Basic cohomology of canonical holomorphic foliations on complex moment-angle manifolds

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Abstract

Battaglia and Zaffran computed the basic Betti numbers for the canonical holomorphic foliation on a moment-angle manifold corresponding to a shellable fan. They conjectured that the basic cohomology ring in the case of any complete simplicial fan has a description similar to the cohomology ring of a complete simplicial toric variety due to Danilov and Jurkiewicz. In this work we prove the conjecture. The proof uses an Eilenberg–Moore spectral sequence argument; the key ingredient is the formality of the Cartan model for the torus action on a moment-angle manifold.

The moment-angle complex

$\mathcal{K}$ an abstract simplicial complex on the set $[m] = \{1, 2, \ldots, m\}$
$I = \{i_1, \ldots, i_k\} \in \mathcal{K}$ a simplex; always assume $\varnothing \in \mathcal{K}$.

Consider the unit m-dimensional polydisc:

$$\mathbb{D}^m = \{(z_1, ..., z_m) \in \mathbb{C}^m : |z_i|^2 \leqslant 1 \text{ for } i = 1, ..., m\}.$$

The moment-angle complex is

$$\mathcal{Z}_{\mathcal{K}} := \bigcup_{I \in \mathcal{K}} \Bigl(\prod_{i \in I} \mathbb{D} \times \prod_{i \notin I} \mathbb{S} \Bigr) \subset \mathbb{D}^m,$$

where $\mathbb{S}$ is the boundary of the unit disk $\mathbb{D}$.

$\mathcal{Z}_{\mathcal{K}}$ has a natural action of the torus

$$T^m = \{(t_1, \ldots, t_m) \in \mathbb{C}^m : |t_i| = 1\}.$$

When $\mathcal{K}$ is simplicial subdivision of a sphere, $\mathcal{Z}_{\mathcal{K}}$ is a topological manifold, called the moment-angle manifold.

We define an open submanifold $U(\mathcal{K}) \subset \mathbb{C}^m$ in a similar way:

$$U(\mathcal{K}) := \bigcup_{I\in\mathcal{K}} \Big(\prod_{i\in I}\mathbb{C} \times \prod_{i\notin I}\mathbb{C}^\times\Big),$$

where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.
$U(\mathcal{K})$ is a toric variety with the corresponding fan given by

$$\Sigma_\mathcal{K} = \{\mathbb{R}_\geqslant\langle \mathbf{e}_i \colon i \in I\rangle \colon I \in \mathcal{K}\},$$

where $\mathbf{e}_i$ denotes the i-th standard basis vector of $\mathbb{R}^m$.

Given a commutative ring R with unit, the face ring of $\mathcal{K}$ is

$$R[\mathcal{K}] := R[v_1, ..., v_m]/I_\mathcal{K},$$

where $R[v_1, ..., v_m]$ is the polynomial algebra, $\deg v_i = 2$, and $I_\mathcal{K}$ is the Stanley–Reisner ideal, generated by those monomials $v_I = \prod_{i\in I} v_i$ for which I is not a simplex of $\mathcal{K}$.

Complex-analytic structures on moment-angle manifolds

Assume that $\mathcal{Z}_\mathcal{K}$ admits a T^m-invariant complex structure. Then the T^m-action extends to a holomorphic action of $(\mathbb{C}^\times)^m$ on $\mathcal{Z}_\mathcal{K}$. The global stabilisers subgroup

$$H = \{g \in (\mathbb{C}^\times)^m \colon g \cdot x = x \text{ for all } x \in \mathcal{Z}_\mathcal{K}\}$$

is a complex-analytic subgroup of $(\mathbb{C}^\times)^m$.

The Lie algebra $\mathfrak{h}$ of H is a complex subalgebra of $\mathrm{Lie}(\mathbb{C}^\times)^m = \mathbb{C}^m$. Furthermore, $\mathfrak{h}$ satisfies

(a) the composite $\mathfrak{h} \hookrightarrow \mathbb{C}^m \xrightarrow{\mathrm{Re}} \mathbb{R}^m$ is injective;

(b) the quotient map $q \colon \mathbb{R}^m \to \mathbb{R}^m / \mathrm{Re}(\mathfrak{h})$ sends the fan $\Sigma_\mathcal{K}$ to a complete fan $q(\Sigma_\mathcal{K})$ in $\mathbb{R}^m / \mathrm{Re}(\mathfrak{h})$.

