$R$ a commutative ring with unit. A formal power series $F(u, v) \in R[[u, v]]$ is called a (commutative one-dimensional) formal group law over $R$ if it satisfies

(a) $F(u, 0) = u$, $F(0, v) = v$;
(b) $F(F(u, v), w) = F(u, F(v, w))$;
(c) $F(u, v) = F(v, u)$.

The original example of a formal group law over a field $k$ is provided by the expansion near the unit of the multiplication map $G \times G \to G$ in a one-dimensional algebraic group over $k$. This also explains the terminology.

A formal group law $F$ over $R$ is linearisable if there exists a coordinate change $u \mapsto g_F(u) = u + \sum_{i \geq 1} g_i u^{i+1} \in R[[u]]$ such that

$$g_F(F(u, v)) = g_F(u) + g_F(v).$$
Theorem

Every formal group law \( F \) is linearisable over \( R \otimes \mathbb{Q} \).

Proof.

Consider the series \( \omega(u) = \left. \frac{\partial F(u,w)}{\partial w} \right|_{w=0} \). Applying \( \left. \frac{\partial}{\partial w} \right|_{w=0} \) to both sides of the identity \( F(F(u,v),w) = F(u,F(v,w)) \) we obtain

\[
\omega(F(u,v)) = \left. \frac{\partial F(F(u,v),w)}{\partial w} \right|_{w=0} = \left. \frac{\partial F(u,F(v,w))}{\partial w} \right|_{w=0} \cdot \left. \frac{\partial F(v,w)}{\partial w} \right|_{w=0} = \left. \frac{\partial F(u,v)}{\partial v} \right|_{w=0} \omega(v).
\]

We therefore have \( \frac{dv}{\omega(v)} = \frac{dF(u,v)}{\omega(F(u,v))} \), where \( u \) is a parameter. Set

\[
g(u) = \int_{0}^{u} \frac{dv}{\omega(v)}.
\]

Integrating the identity \( \frac{dv}{\omega(v)} = \frac{dF(u,v)}{\omega(F(u,v))} \) we obtain

\[
g(w) = \int_{0}^{w} \frac{dv}{\omega(v)} = \int_{0}^{w} \frac{dF(u,v)}{\omega(F(u,v))} = \int_{u}^{F(u,w)} \frac{dt}{\omega(t)} = g(F(u,w)) - g(u),
\]

so that \( g \) is a linearisation of \( F \).
A series $g_F(u) = u + \sum_{i \geq 1} g_i u^{i+1}$ satisfying $g_F(F(u, v)) = g_F(u) + g_F(v)$ is called the logarithm of the formal group law $F$. Its functional inverse series $f_F(t) \in R \otimes \mathbb{Q}[[t]]$ is the exponential of $F$, so we have $F(u, v) = f_F(g_F(u) + g_F(v))$ over $R \otimes \mathbb{Q}$.

If $R$ does not have torsion (i.e. $R \to R \otimes \mathbb{Q}$ is monomorphically), then a formal group law is fully determined by its logarithm.

Example

The multiplicative formal group law is the series

$$F(u, v) = (1 + u)(1 + v) - 1 = u + v + uv.$$  

There is a 1-parameter graded extension given by

$$F_\beta(u, v) = u + v - \beta uv, \quad \deg \beta = -2,$$

with coefficients in $\mathbb{Z}[\beta]$. Its exponential and logarithm are given by

$$f(x) = \frac{1 - e^{-\beta x}}{\beta}, \quad g(u) = -\frac{\ln(1 - \beta u)}{\beta} \in \mathbb{Q}[\beta].$$
Another classical example comes from the theory of elliptic functions. There is a unique meromorphic function $f(x)$ with $f(0) = 0$ and $f'(0) = 1$ satisfying the differential equation

$$(f'(x))^2 = 1 - 2\delta f^2(x) + \varepsilon f^4(x)$$

with $\delta, \varepsilon \in \mathbb{C}$. This function provides a uniformisation for the Jacobi model $y^2 = 1 - 2\delta x^2 + \varepsilon x^4$ of an elliptic curve. When the discriminant

$$\Delta = \varepsilon(\delta^2 - \varepsilon)$$

is nonzero, the elliptic curve is nondegenerate, and $f(x)$ is a doubly periodic function known as the Jacobi elliptic sine and denoted by $\text{sn}(x)$. Its inverse is given by the elliptic integral

