Higher Whitehead products in moment-angle complexes

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Polyhedral products and moment-angle complexes

$(\boldsymbol{X}, \boldsymbol{A}) = \{(X_1, A_1), \dots, (X_m, A_m)\}$ a sequence of pairs of pointed cell complexes, $pt \in A_i \subset X_i$.

$\mathcal{K}$ a simplicial complex on $[m] = \{1, 2, \dots, m\}$, $\qquad \varnothing \in \mathcal{K}$.

Given $I = \{i_1, \dots, i_k\} \subset [m]$, set

$$(\boldsymbol{X}, \boldsymbol{A})^I = Y_1 \times \cdots \times Y_m \qquad \text{where } Y_i = \begin{cases} X_i & \text{if } i \in I, \\ A_i & \text{if } i \notin I. \end{cases}$$

The $\mathcal{K}$-polyhedral product of $(\boldsymbol{X}, \boldsymbol{A})$ is

$$(\boldsymbol{X}, \boldsymbol{A})^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (\boldsymbol{X}, \boldsymbol{A})^I = \bigcup_{I \in \mathcal{K}} \Bigl(\prod_{i \in I} X_i \times \prod_{j \notin I} A_j \Bigr) \subset X_1 \times \cdots \times X_m.$$

Notation: $(X, A)^{\mathcal{K}} = (\boldsymbol{X}, \boldsymbol{A})^{\mathcal{K}}$ when all $(X_i, A_i) = (X, A)$;
$\boldsymbol{X}^{\mathcal{K}} = (\boldsymbol{X}, pt)^{\mathcal{K}}$, $X^{\mathcal{K}} = (X, pt)^{\mathcal{K}}$.

The moment-angle complex

$$\mathcal{Z}_{\mathcal{K}} = (D^2, S^1)^{\mathcal{K}} = \bigcup_{I\in\mathcal{K}} \Bigl(\prod_{i\in I} D^2 \times \prod_{j\notin I} S^1\Bigr)$$

$\mathcal{Z}_{\mathcal{K}}$ is an $(m+n)$-dimensional manifold when $|\mathcal{K}| \cong S^{n-1}$.

$$(\mathbb{C}P^\infty)^{\mathcal{K}} = \bigcup_{I\in\mathcal{K}} (\mathbb{C}P^\infty)^I \subset (\mathbb{C}P^\infty)^m = BT^m$$

is sometimes called the Davis–Januszkiewicz space.

Have a homotopy fibration

$$\mathcal{Z}_{\mathcal{K}} \longrightarrow (\mathbb{C}P^\infty)^{\mathcal{K}} \longrightarrow (\mathbb{C}P^\infty)^m$$

Example

$\mathcal{K} = \bullet\ \bullet \qquad \mathcal{Z}_{\mathcal{K}} = (D^2 \times S^1) \cup (S^1 \times D^2) \cong S^3,$

$$S^3 = \mathrm{hofibre}(\mathbb{C}P^\infty \vee \mathbb{C}P^\infty \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty)$$

There exists a homotopy fibration

$$\begin{array}{ccccc} \mathcal{Z}_{\mathcal{K}} & \longrightarrow & (\mathbb{C}P^\infty)^{\mathcal{K}} & \longrightarrow & (\mathbb{C}P^\infty)^m \\ \| & & \| & & \| \\ (D^2, S^1)^{\mathcal{K}} & & (\mathbb{C}P^\infty, pt)^{\mathcal{K}} & & (\mathbb{C}P^\infty, \mathbb{C}P^\infty)^{\mathcal{K}} \end{array}$$

which splits after looping:

$$\Omega(\mathbb{C}P^\infty)^{\mathcal{K}} \simeq \Omega\mathcal{Z}_{\mathcal{K}} \times T^m$$

Warning: this is not an H-space splitting!

Proposition

There exists an exact sequence of Hopf algebras (over a base ring **k***)*

$$\mathbf{k} \longrightarrow H_*(\Omega\mathcal{Z}_{\mathcal{K}}) \longrightarrow H_*(\Omega(\mathbb{C}P^\infty)^{\mathcal{K}}) \xrightarrow{\mathrm{Ab}} \Lambda[u_1, \ldots, u_m] \longrightarrow 0$$

where $\Lambda[u_1, \ldots, u_m]$ *denotes the exterior algebra and* $\deg u_i = 1$.