Ishida proved that any complex moment-angle manifold $\mathcal{Z}_\mathcal{K}$ is T^m-equivariantly biholomorphic to the quotient manifold $U(\mathcal{K})/H$.

Conversely, P. and Ustinovsky proved that if a complex subspace $\mathfrak{h}$ of $\mathbb{C}^m$ satisfies the conditions (a) and (b) above, then the Lie subgroup H of $(\mathbb{C}^\times)^m$ corresponding to $\mathfrak{h}$ acts on $U(\mathcal{K})$ freely and properly, and the complex manifold $U(\mathcal{K})/H$ is T^m-equivariantly homeomorphic to $\mathcal{Z}_\mathcal{K}$.

We therefore obtain that a moment-angle manifold $\mathcal{Z}_\mathcal{K}$ admits a complex structure if and only if $\mathcal{K}$ is the underlying complex of a complete simplicial fan (that is, $\mathcal{K}$ is a star-shaped sphere triangulation), and any complex structure on such $\mathcal{Z}_\mathcal{K}$ is defined by a choice of a complex subspace $\mathfrak{h} \subset \mathbb{C}^m$ satisfying (a) and (b) above.

A holomorphic foliation on $\mathcal{Z}_{\mathcal{K}}$

Define the Lie subalgebra and the corresponding Lie group

$$\mathfrak{h}' = \mathrm{Re}(\mathfrak{h}) \subset \mathbb{R}^m = \mathfrak{t}, \qquad H' = \exp(\mathfrak{h}') \subset T^m.$$

The restriction of the T^m-action on $U(\mathcal{K})/H$ to $H' \subset T^m$ is almost free. We obtain a *holomorphic* foliation on $\mathcal{Z}_{\mathcal{K}}$ by the orbits of H'.

Remark

If the subspace $\mathfrak{h}' \subset \mathbb{R}^m$ is rational (i. e., generated by integer vectors), then H' is a subtorus of T^m and the complete simplicial fan $\Sigma := q(\Sigma_{\mathcal{K}})$ is rational. The rational fan Σ defines a toric variety

$$V_\Sigma = \mathcal{Z}_{\mathcal{K}}/H' = U(\mathcal{K})/H'_{\mathbb{C}}.$$

The holomorphic foliation of $\mathcal{Z}_{\mathcal{K}}$ by the orbits of H' becomes a holomorphic Seifert fibration over the toric orbifold V_Σ with fibres compact complex tori $H'_{\mathbb{C}}/H$.

Basic cohomology and equivariant cohomology

Let $\mathfrak{g}$ be a Lie algebra. A $\mathfrak{g}^\star$-DGA is a DGA equipped with an action of operators ι_ξ (concatenation) and L_ξ (Lie derivative) for $\xi \in \mathfrak{g}$. For a $\mathfrak{g}^\star$-DGA (A, d_A), the basic subcomplex $A_{\mathrm{bas}\,\mathfrak{g}}$ is given by

$$A_{\mathrm{bas}\,\mathfrak{g}} := \{\omega \in A \colon \iota_\xi\omega = L_\xi\omega = 0 \text{ for any } \xi \in \mathfrak{g}\},$$

The basic cohomology of A is given by $H_{\mathrm{bas}\,\mathfrak{g}}(A) = H(A_{\mathrm{bas}\,\mathfrak{g}}, d_A)$. We omit $\mathfrak{g}$ by writing $H_{\mathrm{bas}}(A)$ for simplicity when $\mathfrak{g}$ is clear from the context.

Let $S(\mathfrak{g}^*)$ denote the symmetric (polynomial) algebra on the dual Lie algebra $\mathfrak{g}^*$ with generators of degree 2, and $\Lambda(\mathfrak{g}^*)$ the exterior algebra with generators of degree 1. The Weil algebra of $\mathfrak{g}$ is the DGA

$$\mathcal{W} = \mathcal{W}(\mathfrak{g}) := \big(\Lambda(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*), d_{\mathcal{W}(\mathfrak{g})}\big)$$

with the standard acyclic (Koszul) differential $d_{\mathcal{W}(\mathfrak{g})}$.

There are two models for equivariant cohomology of A.