$$g(u) = \int_0^u \frac{dt}{\sqrt{1 - 2\delta t^2 + \varepsilon t^4}}.$$
Example

There is the following Euler’s expression for the addition formula for \( \text{sn}(x) \):

\[
F_{\text{ell}}(u, v) = \text{sn}(x + y) = \frac{u \sqrt{1 - 2\delta v^2 + \varepsilon v^4} + v \sqrt{1 - 2\delta u^2 + \varepsilon u^4}}{1 - \varepsilon u^2 v^2},
\]

where \( u = \text{sn} x, \ v = \text{sn} y \). It defines the elliptic formal group law, with exponential \( \text{sn}(x) \) and logarithm \( \text{g}(u) \) as above.

Viewing \( \delta, \varepsilon \) as formal parameters with \( \deg \delta = -4, \deg \varepsilon = -8 \), we obtain the universal elliptic formal group law over the ring \( \mathbb{Z} [\frac{1}{2}][\delta, \varepsilon] \).

Degeneration \( \varepsilon = 0 \) gives the addition formula for \( f(x) = \frac{\sin \sqrt{2\delta x}}{\sqrt{2\delta}} \), while degeneration \( \varepsilon = \delta^2 \) gives the addition formula for \( f(x) = \frac{\tanh \sqrt{\delta} x}{\sqrt{\delta}} \).
Given a ring homomorphism \( r : R \to R' \) and a f.g.l. \( F = \sum_{k,l} a_{kl} u^k v^l \) over \( R \), we obtain a f.g.l. \( r(F) := \sum_{k,l} r(a_{kl}) u^k v^l \in R'[u, v] \) over \( R' \).

A formal group law \( \mathcal{F} \) over a ring \( A \) is **universal** if for any f.g.l. \( F \) over any ring \( R \) there exists a unique homomorphism \( r : A \to R \) such that \( F = r(\mathcal{F}) \).

**Proposition**

A universal formal group law \( \mathcal{F}(u, v) = u + v + \sum_{k \geq 1, l \geq 1} a_{kl} u^k v^l \) exists, and its coefficient ring is the quotient

\[
A = \mathbb{Z}[a_{kl} : k \geq 1, l \geq 1]/\mathcal{I}, \quad \deg a_{kl} = -2(k + l - 1),
\]

of the graded polynomial ring by the graded ‘associativity ideal’ \( \mathcal{I} \), generated by the coefficients of the formal power series

\[
\mathcal{F}(\mathcal{F}(u, v), w) - \mathcal{F}(u, \mathcal{F}(v, w)).
\]

Furthermore, \( \mathcal{F} \) is unique: if \( \mathcal{F}' \) is another universal formal group law over \( A' \), then there is an isomorphism \( r : A \to A' \) such that \( \mathcal{F}' = r(\mathcal{F}) \).
Note that the definition of a formal group law does not assume any grading of the coefficient ring; however, the coefficient ring of the universal formal group law turns out to be naturally graded.

Natural grading: \( \deg u = \deg v = 2, \ \deg a_{kl} = -2(k + l - 1); \)
then the whole expression

\[
\mathcal{F}(u, v) = u + v + \sum_{k \geq 1, l \geq 1} a_{kl} u^k v^l
\]

is homogeneous of degree 2.

**Theorem (Lazard)**

The coefficient ring \( A \) of the universal formal group law \( \mathcal{F} \) is isomorphic to the graded polynomial ring \( \mathbb{Z}[a_1, a_2, \ldots] \) on an infinite number of generators, \( \deg a_i = -2i \).
Construction (geometric cobordisms)

For any cell complex $X$, have $H^2(X) = [X, \mathbb{C}P^\infty]$. Since $\mathbb{C}P^\infty = MU(1)$, every element $x \in H^2(X)$ determines a cobordism class $u_x \in U^2(X)$, a geometric cobordism. Hence, $H^2(X) \subset U^2(X)$ (a subset, not a subgroup!)

When $X$ is a manifold, each $u_x \in U^2(X)$ corresponds to a submanifold $M \subset X$ of codimension 2 with a complex structure on the normal bundle.