The face ring (the Stanley–Reisner ring) of $\mathcal{K}$ is given by

$$\mathbf{k}[\mathcal{K}] = \mathbf{k}[v_1,\ldots,v_m]\big/\big(v_{i_1}\cdots v_{i_k} = 0 \quad \text{if } \{i_1,\ldots,i_k\} \notin \mathcal{K}\big)$$

where $\deg v_i = 2$.

Theorem (Buchstaber-P)

$$\begin{aligned}
H^*((\mathbb{C}P^\infty)^{\mathcal{K}}) &\cong \mathbf{k}[\mathcal{K}] \\
H_*(\Omega(\mathbb{C}P^\infty)^{\mathcal{K}}) &\cong \operatorname{Ext}_{\mathbf{k}[\mathcal{K}]}(\mathbf{k},\mathbf{k}) && \mathbf{k} \text{ is a field} \\
H^*(\mathcal{Z}_{\mathcal{K}}) &\cong \operatorname{Tor}_{\mathbf{k}[v_1,\ldots,v_m]}(\mathbf{k}[\mathcal{K}],\mathbf{k}) \\
&\cong H\big(\Lambda[u_1,\ldots,u_m]\otimes\mathbf{k}[\mathcal{K}], d\big), && du_i = v_i,\ dv_i = 0 \\
&\cong \bigoplus_{I\subset[m]} \widetilde{H}^{*-|I|-1}(\mathcal{K}_I) && \mathcal{K}_I = \mathcal{K}|_I
\end{aligned}$$

Problems

- identifying the homotopy type of the moment-angle complex $\mathcal{Z}_\mathcal{K}$;
- describing the multiplication and higher Massey products in the Tor-algebra $H^*(\mathcal{Z}_\mathcal{K}) = \mathrm{Tor}_{\mathbf{k}[v_1,\ldots,v_m]}(\mathbf{k}[\mathcal{K}],\mathbf{k})$ of the face ring $\mathbf{k}[\mathcal{K}]$;
- describing the Yoneda algebra $\mathrm{Ext}_{\mathbf{k}[\mathcal{K}]}(\mathbf{k},\mathbf{k})$ in terms of generators and relations;
- describing the structure of the Pontryagin algebra $H_*(\Omega(\mathbb{C}P^\infty)^\mathcal{K})$ and its commutator subalgebra $H_*(\Omega\mathcal{Z}_\mathcal{K})$ via iterated and higher Whitehead (Samelson) products;
- identifying the homotopy type of the loop spaces $\Omega(\mathbb{C}P^\infty)^\mathcal{K}$ and $\Omega\mathcal{Z}_\mathcal{K}$.

The case of a flag complex

A missing face of $\mathcal{K}$ is a subset $I \subset [m]$ such that $I \notin \mathcal{K}$, but every proper subset of I is a simplex of $\mathcal{K}$.

$\mathcal{K}$ is a flag complex if one of the following equivalent conditions holds:

- each missing face has two vertices;
- any set of vertices of $\mathcal{K}$ which are pairwise connected by edges spans a simplex;
- $\mathbf{k}[\mathcal{K}]$ is a quadratic algebra (a quotient of a tensor algebra by an ideal generated by quadratic monomials).

$$
\begin{array}{ccc}
\{\text{flag complexes on } [m]\} & \stackrel{1-1}{\longleftrightarrow} & \{\text{simple graphs on } [m]\} \\
\mathcal{K} & \to & \mathcal{K}^1 \quad (\text{one-skeleton}) \\
\mathcal{K}(\Gamma) & \leftarrow & \Gamma
\end{array}
$$

where $\mathcal{K}(\Gamma)$ is the clique complex of Γ.

Theorem

For any flag complex $\mathcal{K}$, there is an isomorphism

$$H_*\big(\Omega(\mathbb{C}P^\infty)^{\mathcal{K}})\big) \cong T\langle\lambda_1,\dots,\lambda_m\rangle\big/\big(\lambda_i^2=0,\ \lambda_i\lambda_j+\lambda_j\lambda_i=0 \text{ for } \{i,j\}\in\mathcal{K}\big)$$

where $T\langle\lambda_1,\dots,\lambda_m\rangle$ is the free algebra on m generators of degree 1.