The Cartan model is

$$\mathcal{C}_{\mathfrak{g}}(A) = ((S(\mathfrak{g}^*) \otimes A)^{\mathfrak{g}}, d_{\mathfrak{g}}),$$

where $(S(\mathfrak{g}^*) \otimes A)^{\mathfrak{g}}$ denotes the $\mathfrak{g}$-invariant subalgebra.

An element $\omega \in \mathcal{C}_{\mathfrak{g}}(A)$ is a "$\mathfrak{g}$-equivariant polynomial map from $\mathfrak{g}$ to A". The differential $d_{\mathfrak{g}}$ is given by

$$d_{\mathfrak{g}}(\omega)(\xi) = d_A(\omega(\xi)) - \iota_{\xi}(\omega(\xi)).$$

The Weil model is

$$\mathcal{W}_{\mathfrak{g}}(A) = ((\mathcal{W} \otimes A)_{\mathrm{bas}}, d),$$

where $d = d_{\mathcal{W}} \otimes 1 + 1 \otimes d_A$.

The Mathai–Quillen isomorphism implies that the Weil model $W_{\mathfrak{g}}(A)$ and the Cartan model $\mathcal{C}_{\mathfrak{g}}(A)$ have the same cohomology $H_{\mathfrak{g}}(A)$.

$H_{\mathfrak{g}}(A)$ is the $\mathfrak{g}$-equivariant cohomology of the $\mathfrak{g}^{\star}$-algebra A.

A $\mathcal{W}^\star$-algebra B is a $\mathfrak{g}^\star$-DGA which is also a $\mathcal{W}$-module.

For a $\mathcal{W}^\star$-algebra B, there are weak equivalences $B_{\mathrm{bas}} \simeq \mathcal{C}_{\mathfrak{g}}(B) \simeq \mathcal{W}_{\mathfrak{g}}(B)$.

In particular, $H_{\mathrm{bas}}(B) \cong H_{\mathfrak{g}}(B)$ if B is a $\mathcal{W}^\star$-algebra.

Now let M be a smooth manifold with an action of a connected Lie group G, and let $\mathfrak{g}$ be the Lie algebra of G. Then the algebra $\Omega(M)$ of differential forms on M is a $\mathcal{W}^\star$-algebra, so we have algebra isomorphisms

$$H_{\mathrm{bas}}(\Omega(M)) \cong H\big(\mathcal{C}_{\mathfrak{g}}(\Omega(M))\big) \cong H\big(\mathcal{W}_{\mathfrak{g}}(\Omega(M))\big).$$

If in addition G is a compact, then the algebra above is isomorphic to the equivariant cohomology $H^*_G(M) := H^*(EG \times_G M)$:

$$H_{\mathrm{bas}}(\Omega(M)) \cong H^*_G(M) \quad \text{for compact } G.$$

$\mathcal{Z}_\mathcal{K}$ a moment-angle manifold with a T^m-invariant complex structure. Want to describe the basic cohomology algebra of $\mathcal{Z}_\mathcal{K}$ with respect to the canonical holomorphic foliation:

$$H^*_{\mathrm{bas}}(\mathcal{Z}_\mathcal{K}) := H_{\mathrm{bas}\,\mathfrak{h}'}(\Omega(\mathcal{Z}_\mathcal{K})).$$

Lemma

Consider the algebra

$$\mathcal{N} := \mathcal{C}_{\mathfrak{h}'}\bigl(\Omega(\mathcal{Z}_\mathcal{K})^{T^m}\bigr) = \bigl(S(\mathfrak{h}'^*) \otimes \Omega(\mathcal{Z}_\mathcal{K})^{T^m}, d_{\mathfrak{h}'}\bigr).$$

Then we have an isomorphism

$$H^*_{\mathrm{bas}}(\mathcal{Z}_\mathcal{K}) \cong H(\mathcal{N}).$$

Proof.

$$H^*_{\mathrm{bas}}(\mathcal{Z}_\mathcal{K}) = H_{\mathrm{bas}\,\mathfrak{h}'}\bigl(\Omega(\mathcal{Z}_\mathcal{K})\bigr) \cong H_{\mathrm{bas}\,\mathfrak{h}'}\bigl(\Omega(\mathcal{Z}_\mathcal{K})^{T^m}\bigr) = H\bigl(\mathcal{C}_{\mathfrak{h}'}\bigl(\Omega(\mathcal{Z}_\mathcal{K})^{T^m}\bigr)\bigr) = H(\mathcal{N}). \quad \square$$

A DGA B is called formal if it is weak equivalent to its cohomology algebra: $(B, d_B) \simeq (H^*(B, d_B), 0)$. (A weak equivalence is the equivalence generated by quasi-isomorphisms; it may not be realised by a single quasi-isomorphism of DGA, but rather by a zigzag of quasi-isomorphisms.)