Indeed, $x \in H^2(X)$ corresponds to a homotopy class of $f_x : X \to \mathbb{C}P^\infty$. May assume $f_x(X)$ is transverse to a hyperplane $H \subset \mathbb{C}P^N \subset \mathbb{C}P^\infty$. Then $M_x = f_x^{-1}(H)$ is a codimension-2 submanifold in $X$.

A homotopy of $f_x$ gives a cobordism of $M_x \to X$.

Conversely, given an embedding $i : M \subset X$ as above, the composite $X \to Th(\nu) \to MU(1) = \mathbb{C}P^\infty$ of the Pontryagin–Thom collapse map and the classifying map for $\nu$ defines an element $x_M \in H^2(X)$, and therefore a geometric cobordism.

If $X$ is oriented, then $i_*\langle M \rangle \in H_*(X)$ is Poincaré dual to $x_M \in H^2(X)$. 
As we have seen, the characteristic number $s_n$ vanishes on decomposable elements of $\Omega^U$. Furthermore, this characteristic number detects indecomposables that may be chosen as polynomial generators:

**Theorem**

A bordism class $[M] \in \Omega_{2n}^U$ may be chosen as a polynomial generator $a_n$ of the ring $\Omega^U$ if and only if

$$s_n[M] = \begin{cases} 
\pm 1 & \text{if } n \neq p^k - 1 \text{ for any prime } p; \\
\pm p & \text{if } n = p^k - 1 \text{ for some prime } p.
\end{cases}$$

There is no universal description of manifolds representing the polynomial generators $a_n \in \Omega^U$. On the other hand, there is a particularly nice family of manifolds whose bordism classes generate the whole ring $\Omega^U$. This family is redundant though, so there are algebraic relations between their bordism classes.
Construction (Milnor hypersurfaces)

The Milnor hypersurface in $\mathbb{C}P^i \times \mathbb{C}P^j$ ($0 \leq i \leq j$) is

$$H_{ij} = \{(z_0 : \cdots : z_i) \times (w_0 : \cdots : w_j) \in \mathbb{C}P^i \times \mathbb{C}P^j : z_0 w_0 + \cdots + z_i w_i = 0\}$$

Note that $H_{0j} \cong \mathbb{C}P^{j-1}$.

More intrinsically, $H_{ij}$ is a hyperplane section of the Segre embedding

$$\sigma : \mathbb{C}P^i \times \mathbb{C}P^j \to \mathbb{C}P^{(i+1)(j+1)-1},$$

$$(z_0 : \cdots : z_i) \times (w_0 : \cdots : w_j) \mapsto (z_0 w_0 : z_0 w_1 : \cdots : z_k w_l : \cdots : z_i w_j),$$

Also, $H_{ij}$ may be identified with the set of pairs $(\ell, \alpha)$, where $\ell$ is a line in $\mathbb{C}^{i+1}$ and $\alpha$ is a hyperplane in $\mathbb{C}^{j+1}$ containing $\ell$.

In particular, $H_{22} = Fl(\mathbb{C}^3)$, the flag manifold.

The projection $H_{ij} \to \mathbb{C}P^i$, $(\ell, \alpha) \mapsto \ell$, is a fibre bundle with fibre $\mathbb{C}P^{j-1}$. 
Denote by $p_1$ and $p_2$ the projections of $\mathbb{C}P^i \times \mathbb{C}P^j$ onto its factors. Then

$$H^*(\mathbb{C}P^i \times \mathbb{C}P^j) = \mathbb{Z}[x, y]/(x^{i+1} = 0, \ y^{j+1} = 0)$$

where $x = p_1^*c_1(\bar{\eta})$, $y = p_2^*c_1(\bar{\eta})$, and $\eta$ the tautological bundle.

**Proposition**

$H_{ij}$ represents the geometric cobordism in $\mathbb{C}P^i \times \mathbb{C}P^j$ corresponding $x + y \in H^2(\mathbb{C}P^i \times \mathbb{C}P^j)$. In particular, the image of the fundamental class $\langle H_{ij} \rangle$ in $H_{2(i+j-1)}(\mathbb{C}P^i \times \mathbb{C}P^j)$ is Poincaré dual to $x + y$.

