relint}\,\mathrm{cone}(\Gamma_{\widehat{J}}) \neq \varnothing$ *for any* $I, J \in \mathcal{K}$.

# Proper actions

A continuous action $G \times X \to X$, $(g, x) \mapsto g\,x$ of a topological group $G$ on a topological space $X$ is proper if the map $h\colon G \times X \to X \times X$, $(g, x) \mapsto (g\,x, x)$ is proper, that is, $h^{-1}(C)$ is compact for any compact $C \subseteq X \times X$.

Properness is a key property for noncompact Lie group actions:

- the quotient $M/G$ of a proper action of a Lie group action $G$ on a manifold $M$ is Hausdorff;
- the quotient $M/G$ of a smooth, free and proper action of a Lie group $G$ on a smooth manifold $M$ is a smooth manifold.

For our action $V \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ we have the following result:

### Theorem

*Let $\Gamma = \{\gamma_1, \dots, \gamma_m\}$ be a vector configuration in $V^*$ defining the action $V \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$, and let $\mathrm{A} = \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_m\}$ be the Gale dual configuration. Let $\mathcal{K}$ be a simplicial complex on $[m]$ such that for any $I \in \mathcal{K}$ the subset $\Gamma_{\widehat{I}}$ spans $V^*$ (equivalently, the subset $\mathrm{A}_I$ is linearly independent). Then*

(1) *the restricted action $V \times U(\mathcal{K}) \to U(\mathcal{K})$ is free;*

(2) *the action $V \times U(\mathcal{K}) \to U(\mathcal{K})$ is proper iff $\{\mathcal{K}, \mathrm{A}\}$ define a fan.*

If $\{\mathcal{K}, \mathrm{A}\}$ define a complete fan in $W^*$ (i. e. the union of all cones is the whole $W^*$), then the quotient $U(\mathcal{K})/V$ is a compact smooth manifold. It is known in toric topology as the real moment-angle manifold corresponding to $\mathcal{K}$.

# Polytopal fans and intersections of quadrics

The normal fan $\Sigma_P$ of a simple convex polytope $P$ in $W$ is an important example of a complete simplicial fan. In this case, the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ are the inward-pointing normals to the facets of $P$, and a subset $A_I$ spans a cone iff the intersection of facets with normals $\boldsymbol{a}_i$, $i \in I$, is nonempty.

Not every complete simplicial fan is a normal fan! In fact, we have

**Theorem**

*Let $\mathrm{A} = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ and $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ be a pair of Gale dual vector configurations. Assume that $\Sigma = \{\mathrm{cone}\,\mathrm{A}_I : I \in \mathcal{K}\}$ is a fan with convex support (respectively, complete fan). The following conditions are equivalent:*

(a) *$\Sigma$ is a normal fan of polyhedron (respectively, polytope);*

(b) $\bigcap_{I \in \mathcal{K}} \mathrm{relint}\,\mathrm{cone}(\Gamma_{\widehat{I}}) \neq \varnothing$.

Therefore, the data $\{\mathcal{K}, \mathrm{A}\}$ define a fan $\Sigma$ iff the relative interiors of Gale dual cones $\operatorname{cone}\Gamma_{\widehat{I}}$ have pairwise nonempty intersections, and $\Sigma$ is the normal fan of a polytope iff all the cones $\operatorname{cone}\Gamma_{\widehat{I}}$ have a common relative interior point.

In the polytopal case, the leaf space $U(\mathcal{K})/V$ can be described as an intersection of quadrics:

## Theorem

*For any $\boldsymbol{c} \in \bigcap_{I\in\mathcal{K}} \operatorname{relint}\operatorname{cone}(\Gamma_{\widehat{I}})$, the quotient $U(\mathcal{K})/V$ is diffeomorphic to*

$$\{\boldsymbol{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m \colon \gamma_1 x_1^2 + \cdots + \gamma_m x_m^2 = \boldsymbol{c}\}.$$

## Idea of proof.

The function $f\colon \mathbb{R}^m \to \mathbb{R}$, $f(\boldsymbol{x}) = \|\gamma_1 x_1^2 + \cdots + \gamma_m x_m^2 - \boldsymbol{c}\|^2$ has a unique minimum at each orbit $V\boldsymbol{x}$, and the set of these minima is the intersection of quadrics above. □

# Holomorphic actions

$V \cong \mathbb{C}^\ell$ a complex space (think of endowing $V \cong \mathbb{R}^k$ with a complex structure, provided that $k = 2\ell$ is even).
$\Gamma = \{\gamma_1, \dots, \gamma_m\}$ a configuration of vectors in $V^*$.

Consider the action of $V$ on $\mathbb{C}^m$ given by

$$V \times \mathbb{C}^m \longrightarrow \mathbb{C}^m$$
$$(\boldsymbol{v}, \boldsymbol{z}) \mapsto \boldsymbol{v} \cdot \boldsymbol{z} = \left(e^{\langle \gamma_1, \boldsymbol{v} \rangle} z_1, \dots, e^{\langle \gamma_m, \boldsymbol{v} \rangle} z_m\right).$$

Provided that the holomorphic action $V \times U(\mathcal{K}) \to U(\mathcal{K})$ is free and proper (the *fan condition*), the quotient $\mathcal{Z}_K = U(\mathcal{K})/V$ is a complex-analytic manifold (the complex moment-angle manifold).

This construction leads to a new family on *non-Kähler* complex manifolds, which includes the classical series of Hopf and Calabi–Eckmann manifolds.

# References (to some earlier works)

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