As T^m is compact, cohomology of the Cartan model

$$\mathcal{C}_{\mathfrak{t}}(\Omega(\mathcal{Z}_{\mathcal{K}})) = \bigl(S(\mathfrak{t}^*) \otimes \Omega(\mathcal{Z}_{\mathcal{K}})^{T^m},\, d_{\mathfrak{t}}\bigr)$$

is the equivariant cohomology $H^*_{T^m}(\mathcal{Z}_{\mathcal{K}})$, which is a module over $S(\mathfrak{t}^*) = H^*_{T^m}(pt) = H^*(BT^m)$.

Lemma

The algebra $\mathcal{C}_{\mathfrak{t}}(\Omega(\mathcal{Z}_{\mathcal{K}}))$ is formal. Furthermore, there is a zigzag of quasi-isomorphisms of DGAs between $\mathcal{C}_{\mathfrak{t}}(\Omega(\mathcal{Z}_{\mathcal{K}}))$ and $H_{T^m}(\mathcal{Z}_{\mathcal{K}})$ which respect the $S(\mathfrak{t}^*)$-module structure.

Proof

In this proof, $\mathcal{W}$ is the Weil algebra $\mathcal{W}(\mathfrak{t})$ of the torus T^m.
$E = EU(m)$ be the space of orthonormal m-frames in $\mathbb{C}^\infty$.

$$
\begin{array}{ccccccc}
 & & \Omega(\mathcal{Z}_\mathcal{K})^{T^m} \otimes \mathcal{W} & \xrightarrow{\ \iota\ } & \Omega(\mathcal{Z}_\mathcal{K})^{T^m} \otimes \Omega(E) & & \\
 & & \uparrow & & \uparrow & & \\
\mathcal{C}_\mathfrak{t}(\Omega(\mathcal{Z}_\mathcal{K})) & \xleftarrow[\varphi]{\simeq} & (\Omega(\mathcal{Z}_\mathcal{K})^{T^m} \otimes \mathcal{W})_{\mathrm{bas}} & \xrightarrow[\iota_{\mathrm{bas}}]{\simeq} & (\Omega(\mathcal{Z}_\mathcal{K})^{T^m} \otimes \Omega(E))_{\mathrm{bas}} & \xrightarrow[\psi]{\cong} & \Omega(\mathcal{Z}_\mathcal{K} \times_{T^m} E) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
S(\mathfrak{t}^*) & = & \mathcal{W}_{\mathrm{bas}} & \xrightarrow{\simeq} & \Omega(E)_{\mathrm{bas}} & \xrightarrow{\cong} & \Omega(BT^m)
\end{array}
$$

Here ι and ι_{bas} are the quasi-isomorphisms induced by the inclusion $\mathcal{W} \hookrightarrow \Omega(E)$ of a free acyclic $\mathcal{W}^\star$-algebra, and the restriction $\mathcal{W}_{\mathrm{bas}} \hookrightarrow \Omega(E)_{\mathrm{bas}}$ is the Chern–Weil homomorphism.
The quasi-isomorphism φ is given by Cartan's Theorem.
The isomorphism ψ follows from the fact that T^m acts freely on E.

The middle line above gives a zigzag of quasi-isomorphisms between $\mathcal{C}_\mathfrak{t}(\Omega(\mathcal{Z}_\mathcal{K}))$ and $\Omega(\mathcal{Z}_\mathcal{K} \times_{T^m} E)$ which respect the $S(\mathfrak{t}^*)$-module structure.

Proof (continued)

Have a weak equivalence between $\mathcal{C}_{\mathfrak{t}}(\Omega(\mathcal{Z}_{\mathcal{K}}))$ and $\Omega(\mathcal{Z}_{\mathcal{K}} \times_{T^m} E)$.

Now, the Borel construction $\mathcal{Z}_{\mathcal{K}} \times_{T^m} E$ is homotopy equivalent to the polyhedral product $(\mathbb{C}P^\infty)^{\mathcal{K}}$, which is a rationally formal space.