Proof. By Adams, $H_*\big(\Omega(\mathbb{C}P^\infty)^{\mathcal{K}}\big) \cong \mathrm{Ext}_{\mathbf{k}[\mathcal{K}]}(\mathbf{k},\mathbf{k})$. By Fröberg, $\mathbf{k}[\mathcal{K}]$ is Koszul, so $\mathrm{Ext}_{\mathbf{k}[\mathcal{K}]}(\mathbf{k},\mathbf{k})$ is its quadratic dual, written above. $\square$

Remember the exact sequence of Hopf algebras

$$\mathbf{k} \longrightarrow H_*(\Omega\mathcal{Z}_{\mathcal{K}}) \longrightarrow H_*(\Omega(\mathbb{C}P^\infty)^{\mathcal{K}}) \xrightarrow{\mathrm{Ab}} \Lambda[u_1,\dots,u_m] \longrightarrow 0$$

Proposition

For any flag complex $\mathcal{K}$, the Poincaré series of $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ is given by

$$P\big(H_*(\Omega\mathcal{Z}_{\mathcal{K}});t\big) = \frac{1}{(1+t)^{m-n}(1-h_1t+\dots+(-1)^nh_nt^n)},$$

where $\boldsymbol{h}(\mathcal{K})=(h_0,h_1\dots,h_n)$ is the h-vector of $\mathcal{K}$.

Whitehead products

The *ith coordinate map*

$$\mu_i\colon (D^2,S^1)\to S^2\cong \mathbb{C}P^1\hookrightarrow (\mathbb{C}P^\infty)^{\vee m}\hookrightarrow (\mathbb{C}P^\infty)^{\mathcal{K}}.$$

Here the second map is the inclusion of the i-th summand in the wedge. The third map is induced by the embedding of m disjoint points into $\mathcal{K}$.

The Whitehead product $[\mu_i,\mu_j]$ of μ_i and μ_j is the homotopy class of

$$S^3\cong\partial D^4\cong\partial(D^2\times D^2)\cong(D^2\times S^1)\cup(S^1\times D^2)\xrightarrow{[\mu_i,\mu_j]}(\mathbb{C}P^\infty)^{\mathcal{K}}$$

where

$$[\mu_i,\mu_j](x,y)=\begin{cases}\mu_i(x) & \text{for } (x,y)\in D^2\times S^1;\\ \mu_j(y) & \text{for } (x,y)\in S^1\times D^2.\end{cases}$$

Every Whitehead product $[\mu_i, \mu_j]$ becomes trivial after composing with the embedding $(\mathbb{C}P^\infty)^{\mathcal{K}} \hookrightarrow (\mathbb{C}P^\infty)^m \simeq K(\mathbb{Z}^m, 2)$. This implies that $[\mu_i, \mu_j]\colon S^3 \to (\mathbb{C}P^\infty)^{\mathcal{K}}$ lifts to the fibre $\mathcal{Z}_{\mathcal{K}}$, as shown next:

$$\begin{tikzcd} \mathcal{Z}_{\mathcal{K}} \arrow[r] & (\mathbb{C}P^\infty)^{\mathcal{K}} \arrow[r] & (\mathbb{C}P^\infty)^m \\ & S^3 \arrow[ul, dashed] \arrow[u, "{[\mu_i,\mu_j]}"'] & \end{tikzcd}$$

We use the same notation $[\mu_i, \mu_j]$ for a lifted map $S^3 \to \mathcal{Z}_{\mathcal{K}}$. Such a lift can be chosen canonically as the inclusion of a subcomplex

$$[\mu_i, \mu_j]\colon S^3 \cong (D^2 \times S^1) \cup (S^1 \times D^2) \hookrightarrow \mathcal{Z}_{\mathcal{K}}.$$

The Whitehead product $[\mu_i, \mu_j]$ is trivial if and only if the map $[\mu_i, \mu_j]\colon S^3 \to \mathcal{Z}_{\mathcal{K}}$ can be extended to a map $D^4 \cong D_i^2 \times D_j^2 \hookrightarrow \mathcal{Z}_{\mathcal{K}}$. This is equivalent to the condition that $\Delta(i,j) = \{i,j\}$ is a 1-simplex of $\mathcal{K}$.

