Lecture 1. Toric Varieties: Basics

Taras Panov

Lomonosov Moscow State University

Summer School
Current Developments in Geometry
Novosibirsk, 27 August–1 September 2018

An algebraic torus is a commutative complex algebraic group isomorphic to a product $(\mathbb{C}^\times)^n$ of copies of the multiplicative group $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. It contains a compact torus T^n as a Lie (but not algebraic) subgroup.

A toric variety is a normal complex algebraic variety V containing an algebraic torus $(\mathbb{C}^\times)^n$ as a Zariski open subset in such a way that the natural action of $(\mathbb{C}^\times)^n$ on itself extends to an action on V.

It follows that $(\mathbb{C}^\times)^n$ acts on V with a dense orbit. Toric varieties originally appeared as equivariant compactifications of an algebraic torus, although non-compact (e.g., affine) examples are now of equal importance.

Example

The algebraic torus $(\mathbb{C}^\times)^n$ and the affine space $\mathbb{C}^n$ are the simplest examples of toric varieties. A compact example is given by the projective space $\mathbb{C}P^n$ on which the torus acts in homogeneous coordinates as

$$(t_1, \ldots, t_n) \cdot (z_0 : z_1 : \ldots : z_n) = (z_0 : t_1 z_1 : \ldots : t_n z_n).$$

Polyhedral cones

A convex polyhedral cone generated by $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k \in \mathbb{R}^n$:

$$\sigma = \mathbb{R}_{\geqslant}\langle \boldsymbol{a}_1, \ldots, \boldsymbol{a}_k \rangle = \{\mu_1 \boldsymbol{a}_1 + \cdots + \mu_k \boldsymbol{a}_k \colon \mu_i \in \mathbb{R}_{\geqslant}\}.$$

Given a cone σ, can choose a *minimal* generating set $\boldsymbol{a}_1, \ldots \boldsymbol{a}_k$.
It is defined up to multiplication of vectors by positive constants.

A cone is rational if its generators $\boldsymbol{a}_i$ can be chosen from the integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$.
Then assume that each $\boldsymbol{a}_i \in \mathbb{Z}^n$ is primitive, i. e. it is the smallest lattice vector in the ray defined by it.

A cone is strongly convex if it does not contain a line.
A cone is simplicial if it is generated by part of a basis of $\mathbb{R}^n$.
A cone is regular if it is generated by part of a basis of $\mathbb{Z}^n$.

Fans

A fan is a finite collection $\Sigma = \{\sigma_1, \dots, \sigma_s\}$ of strongly convex cones in some $\mathbb{R}^n$ such that

- every face of a cone in Σ belongs to Σ, and
- the intersection of any two cones in Σ is a face of each.

A fan Σ is rational (respectively, simplicial, regular) if every cone in Σ is rational (respectively, simplicial, regular).

A fan $\Sigma = \{\sigma_1, \dots, \sigma_s\}$ is called complete if $\sigma_1 \cup \dots \cup \sigma_s = \mathbb{R}^n$.

Algebraic geometry of toric varieties is translated completely into the language of combinatorial and convex geometry.
Namely, there is a bijective correspondence between *rational* fans in n-dimensional space and complex n-dimensional toric varieties.

Under this correspondence,

$$\begin{aligned} \text{cones} &\longleftrightarrow \text{affine varieties} \\ \text{complete fans} &\longleftrightarrow \text{complete (compact) varieties} \\ \text{normal fans of polytopes} &\longleftrightarrow \text{projective varieties} \\ \text{regular fans} &\longleftrightarrow \text{nonsingular varieties} \\ \text{simplicial fans} &\longleftrightarrow \text{orbifolds} \end{aligned}$$

N a lattice of rank n (isomorphic to $\mathbb{Z}^n$),
$N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ its ambient n-dimensional real vector space.
$\mathbb{C}^\times_N = N \otimes_{\mathbb{Z}} \mathbb{C}^\times \cong (\mathbb{C}^\times)^n$ the algebraic torus.
All cones and fans are assumed to be rational.

Affine toric varieties

Given a cone $\sigma \subset N_{\mathbb{R}}$, its dual cone is

$$\sigma^{\vee} = \{ \boldsymbol{x} \in N_{\mathbb{R}}^* \colon \langle \boldsymbol{u}, \boldsymbol{x} \rangle \geqslant 0 \ \text{ for all } \ \boldsymbol{u} \in \sigma \}.$$

We have $(\sigma^{\vee})^{\vee} = \sigma$, and $\sigma^{\vee}$ is strongly convex iff $\dim \sigma = n$.