Rational formality implies a zigzag of quasi-isomorphisms between $\Omega(\mathcal{Z}_{\mathcal{K}} \times_{T^m} E)$ and $H^*_{T^m}(\mathcal{Z}_{\mathcal{K}}) = H^*(\mathcal{Z}_{\mathcal{K}} \times_{T^m} E)$, as the de Rham forms $\Omega(\mathcal{Z}_{\mathcal{K}} \times_{T^m} E)$ is a commutative cochain model. This zigzag can be chosen to respect the $H^*(BT^m)$-module structure. $\square$

We have the following extended functoriality property of Tor in the category of DGAs, which is a standard corollary of the Eilenberg-Moore spectral sequence:

Lemma (Eilenberg–Moore)

Let A and B be DGAs, let L, L' be a pair of A-modules and let M, M' be a pair of B-modules given together with morphisms

$$f\colon A \to B, \quad g\colon L \to M, \quad g'\colon L' \to M'$$

where g and g' are f-linear. If f, g and g' are quasi-isomorphisms, then

$$\mathrm{Tor}_f(g, g')\colon \mathrm{Tor}_A(L, L') \to \mathrm{Tor}_B(M, M')$$

is an isomorphism.

Main result

Theorem

There is an isomorphism of algebras:

$$H^*_{\mathrm{bas}}(\mathcal{Z}_\mathcal{K}) \cong \mathbb{R}[v_1, ..., v_m]/(I_\mathcal{K} + J),$$

where $I_\mathcal{K}$ is the Stanley–Reisner ideal of $\mathcal{K}$, generated by the monomials

$$v_{i_1} \cdots v_{i_k} \quad \text{with } \{i_1, \ldots, i_k\} \notin \mathcal{K},$$

and J is the ideal generated by the linear forms

$$\sum_{i=1}^{m} \langle \mathbf{u}, q(\mathbf{e}_i)\rangle v_i \quad \text{with } \mathbf{u} \in (\mathfrak{t}/\mathfrak{h}')^*.$$

Here $q\colon \mathfrak{t} \to \mathfrak{t}/\mathfrak{h}'$ is the projection, and $\mathfrak{t} = \mathbb{R}^m$.

Proof

Denote $\mathfrak{g}' := \mathfrak{t}/\mathfrak{h}'$. We have a splitting $\mathfrak{t} \cong \mathfrak{g}' \oplus \mathfrak{h}'$. Hence, $S(\mathfrak{t}^*) \cong S(\mathfrak{g}'^*) \otimes S(\mathfrak{h}'^*)$, and $S(\mathfrak{t}^*)$ is an $S(\mathfrak{g}'^*)$-module via the linear monomorphism $q^* \colon \mathfrak{g}'^* \to \mathfrak{t}^*$. We also obtain a DGA isomorphism

$$\mathcal{C}_{\mathfrak{t}}(\Omega(\mathcal{Z}_{\mathcal{K}})) \cong S(\mathfrak{g}'^*) \otimes \mathcal{N}, \quad (1)$$

where $\mathcal{N} = \mathcal{C}_{\mathfrak{h}'}\big(\Omega(\mathcal{Z}_{\mathcal{K}})^{T^m}\big)$ and the right hand side is understood as the Cartan model of $\mathcal{N}$ with respect to the Lie algebra $\mathfrak{g}'$.

Recall that $H^*_{T^m}(\mathcal{Z}_{\mathcal{K}}) \cong \mathbb{R}[\mathcal{K}]$. Since the fan $\Sigma = q(\Sigma_{\mathcal{K}})$ is complete, the Stanley–Reisner ring $\mathbb{R}[\mathcal{K}]$ is Cohen–Macaulay, that is, it is a finitely-generated free module over its polynomial subalgebra.

Furthermore, the composite $\mathfrak{g}'^* \hookrightarrow \mathfrak{t}^* \to \mathfrak{t}^*_I$ is onto for any $I \in \mathcal{K}$, where $\mathfrak{t}_I$ is the coordinate subspace generated by all $\mathbf{e}_i$ with $i \in I$. Therefore, a criterion applies to show that $\mathbb{R}[\mathcal{K}]$ is a free module over $S(\mathfrak{g}'^*)$.