Higher Whitehead products are defined inductively as follows.
Assume that the $(n-1)$-fold product

$$[\mu_{i_1}, \ldots, \widehat{\mu_{i_k}}, \ldots, \mu_{i_n}]\colon S^{2(n-1)-1} \to (\mathbb{C}P^\infty)^{\mathcal{K}}$$

is trivial for any k. Then there exists a canonical extension

$$\overline{[\mu_{i_1}, \ldots, \widehat{\mu_{i_k}}, \ldots, \mu_{i_n}]}\colon D^2_{i_1} \times \cdots \times D^2_{i_{k-1}} \times D^2_{i_{k+1}} \times \cdots \times D^2_{i_n} \hookrightarrow \mathcal{Z}_{\mathcal{K}} \to (\mathbb{C}P^\infty)^{\mathcal{K}}.$$

The n-fold product $[\mu_{i_1}, \ldots, \mu_{i_n}]$ the homotopy class of the map

$$S^{2n-1} \cong \partial(D^2_{i_1} \times \cdots \times D^2_{i_n}) \cong \bigcup_{k=1}^{n} (D^2_{i_1} \times \cdots \times S^1_{i_k} \times \cdots \times D^2_{i_n}) \to (\mathbb{C}P^\infty)^{\mathcal{K}}$$

given by

$$[\mu_{i_1}, \ldots, \mu_{i_n}](x_1, \ldots, x_n) = \overline{[\mu_{i_1}, \ldots, \widehat{\mu}_{i_k}, \ldots, \mu_{i_n}]}(x_1, \ldots, \widehat{x}_k, \ldots, x_n) \quad \text{if } \ x_k \in S^1_{i_k}.$$

Example

$[\mu_{i_1},\ldots,\mu_{i_p}]$ is defined in $\pi_{2p-1}((\mathbb{C}P^\infty)^{\mathcal{K}})$ iff $\partial\Delta(i_1,\ldots,i_p)\subset\mathcal{K}$, and $[\mu_{i_1},\ldots,\mu_{i_p}]$ is trivial iff $\Delta(i_1,\ldots,i_p)\subset\mathcal{K}$.

We consider general iterated higher Whitehead products. For example,

$$\Big[\mu_1,\mu_2,[\mu_3,\mu_4,\mu_5],\big[\mu_6,\mu_{13},[\mu_7,\mu_8,\mu_9],\mu_{10}\big],[\mu_{11},\mu_{12}]\Big].$$

Nested products have the form

$$w=\Big[\ldots\big[[\mu_{i_{11}},\ldots,\mu_{i_{1p_1}}],\mu_{i_{21}},\ldots,\mu_{i_{2p_2}}\big]\ldots,\mu_{i_{np_n}}\Big]\colon S^{d(w)}\to(\mathbb{C}P^\infty)^{\mathcal{K}}$$

Any iterated higher Whitehead product lifts to a map $S^{d(w)}\to\mathcal{Z}_{\mathcal{K}}$.

We say that a simplicial complex $\mathcal{K}$ realises a higher iterated Whitehead product w if w is a nontrivial element of $\pi_*(\mathcal{Z}_{\mathcal{K}})$.

For example, the complex $\partial\Delta(i_1,\ldots,i_p)$ realises the single (non-iterated) higher Whitehead product $[\mu_{i_1},\ldots,\mu_{i_p}]$.