$S_\sigma := \sigma^{\vee} \cap N^*$ a finitely generated semigroup (with respect to addition).

$A_\sigma = \mathbb{C}[S_\sigma]$ the semigroup ring of S_σ.
It is a commutative finitely generated $\mathbb{C}$-algebra, with a $\mathbb{C}$-vector space basis $\{\chi^{\boldsymbol{u}} \colon \boldsymbol{u} \in S_\sigma\}$, and multiplication $\chi^{\boldsymbol{u}} \cdot \chi^{\boldsymbol{u}'} = \chi^{\boldsymbol{u}+\boldsymbol{u}'}$, so $\chi^{\boldsymbol{0}}$ is the unit.

$V_\sigma := \operatorname{Spec}(A_\sigma)$ the affine toric variety corresponding to σ,
$A_\sigma = \mathbb{C}[V_\sigma]$ is the algebra of regular functions on V_σ.

Example

Let $N = \mathbb{Z}^n$ and $\sigma = \mathbb{R}_{\geqslant}\langle \boldsymbol{e}_1, \ldots, \boldsymbol{e}_k \rangle$, where $0 \leqslant k \leqslant n$.
Then $S_\sigma = \sigma^\vee \cap N^*$ is generated by $\boldsymbol{e}_1^*, \ldots, \boldsymbol{e}_k^*$ and $\pm\boldsymbol{e}_{k+1}^*, \ldots, \pm\boldsymbol{e}_n^*$, and

$$A_\sigma \cong \mathbb{C}[x_1, \ldots, x_k, x_{k+1}, x_{k+1}^{-1}, \ldots, x_n, x_n^{-1}],$$

where $x_i := \chi^{\boldsymbol{e}_i^*}$. The corresponding affine variety is

$$V_\sigma \cong \mathbb{C} \times \cdots \times \mathbb{C} \times \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times = \mathbb{C}^k \times (\mathbb{C}^\times)^{n-k}.$$

In particular, for $k = n$ we obtain $V_\sigma = \mathbb{C}^n$,
and for $k = 0$ (i. e. $\sigma = \{\mathbf{0}\}$) we obtain $V_\sigma = (\mathbb{C}^\times)^n$, the torus.

By choosing a multiplicative generator set in A_σ we represent it as

$$A_\sigma = \mathbb{C}[x_1, \ldots, x_r]/\mathcal{I};$$

then the variety V_σ is the common zero set of polynomials from the ideal $\mathcal{I}$.

Example

Let $\sigma = \mathbb{R}_\geqslant\langle \boldsymbol{e}_2, 2\boldsymbol{e}_1 - \boldsymbol{e}_2\rangle$ in $\mathbb{R}^2$.
The two vectors do not span $\mathbb{Z}^2$, so the cone is not regular.

Then $\sigma^\vee = \mathbb{R}_\geqslant\langle \boldsymbol{e}_1^*, \boldsymbol{e}_1^* + 2\boldsymbol{e}_2^*\rangle$. The semigroup S_σ is generated by $\boldsymbol{e}_1^*$, $\boldsymbol{e}_1^* + \boldsymbol{e}_2^*$ and $\boldsymbol{e}_1^* + 2\boldsymbol{e}_2^*$, with one relation among them.Therefore,

$$A_\sigma = \mathbb{C}[x, xy, xy^2] \cong \mathbb{C}[u, v, w]/(v^2 - uw)$$

and V_σ is a quadratic cone (a singular variety).

Toric varieties from fans

If τ is a face of σ, then $\sigma^\vee \subset \tau^\vee$, and the inclusion of algebras $\mathbb{C}[S_\sigma] \to \mathbb{C}[S_\tau]$ induces an inclusion $V_\tau \to V_\sigma$ of a Zariski open subset.

This allows us to glue the affine varieties V_σ corresponding to all cones σ in a fan Σ into a toric variety $V_\Sigma = \operatorname{colim}_{\sigma \in \Sigma} V_\sigma$.

Here is the crucial point: the fact that the cones σ patch into a fan Σ guarantees that the variety V_Σ obtained by gluing the pieces V_σ is Hausdorff in the usual topology. In algebraic geometry, the Hausdorffness is replaced by the related notion of separatedness: a variety V is separated if the image of the diagonal map $\Delta\colon V \to V \times V$ is Zariski closed.