Proof (continued)

We have a sequence of algebra isomorphisms:

$$\begin{aligned}\mathrm{Tor}_{S(\mathfrak{g}'^*)}\bigl(\mathbb{R}, S^*(\mathfrak{g}'^*)\otimes\mathcal{N}\bigr) &\cong \mathrm{Tor}_{S(\mathfrak{g}'^*)}\bigl(\mathbb{R},\mathcal{C}_{\mathfrak{t}}(\Omega(\mathcal{Z}_{\mathcal{K}}))\bigr)\\ &\cong \mathrm{Tor}_{S(\mathfrak{g}'^*)}\bigl(\mathbb{R}, H^*_{T^m}(\mathcal{Z}_{\mathcal{K}})\bigr) \cong \mathrm{Tor}^0_{S(\mathfrak{g}'^*)}\bigl(\mathbb{R}, H^*_{T^m}(\mathcal{Z}_{\mathcal{K}})\bigr)\\ &\cong \mathbb{R}\otimes_{S(\mathfrak{g}'^*)} H^*_{T^m}(\mathcal{Z}_{\mathcal{K}}) \cong H^*_{T^m}(\mathcal{Z}_{\mathcal{K}})/S^+(\mathfrak{g}'^*)\\ &\cong \mathbb{R}[v_1,\ldots,v_m]/(I_{\mathcal{K}}+J). \qquad (2)\end{aligned}$$

The first isomorphism follows from (1). The second isomorphism is by formality of the Cartan model $\mathcal{C}_{\mathfrak{t}}(\Omega(\mathcal{Z}_{\mathcal{K}}))$. In the third isomorphism, the higher Tor vanish because $H^*_{T^m}(\mathcal{Z}_{\mathcal{K}})$ is a free module over $S^*(\mathfrak{g}'^*)$. The fourth and fifth isomorphisms are clear. For the last isomorphism, recall that $q\colon \mathfrak{g}\to\mathfrak{g}/\mathfrak{h}'=\mathfrak{g}'$ is the quotient projection, so that $q^*(\mathbf{u})=\sum_{i=1}^m\langle\mathbf{u},\mathbf{a}_i\rangle v_i$ for any $\mathbf{u}\in\mathfrak{g}'^*$.

Proof (continued)

On the other hand, we have a sequence of isomorphisms

$$\begin{aligned}\mathrm{Tor}_{S(\mathfrak{g}'^*)}\big(\mathbb{R}, S(\mathfrak{g}'^*)\otimes\mathcal{N}\big) &\cong \mathrm{Tor}^0_{S(\mathfrak{g}'^*)}\big(\mathbb{R}, S(\mathfrak{g}'^*)\otimes\mathcal{N}\big)\\ &\cong H(\mathbb{R})\otimes_{S(\mathfrak{g}'^*)} H\big(S(\mathfrak{g}'^*)\otimes\mathcal{N}\big) \cong H\big(\mathbb{R}\otimes_{S(\mathfrak{g}'^*)}(S(\mathfrak{g}'^*)\otimes\mathcal{N})\big)\\ &\cong H(\mathcal{N}) \cong H^*_{\mathrm{bas}}(\mathcal{Z}_{\mathcal{K}}). \qquad (3)\end{aligned}$$

For the first isomorphism, the higher Tor vanish by the previous page. The second isomorphism is by definition of Tor^0. The third isomorphism follows from the Künneth Theorem, since $H\big(S(\mathfrak{g}'^*)\otimes\mathcal{N}\big) = H^*_{T^m}(\mathcal{Z}_{\mathcal{K}})$ is a free module over $S^*(\mathfrak{g}'^*)$. The fourth isomorphism is clear. The last isomorphism is a lemma above.

The theorem follows from (2) and (3). □

Using the notion of transverse equivalence, the theorem above can be generalised to arbitrary complex manifolds with holomorphic maximal torus actions. These include moment-angle manifolds and LVMB-manifolds (named after Lopez de Medrano, Verjovsky, Meersseman and Bosio.)

The conjecture of Battaglia and Zaffran is therefore proved completely.

This is the subject of the next talk by Hiroaki Ishida.

Reference:

[1] Ishida, Hiroaki; Krutowski, Roman; Panov, Taras. *Basic cohomology of canonical holomorphic foliations on complex moment-angle manifolds*. Preprint (2018), arXiv:1811.12038.