Have adjoints of the coordinate maps

$$\overline{\mu}_i\colon S^1 \to \Omega(\mathbb{C}P^\infty)^{\mathcal{K}}$$

and Samelson products

$$[\overline{\mu}_i, \overline{\mu}_j]\colon S^2 \to \Omega(\mathbb{C}P^\infty)^{\mathcal{K}}.$$

Theorem

For any simplicial complex $\mathcal{K}$, there is an injective homomorphism

$$T\langle\lambda_1,\ldots,\lambda_m\rangle\big/\big(\lambda_i^2=0,\ \lambda_i\lambda_j+\lambda_j\lambda_i=0 \text{ for } \{i,j\}\in\mathcal{K}\big) \hookrightarrow H_*\big(\Omega(\mathbb{C}P^\infty)^{\mathcal{K}}\big)$$

which becomes an isomorphism when $\mathcal{K}$ is a flag complex.
Here $\lambda_i \in H_1(\Omega(\mathbb{C}P^\infty)^{\mathcal{K}})$ is the Hurewicz image of $\overline{\mu}_i \in \pi_1(\Omega(\mathbb{C}P^\infty)^{\mathcal{K}})$, and the graded commutator $\lambda_i\lambda_j + \lambda_j\lambda_i$ is the Hurewicz image of the Samelson product $[\overline{\mu}_i, \overline{\mu}_j] \in \pi_2(\Omega(\mathbb{C}P^\infty)^{\mathcal{K}})$.

When $\mathcal{K}$ is not a flag complex, higher Whitehead (Samelson) products appear in $\pi_*((\mathbb{C}P^\infty)^\mathcal{K})$ (resp. in $\pi_*(\Omega(\mathbb{C}P^\infty)^\mathcal{K})$ and lift to $\pi_*(\mathcal{Z}_\mathcal{K})$ (resp. to $\pi_*(\Omega\mathcal{Z}_\mathcal{K})$).

Lifts $S^p \to \mathcal{Z}_\mathcal{K}$ of higher iterated Whitehead products of the μ_i provide an important family of spherical classes in $H_*(\mathcal{Z}_\mathcal{K})$. We may ask the following question:

Question

Assume that $\mathcal{Z}_\mathcal{K}$ is homotopy equivalent to a wedge of spheres. Is it true that all wedge summands are represented by lifts $S^p \to \mathcal{Z}_\mathcal{K}$ of higher iterated Whitehead products of the canonical maps $\mu_i\colon S^2 \to (\mathbb{C}P^\infty)^\mathcal{K}$?

This is the case for many important classes of simplicial complexes: flag, shifted, shellable, totally fillable. However, in general the answer in negative. A counterexample was found by S. Abramyan in 2018.

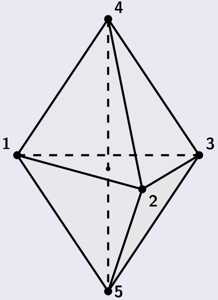
Substitution of simplicial complexes

Let $\mathcal{K}$ be a simplicial complex on the set $[m]$, and let $\mathcal{K}_1, \ldots, \mathcal{K}_m$ be a set of m simplicial complexes. We refer to the simplicial complex

$$\begin{aligned}\mathcal{K}(\mathcal{K}_1, \ldots, \mathcal{K}_m) \\ &= \{ I_{j_1} \sqcup \cdots \sqcup I_{j_k} \mid I_{j_l} \in \mathcal{K}_{j_l},\ l = 1, \ldots, k \quad \text{and} \quad \{j_1, \ldots, j_k\} \in \mathcal{K} \}\end{aligned}$$

as the substitution of $\mathcal{K}_1, \ldots, \mathcal{K}_m$ into $\mathcal{K}$.

Substitution complex $\partial\Delta(\partial\Delta(1,2,3),4,5)$



Next we describe inductively the canonical simplicial complex $\partial\Delta_w$ associated with a general iterated higher Whitehead product w.

Start with the boundary of simplex $\partial\Delta(i_1,\ldots,i_m)$, corresponding to a single higher Whitehead product $[\mu_{i_1},\ldots,\mu_{i_m}]$.
Now write a general iterated higher Whitehead product recursively as

$$w=[w_1,\ldots,w_q,\mu_{i_1},\ldots,\mu_{i_p}]\in\pi_*(\mathcal{Z}_\mathcal{K}),$$

where $w_1,\ldots,w_q$ are nontrivial general iterated higher Whitehead products, $q\geq 0$.

Assign to w the substitution complex

$$\partial\Delta_w\stackrel{\mathrm{def}}{=}\partial\Delta(\partial\Delta_{w_1},\ldots,\partial\Delta_{w_q},i_1,\ldots,i_p).$$

Proposition

The complex $\partial\Delta_w$ is homotopy equivalent to a wedge of spheres.