Lemma

If $\Sigma = \{\sigma\}$ is a fan, then the variety $V_\Sigma = \operatorname{colim}_{\sigma \in \Sigma} V_\sigma$ is separated.

The torus action

The variety V_σ carries an algebraic action of the torus $\mathbb{C}^\times_N = N \otimes_{\mathbb{Z}} \mathbb{C}^\times$

$$\mathbb{C}^\times_N \times V_\sigma \to V_\sigma, \quad (\boldsymbol{t}, x) \mapsto \boldsymbol{t} \cdot x$$

which is defined as follows.

A point $\boldsymbol{t} \in \mathbb{C}^\times_N$ gives a homomorphism

$$N^* \to \mathbb{C}^\times, \quad \boldsymbol{u} = (u_1, \ldots, u_n) \mapsto \boldsymbol{t}(\boldsymbol{u}) = t_1^{u_1} \cdots t_n^{u_n}.$$

Points of V_σ correspond to algebra morphisms $A_\sigma \to \mathbb{C}$, or to *semigroup homomorphisms* $S_\sigma \to \mathbb{C}_{\mathrm{m}}$, where $\mathbb{C}_{\mathrm{m}} = \mathbb{C}^\times \cup \{0\}$ is the multiplicative semigroup of complex numbers.

Then define $\boldsymbol{t} \cdot x$ as the point in V_σ corresponding to the semigroup homomorphism $S_\sigma \to \mathbb{C}_{\mathrm{m}}$ given by

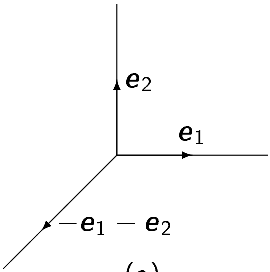
$$\boldsymbol{u} \mapsto t(\boldsymbol{u}) x(\boldsymbol{u}).$$

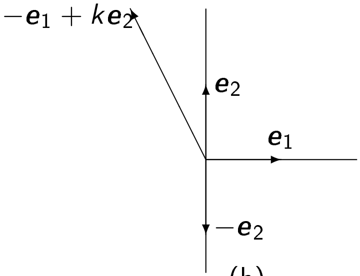
The morphism of algebras dual to the action $\mathbb{C}_N^\times \times V_\sigma \to V_\sigma$ is given by

$$A_\sigma \to \mathbb{C}[N^*] \otimes A_\sigma, \quad \chi^{\boldsymbol{u}} \mapsto \chi^{\boldsymbol{u}} \otimes \chi^{\boldsymbol{u}} \quad \text{for } \boldsymbol{u} \in S_\sigma.$$

If $\sigma = \{\mathbf{0}\}$, then we obtain the multiplication in the algebraic group $\mathbb{C}_N^\times$.

The actions on the varieties V_σ are compatible with the inclusions of open sets $V_\tau \to V_\sigma$ corresponding to the inclusions of faces $\tau \subset \sigma$.
Therefore, for each fan Σ we obtain a $\mathbb{C}_N^\times$-action on the variety V_Σ, which extends the $\mathbb{C}_N^\times$-action on itself.





Example (complex projective plane $\mathbb{C}P^2$)

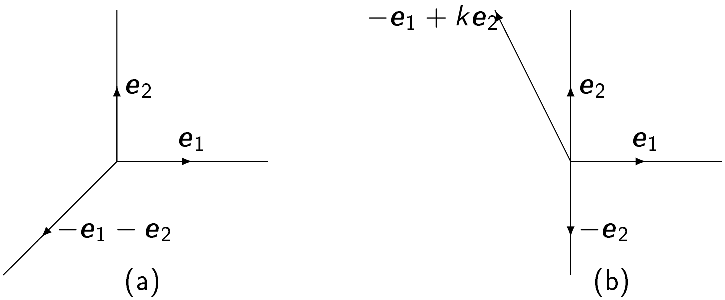
Σ has three maximal cones:

$$\sigma_0 = \mathbb{R}_\geqslant\langle \boldsymbol{e}_1, \boldsymbol{e}_2\rangle, \quad \sigma_1 = \mathbb{R}_\geqslant\langle \boldsymbol{e}_2, -\boldsymbol{e}_1 - \boldsymbol{e}_2\rangle, \quad \sigma_2 = \mathbb{R}_\geqslant\langle -\boldsymbol{e}_1 - \boldsymbol{e}_2, \boldsymbol{e}_1\rangle,$$

see Figure (a). Then each V_{σ_i} is $\mathbb{C}^2$, with coordinates

$$(x, y) \quad \text{for } \sigma_0, \quad (x^{-1}, x^{-1}y) \quad \text{for } \sigma_1, \quad (y^{-1}, xy^{-1}) \quad \text{for } \sigma_2.$$

These three affine charts glue together into $V_\Sigma = \mathbb{C}P^2$ with homogeneous coordinates $[z_0 : z_1 : z_2]$, $x = z_1/z_0$, $y = z_2/z_0$.