**Proof.**

We have $x + y = c_1(p_1^*(\bar{\eta}) \otimes p_2^*(\bar{\eta}))$. The classifying map for $p_1^*(\bar{\eta}) \otimes p_2^*(\bar{\eta})$ is the Segre embedding $\sigma : \mathbb{C}P^i \times \mathbb{C}P^j \to \mathbb{C}P^{(i+1)(j+1)-1} \to \mathbb{C}P^\infty$.

The codimension-2 submanifold in $\mathbb{C}P^i \times \mathbb{C}P^j$ corresponding to $x + y$ is the preimage $\sigma^{-1}(H)$ of a generally positioned hyperplane in $\mathbb{C}P^{(i+1)(j+1)-1}$. The Milnor hypersurface $H_{ij}$ is exactly $\sigma^{-1}(H)$ for one such hyperplane $H$. 

\vspace{1cm}

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Lemma

\[ s_{i+j-1}[H_{ij}] = \begin{cases} 
  j & \text{if } i = 0; \\
  -(i+j) & \text{if } i > 1.
\end{cases} \]

Proof.

The stably complex structure on \( H_{0j} = \mathbb{C}P^{j-1} \) comes from the isomorphism \( T(\mathbb{C}P^{j-1}) \oplus \mathbb{C} \cong \bar{\eta} \oplus \cdots \oplus \bar{\eta} \) (\( j \) summands) and \( x = c_1(\bar{\eta}) \), so have

\[ s_{j-1}[\mathbb{C}P^{j-1}] = jx^{j-1}\langle \mathbb{C}P^{j-1} \rangle = j. \]

Denote by \( \nu \) the normal bundle of \( \iota: H_{ij} \hookrightarrow \mathbb{C}P^{i} \times \mathbb{C}P^{j} \). Then

\[ T(H_{ij}) \oplus \nu = \iota^*(T(\mathbb{C}P^{i} \times \mathbb{C}P^{j})). \]

We have \( s_{i+j-1}(\nu) = \iota^*(x+y)^{i+j-1} \) and

\[ s_{i+j-1}(T(\mathbb{C}P^{i} \times \mathbb{C}P^{j})) = (i+1)x^{i+j-1} + (j+1)y^{i+j-1} = 0 \text{ for } i > 1, \]

so

\[ s_{i+j-1}[H_{ij}] = -s_{i+j-1}(\nu)\langle H_{ij} \rangle = -\iota^*(x+y)^{i+j-1}\langle H_{ij} \rangle = -(x+y)^{i+j}\langle \mathbb{C}P^{i} \times \mathbb{C}P^{j} \rangle = -(i+j). \]
The bordism classes \( \{[H_{ij}], 0 \leq i \leq j \} \) generate the ring \( \Omega^U \).

**Proof.**

\[
\text{g.c.d.} \left( \binom{n+1}{i}, \ 1 \leq i \leq n \right) = \begin{cases} p & \text{if } n = p^k - 1, \\ 1 & \text{otherwise.} \end{cases}
\]

Now the previous calculation of \( s_{i+j-1}[H_{ij}] \) implies that a certain integer linear combination of bordism classes \([H_{ij}]\) with \( i + j = n + 1 \) has \( s_{i+j-1} \) equal \( p \) or \( 1 \), as needed for the polynomial generator \( a_n \).

**Example**

- \( \Omega^U_2 = \mathbb{Z} \), generated by \([\mathbb{C}P^1]\), as \( 1 = 2^1 - 1 \) and \( s_1[\mathbb{C}P^1] = 2 \);
- \( \Omega^U_4 = \mathbb{Z} \oplus \mathbb{Z} \), generated by \([\mathbb{C}P^1 \times \mathbb{C}P^1]\) and \([\mathbb{C}P^2]\), as \( 2 = 3^1 - 1 \) and \( s_2[\mathbb{C}P^2] = 3 \);
- \([\mathbb{C}P^3]\) cannot be taken as the polynomial generator \( a_3 \in \Omega^U_6 \), since \( s_3[\mathbb{C}P^3] = 4 \), while \( s_3(a_3) = \pm 2 \). We have \( a_3 = [H_{22}] + [\mathbb{C}P^3] \).
The applications of formal group laws in cobordism theory build upon the following fundamental construction due to Novikov.

