Lecture 4. Equivariant genera, rigidity and fibre multiplicativity

Taras Panov

Lomonosov Moscow State University

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Reminder: the universal toric genus

The universal toric genus

$$\varPhi\colon \varOmega_{U:T^k} \longrightarrow U^*(BT^k) = \varOmega_U[[u_1, \dots, u_k]],$$

assigns to a geometric cobordism class $[M, c_T] \in \varOmega^{-2n}_{U:T^k}$ of a $2n$-dimensional stably complex T^k-manifold the 'cobordism class' of the map $ET^k \times_{T^k} M \to BT^k$.

We may write an expansion

$$\varPhi(M) = \sum_{\omega} g_\omega(M) u^\omega,$$

where $\omega = (\omega_1, \dots, \omega_k) \in \mathbb{N}^k$, $u^\omega = u_1^{\omega_1} \cdots u_k^{\omega_k}$, $g_\omega(M) \in \varOmega_U^{-2(|\omega|+n)}$.

We have $g_0(M) = [M] \in \varOmega_U^{-2n}$.
How to express the other coefficients $g_\omega(M)$?

Bounded flag manifolds (after N. Ray)

A bounded flag in $\mathbb{C}^{n+1}$ is a complete flag

$$\mathcal{U} = \{U_1 \subset U_2 \subset \cdots \subset U_{n+1} = \mathbb{C}^{n+1}, \quad \dim U_i = i\}$$

for which U_k contains the subspace $\mathbb{C}^{k-1} = \langle \boldsymbol{e}_1, \ldots, \boldsymbol{e}_{k-1} \rangle$, $2 \leqslant k \leqslant n$.
BF_n the set of all bounded flags in $\mathbb{C}^{n+1}$.

BF_n is a complete projective toric variety. It is also a Bott tower, i. e. the total space of an n-fold iterated $\mathbb{C}P^1$-bundle over $B_0 = pt$.

The toric quotient construction describes BF_n as the quotient of

$$(S^3)^n = \{(z_1,\ldots,z_{2n}) \in \mathbb{C}^{2n} \colon |z_k|^2 + |z_{k+n}|^2 = 1,\ 1 \leqslant k \leqslant n\}$$

by the free action of the torus $T^n = \{(t_1,\ldots,t_n)\}$ given by

$$(z_1,\ldots,z_{2n}) \mapsto (t_1z_1,\ t_1^{-1}t_2z_2,\ \ldots,\ t_{n-1}^{-1}t_nz_n,\ t_1z_{n+1},\ t_2z_{n+2},\ \ldots,\ t_nz_{2n})$$

For $1 \leqslant i \leqslant n$ there are complex line bundles

$$\xi_i \colon (S^3)^n \times_{T^n} \mathbb{C} \longrightarrow BF_n$$

via the action $(t_1,\ldots,t_n)\cdot z = t_i^{-1}z$ for $z \in \mathbb{C}$.

The tangent bundle of BF_n stably splits as

$$\mathcal{T}(BF_n) \oplus \underline{\mathbb{C}}^n \cong \bar{\xi}_1 \oplus \xi_1\bar{\xi}_2 \oplus \cdots \oplus \xi_{n-1}\bar{\xi}_n \oplus \bar{\xi}_1 \oplus \bar{\xi}_2 \oplus \cdots \oplus \bar{\xi}_n.$$

When $n = 1$ we obtain the standard isomorphism $\mathcal{T}\mathbb{C}P^1 \oplus \underline{\mathbb{C}} \cong \bar{\eta} \oplus \bar{\eta}$, as $\xi_1 = \eta$ is the tautological line bundle.

We shall change the stably complex structure on BF_n so that the resulting bordism class in Ω^U_{2n} will be zero. To see that this is possible, we regard BF_n as a *sphere bundle* over BF_{n-1}. If a stably complex structure $c_{\mathcal T}$ on BF_n restricts to a trivial stably complex structure on each fibre S^2, then $c_{\mathcal T}$ extends over the associated 3-disk bundle, so it is cobordant to zero.

We decompose $(S^3)^n$ as $(S^3)^{n-1} \times S^3$ and T^n as $T^{n-1} \times T^1$; then T^1 acts trivially on $(S^3)^{n-1}$, and we obtain

$$BF_n = (S^3)^n / T^n = \big((S^3)^{n-1} \times (S^3/T^1)\big)/T^{n-1} = BF_{n-1} \times_{T^{n-1}} S^2.$$

Here T^1 acts on S^3 diagonally, so the stably complex structure on S^2 is the standard structure of $\mathbb{C}P^1$.