Example (Hirzebruch surfaces)

Fix $k \in \mathbb{Z}$ and consider the complete fan in $\mathbb{R}^2$ with the four two-dimensional cones generated by

$$(e_1, e_2), \quad (e_1, -e_2), \quad (-e_1 + ke_2, -e_2), \quad (-e_1 + ke_2, e_2),$$

see Figure (b). Tthe corresponding toric variety F_k is the projectivisation $\mathbb{C}P(\underline{\mathbb{C}} \oplus \mathcal{O}(k))$ of the sum of a trivial line bundle $\underline{\mathbb{C}}$ and the kth power $\mathcal{O}(k) = \bar{\eta}^{\otimes k}$ of the conjugate tautological line bundle $\bar{\eta}$ over $\mathbb{C}P^1$. These 2-dimensional complex varieties F_k are known as Hirzebruch surfaces.

Orbits and invariant subvarieties

A toric variety V_Σ is a disjoint union of its orbits by the action of the algebraic torus $\mathbb{C}_N^\times$.
There is one such orbit O_σ for each cone $\sigma \in \Sigma$, and we have $O_\sigma \cong (\mathbb{C}^\times)^{n-k}$ if $\dim \sigma = k$.
In particular, n-dimensional cones correspond to fixed points, and the apex (the zero cone) corresponds to the dense orbit $\mathbb{C}_N^\times$.

The orbit closure $\overline{O}_\sigma$ is a closed irreducible $\mathbb{C}_N^\times$-invariant subvariety of V_σ, and it is itself a toric variety.
In fact, $\overline{O}_\sigma$ consists of those orbits O_τ for which τ contains σ as a face.
Any irreducible invariant subvariety of V_Σ can be obtained in this way.
In particular, irreducible invariant divisors $D_1, \ldots, D_m$ of V_Σ correspond to edges of Σ.

Polytopes and normal fans

P a convex polytope with vertices in the lattice N^* (a lattice polytope):

$$P = \{\boldsymbol{x} \in N^*_{\mathbb{R}} \colon \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle + b_i \geqslant 0 \quad \text{for } i = 1, \ldots, m\}$$

where $b_i \in \mathbb{Z}$ and $\boldsymbol{a}_i \in N$ is the primitive normal to the facet

$$F_i = \{\boldsymbol{x} \in P \colon \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle + b_i = 0\}, \quad i = 1, \ldots, m.$$

Given a face $Q \subset P$, define the cone

$$\sigma_Q = \{\boldsymbol{u} \in N_{\mathbb{R}} \colon \langle \boldsymbol{u}, \boldsymbol{x}' \rangle \leqslant \langle \boldsymbol{u}, \boldsymbol{x} \rangle \text{ for all } \boldsymbol{x}' \in Q \text{ and } \boldsymbol{x} \in P\}.$$

The dual cone $\sigma_Q^\vee$ is the 'polyhedral angle' at the face Q; it is generated by all vectors $\boldsymbol{x} - \boldsymbol{x}'$ pointing from $\boldsymbol{x}' \in Q$ to $\boldsymbol{x} \in P$.

The cone σ_Q is generated by those $\boldsymbol{a}_i$ which are normal to Q. Then

$$\Sigma_P = \{\sigma_Q \colon Q \text{ is a face of } P\}$$

is a complete fan Σ_P in $N_{\mathbb{R}}$, called the normal fan of P.

If $\boldsymbol{0} \in \operatorname{int} P$, then Σ_P consists of cones over the faces of the polar P^*.

The normal fan Σ_P has a maximal cone σ_v for each vertex $v \in P$. The dual cone σ_v^* is the 'vertex cone' at v, generated by all vectors pointing from v to other points of P.

The normal fan Σ_P is simplicial if and only if P is simple, i. e. there are precisely $n = \dim P$ facets meeting at each vertex of P.
In this case, the cones of Σ_P are generated by those sets of normals $\{\boldsymbol{a}_{i_1}, \ldots, \boldsymbol{a}_{i_k}\}$ for which the intersection of facets $F_{i_1} \cap \cdots \cap F_{i_k}$ is nonempty.