Let $X$ be a cell complex and $u, v \in U^2(X)$ two geometric cobordisms corresponding to elements $x, y \in H^2(X)$ respectively. Denote by $u +_H v$ the geometric cobordism corresponding to the cohomology class $x + y$.

**Proposition**

The following relation holds in $U^2(X)$:

$$u +_H v = F_U(u, v) = u + v + \sum_{k \geq 1, l \geq 1} \alpha_{kl} u^k v^l,$$

where the coefficients $\alpha_{kl} \in \Omega^{-2(k+l-1)}_U$ do not depend on $X$. The series $F_U(u, v)$ is a formal group law over the complex cobordism ring $\Omega_U$. 

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Proof.

We first calculate on the universal example $X = \mathbb{C}P^\infty \times \mathbb{C}P^\infty$. Then

$$U^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = \Omega_U[[u, v]],$$

where $u, v$ are canonical geometric cobordisms given by the projections of $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$ onto its factors.

We therefore have the following relation in $U^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$:

$$u +_H v = \sum_{k,l \geq 0} \alpha_{kl} u^k v^l,$$

where $\alpha_{kl} \in \Omega^{-2(k+l-1)}_U$.

Now let the geometric cobordisms $u, v \in U^2(X)$ be given by maps $f_u, f_v : X \to \mathbb{C}P^\infty$ respectively. Then $u = (f_u \times f_v)^*(u)$, $v = (f_u \times f_v)^*(v)$ and $u +_H v = (f_u \times f_v)^*(u +_H v)$, where $f_u \times f_v : X \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty$.

Applying the $\Omega_U$-module map $(f_u \times f_v)^*$ to the above expression gives the required formula.
The series \( u +_H v = F_U(u, v) \) is called the formal group law of geometric cobordisms, or simply the formal group law of complex cobordism.

By definition, the geometric cobordism \( u \in U^2(X) \) is the first Conner–Floyd Chern class \( c_1^U(\xi) \) of the complex line bundle \( \xi \) over \( X \) obtained by pulling back the conjugate tautological bundle along the map \( f_u : X \to \mathbb{C}P^\infty \).

It follows that the formal group law of geometric cobordisms gives an expression of \( c_1^U(\xi \otimes \eta) \in U^2(X) \) in terms of the classes \( u = c_1^U(\xi) \) and \( v = c_1^U(\eta) \) of the factors:

\[
c_1^U(\xi \otimes \eta) = F_U(u, v).
\]
Theorem (Buchstaber)

\[
F_U(u, v) = \frac{\sum_{i,j \geq 0} [H_{ij}] u^i v^j}{(\sum_{r \geq 0} [\mathbb{CP}^r] u^r) (\sum_{s \geq 0} [\mathbb{CP}^s] v^s)},
\]

where \( H_{ij} \) (\( 0 \leq i \leq j \)) are Milnor hypersurfaces and \( H_{ji} = H_{ij} \).

Proof.

Consider the Poincaré–Atiyah duality map

\[
D: U^2(\mathbb{CP}^i \times \mathbb{CP}^j) \to U_{2(i+j)-2}(\mathbb{CP}^i \times \mathbb{CP}^j)
\]

and the augmentation

\[
\varepsilon: U_*(\mathbb{CP}^i \times \mathbb{CP}^j) \to U_*(pt) = \Omega^U.
\]

The composite \( \varepsilon D: U^2(\mathbb{CP}^i \times \mathbb{CP}^j) \to \Omega^U_{2(i+j)-2} \) takes geometric cobordisms to the bordism classes of the corresponding submanifolds.

In particular, \( \varepsilon D(u +_H v) = [H_{ij}] \), \( \varepsilon D(u^k v^l) = [\mathbb{CP}^{i-k}][\mathbb{CP}^{j-l}] \). Applying \( \varepsilon D \) to \( u +_H v = F_U(u, v) \) we get \( [H_{ij}] = \sum_{k,l} \alpha_{kl} [\mathbb{CP}^{i-k}][\mathbb{CP}^{j-l}] \). Therefore,

\[
\sum_{i,j} [H_{ij}] u^i v^j = \left( \sum_{k,l} \alpha_{kl} u^k v^l \right) \left( \sum_{i \geq k} [\mathbb{CP}^{i-k}] u^{i-k} \right) \left( \sum_{j \geq l} [\mathbb{CP}^{j-l}] v^{j-l} \right).
\]

\( \square \)
**Corollary**

The coefficients of the formal group law of geometric cobordisms generate the complex cobordism ring $\Omega_U$.