Now change the torus action on

$$(S^3)^n = \{(z_1, \ldots, z_{2n}) \in \mathbb{C}^{2n} \colon |z_k|^2 + |z_{k+n}|^2 = 1,\ 1 \leqslant k \leqslant n\}$$

to the following:

$$(z_1, \ldots, z_{2n}) \mapsto (t_1 z_1, t_1^{-1} t_2 z_2, \ldots, t_{n-1}^{-1} t_n z_n, t_1^{-1} z_{n+1}, t_2^{-1} z_{n+2}, \ldots, t_n^{-1} z_{2n}).$$

Then each T^1 acts on the corresponding S^3 antidiagonally, so the stably complex structure on S^2 is trivial, as needed. The resulting stably complex structure on BF_n is given by the isomorphism

$$\mathcal{T}(BF_n) \oplus \underline{\mathbb{R}}^{2n} \cong \bar{\xi}_1 \oplus \xi_1 \bar{\xi}_2 \oplus \cdots \oplus \xi_{n-1} \bar{\xi}_n \oplus \xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_n,$$

and its cobordism class in Ω^U_{2n} is zero.

We denote by B_n the manifold BF_n with the zero-cobordant stably complex structure above.

The 'tautological' line bundle ξ_n is classified by a map $B_n \to \mathbb{C}P^\infty$ and therefore defines a bordism class $\beta_n \in U_{2n}(\mathbb{C}P^\infty)$. We also set $\beta_0 = 1$.

$\{\beta_n \colon n \geqslant 0\}$ Ray's basis of the Ω_U-module $U_*(\mathbb{C}P^\infty)$.

Proposition (Ray, 1986)

The classes $\{\beta_n \colon n \geqslant 0\}$ form a basis of the free Ω_U-module $U_*(\mathbb{C}P^\infty)$ which is dual to the basis $\{u^k \colon k \geqslant 0\}$ of $U^*(\mathbb{C}P^\infty) = \Omega_U[[u]]$. Here $u = c_1^U(\bar\eta)$ is the cobordism first Chern class of the conjugate tautological line bundle over $\mathbb{C}P^\infty$ (represented by $\mathbb{C}P^{\infty-1} \subset \mathbb{C}P^\infty$).

Proof. As $[B_n] = 0$ in Ω_U, we have $\beta_n \in \widetilde{U}_{2n}(\mathbb{C}P^\infty)$ for $n > 0$. To show that $\{\beta_n \colon n \geqslant 0\}$ and $\{u^k \colon k \geqslant 0\}$ are dual bases it is enough to verify that $u \frown \beta_n = \beta_{n-1}$. The bordism class $u \frown \beta_n$ is obtained by making the map $B_n \to \mathbb{C}P^\infty$ transverse to the zero section $\mathbb{C}P^{\infty-1}$ of $\mathbb{C}P^\infty = MU(1)$ or, equivalently, by restricting $B_n \to \mathbb{C}P^\infty$ to the zero set of a transverse section of ξ_n. This gives precisely B_{n-1}. □

For a vector $\omega = (\omega_1, \dots, \omega_k)$ of nonnegative integers, define the manifold $B_\omega = B_{\omega_1} \times \dots \times B_{\omega_k}$ and the corresponding bordism class $\beta_\omega \in U_{2|\omega|}(BT^k)$.

Corollary

The set $\{\beta_\omega\}$ is a basis of the free Ω_U-module $U_*(BT^k)$; this basis is dual to the basis $\{u^\omega\}$ of $U^*(BT^k) = \Omega_U[[u_1, \dots, u_k]]$.

M a tangentially stably complex T^k-manifold M.
$(S^3)^\omega = (S^3)^{\omega_1} \times \cdots \times (S^3)^{\omega_k}$, on which $T^\omega = T^{\omega_1} \times \cdots \times T^{\omega_k}$ acts coordinatewise with quotient B_ω. Define

$$G_\omega(M) = (S^3)^\omega \times_{T^\omega} M,$$

where T^ω acts on M via the representation

$$(t_{1,1}, \ldots, t_{1,\omega_1}; \ldots ; t_{k,1}, \ldots, t_{k,\omega_k}) \longmapsto (t_{1,\omega_1}, \ldots, t_{k,\omega_k}).$$

Theorem

The manifold $G_\omega(M)$ represents the bordism class of the coefficient $g_\omega(M) \in \Omega_U^{-2(|\omega|+n)}$ in the expansion $\Phi(M) = \sum_\omega g_\omega(M) u^\omega$.