The normal fan Σ_P of a polytope P contains the information about the normals to the facets (the generators $\boldsymbol{a}_i$ of the edges of Σ_P) and the combinatorial structure of P (which sets of vectors $\boldsymbol{a}_i$ span a cone of Σ_P is determined by which facets intersect at a face).
However the scalars b_i in the presentation of P by inequalities are lost. Not every complete fan can be obtained by 'forgetting the numbers b_i' from a presentation of a polytope by inequalities, i. e. not every complete fan is a normal fan. This is fails even for regular fans.

Projective toric varieties

Given a lattice polytope $P \subset N^*_{\mathbb{R}}$, define the toric variety $V_P = V_{\Sigma_P}$.

Since the normal fan Σ_P does not depend on the linear size of the polytope, we may assume that for each vertex v the semigroup S_{σ_v} is generated by the lattice points of the polytope (this can always be achieved by replacing P by kP with sufficiently large k).

Since $N^* = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{C}^\times_N, \mathbb{C}^\times)$, the lattice points of the polytope $P \subset N^*$ define an embedding

$$i_P \colon \mathbb{C}^\times_N \to (\mathbb{C}^\times)^{|N^* \cap P|},$$

where $|N^* \cap P|$ is the number of lattice points in P.

Proposition

The toric variety V_P is identified with the projective closure $\overline{i_P(\mathbb{C}^\times_N)} \subset \mathbb{C}P^{|N^* \cap P|}$.

It follows that toric varieties V_P arising from lattice polytopes P are projective, i. e. can be defined by a set of homogeneous equations in a projective space. The converse is also true: the fan corresponding to a projective toric variety is the normal fan of a lattice polytope.

A polytope carries more geometric information than its normal fan: different lattice polytopes with the same normal fan Σ correspond to different projective embeddings of the toric variety V_Σ.

If $D_1, \ldots, D_m$ are the invariant divisors corresponding to the facets of P, then $D_P = b_1 D_1 + \cdots + b_m D_m$ is an ample divisor on V_P.
This means that, when k is sufficiently large, kD_P is a hyperplane section divisor for a projective embedding $V_P \subset \mathbb{C}P^r$.
The space of sections $H^0(V_P, kD_P)$ of (the line bundle corresponding to) kD_P has basis corresponding to the lattice points in kP.
The embedding of V_P into the projectivisation of $H^0(V_P, kD_P)$ is exactly the projective embedding described above.

Cohomology of toric manifolds

A toric manifold is a smooth complete (compact) toric variety. Toric manifolds V_Σ correspond to complete regular fans Σ. Projective toric manifolds V_P correspond to lattice polytopes P whose normal fans are regular.

The cohomology of a toric manifold V_Σ can be calculated effectively from the fan Σ. The Betti numbers are determined by the combinatorics of Σ only, while the ring structure of $H^*(V_\Sigma)$ depends on the geometric data. The latter consist of the primitive generators $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ of one-dimensional cones (edges) of Σ.

$f_i = f_i(\Sigma)$ the number of $(i+1)$-dimensional cones of Σ.

If $\Sigma = \Sigma_P$ is the normal fan of an n-dimensional polytope P, then f_i is the number of $(n-i-1)$-dimensional faces of P.

$f_{-1} = 1$,
f_0 is the number of edges of Σ (facets of P),
f_{n-1} is the number of maximal cones of Σ (vertices of P).

$\boldsymbol{f}(\Sigma) = (f_0, f_1, \ldots, f_{n-1})$ the f-vector of Σ.

The h-vector $\boldsymbol{h}(\Sigma) = (h_0, h_1 \ldots, h_n)$ is defined from the identity

$$h_0 t^n + h_1 t^{n-1} + \cdots + h_n = (t-1)^n + f_0 (t-1)^{n-1} + \cdots + f_{n-1}.$$

Theorem (Danilov–Jurkiewicz)

Let V_Σ be the toric manifold corresponding to a complete regular fan Σ in $N_\mathbb{R}$. The cohomology ring of V_Σ is given by

$$H^*(V_\Sigma) \cong \mathbb{Z}[v_1, \ldots, v_m]/\mathcal{I},$$

where $v_1, \ldots, v_m \in H^2(V_\Sigma)$ are the cohomology classes dual to the invariant divisors corresponding to the one-dimensional cones of Σ, and $\mathcal{I}$ is the ideal generated by elements of the following two types:

(a) $v_{i_1} \cdots v_{i_k}$, *where $\boldsymbol{a}_{i_1}, \ldots, \boldsymbol{a}_{i_k}$ do not span a cone of Σ;*

(b) $\displaystyle\sum_{j=1}^{m} \langle \boldsymbol{a}_j, \boldsymbol{u} \rangle v_j$, *for any $\boldsymbol{u} \in N^*$.*

The homology groups of V_Σ vanish in odd dimensions, and are free abelian in even dimensions, with ranks given by

$$b_{2i}(V_\Sigma) = h_i,$$

where h_i, $i = 0, 1, \ldots, n$, are the components of the h-vector of Σ.

To obtain an explicit presentation of the ring $H^*(V_\Sigma)$ we choose a basis of N write the coordinates of $\boldsymbol{a}_i$ in the columns of the integer $n \times m$-matrix

$$\Lambda = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$$

Then the n linear forms $a_{j1}v_1 + \cdots + a_{jm}v_m$ corresponding to the rows of Λ vanish in $H^*(V; \mathbb{Z})$.

Example (complex projective space $\mathbb{C}P^n$)

The corresponding polytope is an n-simplex $P = \Delta^n$, and

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

The cohomology ring $H^*(\mathbb{C}P^n)$ is given by

$$\mathbb{Z}[v_1, \ldots, v_{n+1}]/(v_1 \cdots v_{n+1}, v_1 - v_{n+1}, \ldots, v_n - v_{n+1}) \cong \mathbb{Z}[v]/(v^{n+1})$$

The Danilov–Jurkiewicz Theorem remains valid for complete simplicial fans and corresponding toric orbifolds if the integer coefficients are replaced by the rationals. The integral cohomology of toric orbifolds often has torsion, and the integer ring structure is subtle even in the simplest case of weighted projective spaces.

The cohomology ring of a toric manifold V_Σ is generated by two-dimensional classes. This is the first property to check if one wishes to determine whether a given algebraic variety or smooth manifold has a structure of a toric manifold. For instance, this rules out flag varieties and Grassmannians different from projective spaces.

Theorem (Hard Lefschetz Theorem for toric orbifolds)

Let P be a lattice simple polytope, V_P the corresponding projective toric variety with ample divisor $b_1D_1 + \cdots + b_mD_m$, and $\omega = b_1v_1 + \cdots + b_mv_m \in H^2(V_P;\mathbb{C})$ the corresponding cohomology class. Then the maps

$$H^{n-i}(V_P;\mathbb{C}) \xrightarrow{\omega^i} H^{n+i}(V_P;\mathbb{C})$$

are isomorphisms for all $i = 1, \ldots, n$.

If V_P is smooth, then it is Kähler, and ω is the class of the Kähler 2-form.

Question

How to characterise the f-vectors (or h-vectors) for simplicial polytopes, simplicial fans or triangulated spheres?

Theorem (Billera–Lee, Stanley)

The following conditions are necessary and sufficient for a collection $(f_0, f_1, \ldots, f_{n-1})$ to be the f-vector of a simplicial polytope:

(a) $h_i = h_{n-i} \qquad$ *for* $i = 0, \ldots, n;$

(b) $h_0 \leqslant h_1 \leqslant h_2 \leqslant \cdots \leqslant h_{[n/2]};$

(c) $\ldots$ *(a restriction on the growth of* h_i*).*

Stanley's argument: realise the dual simple polytope as a lattice polytope P and consider the projective toric variety V_P. We have

$$\dim H^{2i}(V_P, \mathbb{Q}) = h_i.$$

Then (a) is Poincaré duality, while (b) and (c) follow from the Hard Lefschetz Theorem.

Conjecture (McMullen)

Is it true that the same conditions (a)–(c) characterise the f-vectors of triangulated spheres?

This is open even for complete simplicial fans (star-shaped spheres).

Literature

[1] Victor Buchstaber and Taras Panov. *Toric Topology.* Mathematical Surveys and Monographs, vol. 204, Amer. Math. Soc., Providence, RI, 2015.

[2] David A. Cox, John B. Little and Henry K. Schenck. *Toric varieties.* Graduate Studies in Mathematics, 124. Amer. Math. Soc., Providence, RI, 2011.

[3] William Fulton. *Introduction to Toric Varieties*. Annals of Mathematics Studies, 131. Princeton Univ. Press, Princeton, NJ, 1993.