Realisation of higher Whitehead products

Recall that a simplicial complex $\mathcal{K}$ realises a higher iterated Whitehead product w if w is a nontrivial element of $\pi_*(\mathcal{Z}_\mathcal{K})$.

For example, the complex $\mathcal{K} = \partial\Delta(i_1, \dots, i_p)$ realises the single (non-iterated) higher Whitehead product $[\mu_{i_1}, \dots, \mu_{i_p}] \in \pi_{2p+1}(\mathcal{Z}_\mathcal{K}) = \pi_{2p+1}(S^{2p+1})$.

Theorem (Abramyan–P)

Let $w_1, \dots, w_q$ be nontrivial iterated higher Whitehead products. The substitution complex $\partial\Delta_w$ realises the iterated higher Whitehead product

$$w = [w_1, \dots, w_q, \mu_{i_1}, \dots, \mu_{i_p}].$$

For a particular configuration of brackets, a more precise statement holds.

Theorem (Abramyan–P)

Let $w_j = [\mu_{j_1}, \ldots, \mu_{j_{p_j}}]$, $j = 1, \ldots, q$, be nontrivial single higher Whitehead products. Consider an iterated higher Whitehead product

$$w = [w_1, \ldots, w_q, \mu_{i_1}, \ldots, \mu_{i_p}].$$

Then the product w is

(a) *defined in $\pi_*(\mathcal{Z}_\mathcal{K})$ if and only if $\mathcal{K}$ contains $\partial\Delta_w = \partial\Delta\big(\partial\Delta_{w_1}, \ldots, \partial\Delta_{w_q}, i_1, \ldots, i_p\big)$ as a subcomplex, where $\partial\Delta_{w_j} = \partial\Delta(j_1, \ldots, j_{p_j})$, $j = 1, \ldots, q$;*

(b) *trivial in $\pi_*(\mathcal{Z}_\mathcal{K})$ if and only if $\mathcal{K}$ contains a subcomplex*

$$\Delta\big(\partial\Delta_{w_1}, \ldots, \partial\Delta_{w_q}, i_1, \ldots, i_p\big) = \partial\Delta_{w_1} * \cdots * \partial\Delta_{w_q} * \Delta(i_1, \ldots, i_p).$$

Note that assertion (a) implies that $\partial\Delta_w$ is the smallest simplicial complex realising the Whitehead product w.

Example

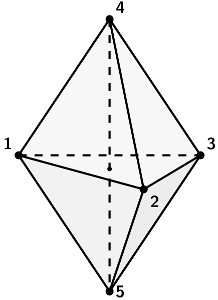
Take $w = \big[[\mu_1, \mu_2, \mu_3], \mu_4, \mu_5\big]$.

For the existence of w it is necessary that the brackets $\big[[\mu_1, \mu_2, \mu_3], \mu_4\big]$, $\big[[\mu_1, \mu_2, \mu_3], \mu_5\big]$ and $[\mu_4, \mu_5]$ vanish.

This implies that $\mathcal{K}$ contains subcomplexes $\partial\Delta(1,2,3) * \Delta(4)$, $\partial\Delta(1,2,3) * \Delta(5)$ and $\Delta(4,5)$.

Hence, $\mathcal{K}$ contains the complex $\partial\Delta\big(\partial\Delta(1,2,3), 4, 5\big)$ shown right.

Therefore, the latter is the smallest complex realising the Whitehead bracket $w = \big[[\mu_1, \mu_2, \mu_3], \mu_4, \mu_5\big]$.



For $\mathcal{K} = \partial\Delta\big(\partial\Delta(1,2,3), 4, 5\big)$, we have $\mathcal{Z}_\mathcal{K} \simeq (S^5)^{\vee 4} \vee (S^6)^{\vee 3} \vee S^7 \vee S^8$, and each sphere is a Whitehead product.

For example, S^7 corresponds to $w = \Big[\big[[\mu_3, \mu_4, \mu_5], \mu_1\big], \mu_2\Big]$, and S^8 corresponds to $w = \big[[\mu_1, \mu_2, \mu_3], \mu_4, \mu_5\big]$.

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