Idea of proof. We have $g_\omega(M) = \langle \Phi(M), \beta_\omega \rangle$. The latter Kronecker product is represented on the pullback of the diagram

$$B_\omega \longrightarrow BT^k \longleftarrow ET^k \times_{T^k} M,$$

which is exactly $G_\omega(M)$. □

Equivariant genera

Historically, equivariant extensions of genera were first considered by Atiyah and Hirzebruch in 1970, who established the rigidity property of the χ_y-genus of S^1-manifolds. The origins of these concepts lie in the Atiyah–Bott fixed point formula, which also acted as a catalyst for the development of equivariant index theory. This development culminated in the celebrated result of Bott and Taubes establishing the rigidity of the Ochanine–Witten elliptic genus on spin S^1-manifolds.

Here we develop an approach to equivariant genera and rigidity based solely on complex cobordism theory. It allows us to define an equivariant extension and the appropriate concept of rigidity for an arbitrary Hirzebruch genus, and agrees with the classical index-theoretical approach when the genus is the index of an elliptic complex.

Our definition of an equivariant genus uses a universal transformation of cohomology theories, introduced by Buchstaber in 1970:

Construction (Chern–Dold character)

Consider multiplicative transformations of cohomology theories

$$h\colon U^*(X) \to H^*(X;\Omega_U \otimes \mathbb{Q}).$$

Such an h is determined uniquely by a power series
$h(u) \in H^2(\mathbb{C}P^\infty;\Omega_U \otimes \mathbb{Q}) = \Omega_U \otimes \mathbb{Q}[[x]]$,
where $u = c_1^U(\bar\eta) \in U^2(\mathbb{C}P^\infty)$ and $x = c_1^H(\bar\eta) \in H^2(\mathbb{C}P^\infty)$.

The Chern–Dold character is the unique multiplicative transformation

$$\mathrm{ch}_U\colon U^*(X) \to H^*(X;\Omega_U \otimes \mathbb{Q})$$

which reduces to the canonical inclusion $\Omega_U \to \Omega_U \otimes \mathbb{Q}$ in the case $X = pt$.

Proposition

The Chern–Dold character satisfies

$$\mathrm{ch}_U(u) = f_U(x),$$

where $f_U(x)$ is the exponential of the formal group law F_U in cobordism.

Proof. Since ch_U acts identically on Ω_U, it follows that

$$\mathrm{ch}_U F_U(v_1, v_2) = F_U(\mathrm{ch}_U(v_1), \mathrm{ch}_U(v_2)) \tag{1}$$

for any $v_1, v_2 \in U^2(\mathbb{C}P^\infty)$. Set $f(x) := \mathrm{ch}_U(u) \in \Omega_U \otimes \mathbb{Q}[[x]]$. Let $v_i = c_1^U(\xi_i)$ and $x_i = c_1^H(\xi_i)$ for some line bundles ξ_i, $i = 1, 2$. Then

$$\mathrm{ch}_U F_U(v_1, v_2) = \mathrm{ch}_U\bigl(c_1^U(\xi_1 \otimes \xi_2)\bigr) = f\bigl(c_1^H(\xi_1 \otimes \xi_2)\bigr) = f(x_1 + x_2),$$
$$\mathrm{ch}_U(v_1) = f(x_1), \quad \mathrm{ch}_U(v_2) = f(x_2).$$

Substituting these expressions in (1) we get $f(x_1 + x_2) = F_U(f(x_1), f(x_2))$, which means that $f(x)$ is the exponential of F_U. □

Take a genus $\varphi\colon \Omega_U \to R$ with torsion-free R; such φ is uniquely determined by $f(x) = x + \cdots \in R \otimes \mathbb{Q}[[x]]$ via Hirzebruch's correspondence.

Define a multiplicative transformation

$$h_\varphi\colon U^*(X) \xrightarrow{\mathrm{ch}_U} H^*(X; \Omega_U \otimes \mathbb{Q}) \xrightarrow{\varphi} H^*(X; R \otimes \mathbb{Q})$$

where the second homomorphism acts by φ on the coefficients only. In the case $X = BT^k$ we obtain a homomorphism

$$h_\varphi\colon \Omega_U[[u_1, \ldots, u_k]] \to R \otimes \mathbb{Q}[[x_1, \ldots, x_k]]$$

which acts on the coefficients as φ and sends u_i to $f(x_i)$ for $1 \leqslant i \leqslant k$.

Construction (equivariant genus)

The T^k-equivariant extension of φ is the ring homomorphism

$$\varphi^T\colon \Omega_{U:T^k} \to R \otimes \mathbb{Q}[[x_1, \ldots, x_k]]$$

defined as the composition $h_\varphi \cdot \Phi$ with the universal toric genus.

Rigidity

We have

$$\varphi^T(M) = \varphi(M) + \sum_{|\omega|>0} \varphi(g_\omega(M))\, f(x)^\omega.$$

In particular, the T^k-equivariant extension of the universal genus $\mathrm{id}\colon \Omega^U_* \to \Omega^U_*$ is $\varPhi$; hence the name "universal toric genus".

Definition

A genus $\varphi\colon \Omega_U \to R$ is T^k-rigid on a stably complex T^k-manifold M whenever $\varphi^T\colon \Omega_{U:T^k} \to R \otimes \mathbb{Q}[[u_1,\ldots,u_k]]$ satisfies $\varphi^T(M) = \varphi(M)$; if this holds for every M, then φ is T^k-rigid.

Other definitions of rigidity:

Atiyah–Hirzebruch: Assume $\varphi(M) = \mathrm{ind}(\mathcal{E})$, the index of an elliptic complex $\mathcal{E}$ of complex vector bundles over M.
For any T^k-manifold M this index has a T^k-equivariant extension $\mathrm{ind}^T(\mathcal{E})$, which is an element of the complex representation ring $R_U(T^k)$.
Then φ is rigid if φ^T takes values in $\mathbb{Z} \subset R_U(T^k)$ (trivial representations).

Krichever: Considered rational genera $\varphi\colon \varOmega_U \to \mathbb{Q}$ and equivariant extensions $\varphi^K\colon \varOmega_{U:T^k} \to K^0(BT^k) \otimes \mathbb{Q}$, where $K^0(BT^k) = \widehat{R_U(T^k)}$.
Then φ is rigid if φ^K takes values in $\mathbb{Q} \subset K^0(BT^k) \otimes \mathbb{Q}$.

Krichever's $\varphi^K\colon \varOmega_{U:T^k} \to K^0(BT^k) \otimes \mathbb{Q}$ is related to ours $\varphi^T\colon \varOmega_{U:T^k} \to H^{even}(BT^k;\mathbb{Q}) = \mathbb{Q}[[x_1,\ldots,x_k]]$ via the Chern character $\mathrm{ch}\colon K^0(X) \otimes \mathbb{Q} \xrightarrow{\cong} H^{even}(X;\mathbb{Q})$. It follows that a rational genus is T^k-rigid if and only if it is rigid in the sense of Krichever (and therefore in the original index-theoretical sense of Atiyah–Hirzebruch if φ is an index).

Fibre multiplicativity

Consider fibre bundles $M \to E \times_G M \xrightarrow{\pi} B$,
where M and B are connected and stably tangentially complex,
G a compact Lie group of positive rank whose action preserves the stably complex structure on M,
$E \to B$ is a principal G-bundle.

Then $N := E \times_G M$ inherits a canonical stably complex structure.

Definition

A genus $\varphi\colon \Omega_U \to R$ is fibre multiplicative with respect to the stably complex manifold M whenever $\varphi(N) = \varphi(M)\varphi(B)$ for any such bundle π with fibre M; if this holds for every M, then φ is fibre multiplicative.

Theorem (Buchstaber-P-Ray)

If a genus φ is T^k-rigid on M, then it is fibre multiplicative with respect to M for bundles whose structure group G has $U^*(BG)$ torsion-free.

On the other hand, if φ is fibre multiplicative with respect to M, then it is T^k-rigid on M.

Proof. Let φ be fibre multiplicative. Consider $G_\omega(M) = (S^3)^\omega \times_{T^\omega} M$. Apply φ to the bundle $M \to G_\omega(M) \to B_\omega$. Since B_ω bounds for $|\omega| > 0$, we get $\varphi(G_\omega(M)) = 0$. So φ is T^k-rigid.

The other direction is proved by considering the pullback squares

$$\begin{array}{ccccc} E \times_G M & \xrightarrow{f'} & EG \times_G M & \xleftarrow{i'} & ET^k \times_{T^k} M \\ \pi \big\downarrow & & \pi^G \big\downarrow & & \pi^{T^k} \big\downarrow \\ B & \xrightarrow{f} & BG & \xleftarrow{i} & BT^k \end{array}$$

and using the properties of the Gysin homomorphism. □

Example

The signature (or the L-genus) is fibre multiplicative over any simply connected base by [Chern–Hirzebruch–Serre], and so is rigid.

Literature

[1] Victor M. Buchstaber, Taras Panov and Nigel Ray. *Toric genera*. Internat. Math. Res. Notices 2010, no. 16, 3207–3262.

[2] Victor Buchstaber and Taras Panov. *Toric Topology.* Mathematical Surveys and Monographs, vol. 204, Amer. Math. Soc., Providence, RI, 2015.