**Theorem (Mishchenko)**

The logarithm of the formal group law of geometric cobordisms is given by

$$g_U(u) = u + \sum_{k \geq 1} [\mathbb{C}P^k] \frac{u^{k+1}}{k+1} \in \Omega_U \otimes \mathbb{Q}[[u]].$$

**Proof.**

$$\frac{dg_U(u)}{du} = 1 + \frac{1}{\frac{\partial F_U(u,v)}{\partial v} \bigg|_{v=0}} = \frac{1 + \sum_{k>0} [\mathbb{C}P^k] u^k}{1 + \sum_{i>0} ([H_{i1}] - [\mathbb{C}P^1][\mathbb{C}P^{i-1}]) u^i}.$$

Now $[H_{i1}] = [\mathbb{C}P^1][\mathbb{C}P^{i-1}]$ (by calculating the Chern numbers), which gives

$$\frac{dg_U(u)}{du} = 1 + \sum_{k>0} [\mathbb{C}P^k] u^k.$$
Theorem (Quillen)

The formal group law $F_U$ of geometric cobordisms is universal.

Proof.

Let $F$ be the universal formal group law over a ring $A$. Then there is a homomorphism $r : A \to \Omega_U$ which takes $F$ to $F_U$.

The series $F$, viewed as a f.g.l. over the ring $A \otimes \mathbb{Q}$, has the universality property for all f.g.l. over $\mathbb{Q}$-algebras. Writing the logarithm of $F$ as $\sum b_k \frac{u^{k+1}}{k+1}$ we obtain that $A \otimes \mathbb{Q} = \mathbb{Q}[b_1, b_2, \ldots]$.

By Mishchenko’s formula for the logarithm, $r(b_k) = [CP^k] \in \Omega_U$. Since $\Omega_U \otimes \mathbb{Q} \cong \mathbb{Q}[[CP^1], [CP^2], \ldots]$, this implies that $r \otimes \mathbb{Q}$ is an isomorphism.

By Lazard’s Theorem the ring $A$ does not have torsion, so $r$ is a monomorphism. On the other hand, Buchstaber’s formula for $F_U(u, v)$ implies that the image $r(A)$ contains the bordism classes $[H_{ij}] \in \Omega_U$, $0 \leq i \leq j$. Since these classes generate the whole ring $\Omega_U$, the map $r$ is onto and thus an isomorphism.
Every homomorphism $\varphi: \Omega^U \to R$ from the complex bordism ring to a commutative ring $R$ with unit can be regarded as a multiplicative characteristic of manifolds which is an invariant of bordism classes. Such a homomorphism is called a (complex) $R$-genus.

Assume that the ring $R$ does not have additive torsion. Then every $R$-genus $\varphi$ is fully determined by the corresponding homomorphism $\Omega^U \otimes \mathbb{Q} \to R \otimes \mathbb{Q}$, which we shall also denote by $\varphi$. A construction due to Hirzebruch describes homomorphisms $\varphi: \Omega^U \otimes \mathbb{Q} \to R \otimes \mathbb{Q}$ by means of universal $R$-valued characteristic classes of special type.
Consider the evaluation homomorphism $e : \Omega^U \to H_*(BU)$ for tangential characteristic numbers. Then $e$ is a monomorphism, and $e \otimes \mathbb{Q} : \Omega^U \otimes \mathbb{Q} \to H_*(BU; \mathbb{Q})$ is an isomorphism.

It follows that every homomorphism $\varphi : \Omega^U \otimes \mathbb{Q} \to R \otimes \mathbb{Q}$ can be interpreted as an element of $\text{Hom}_\mathbb{Q}(H_*(BU; \mathbb{Q}), R \otimes \mathbb{Q}) = H^*(BU; \mathbb{Q}) \otimes R$, or as a sequence of polynomials $\{K_i(c_1, \ldots, c_i), \ i \geq 0\}$, $\text{deg} \ K_i = 2i$.

The fact that $\varphi$ is a ring homomorphism imposes certain conditions on the sequence $\{K_i\}$. These conditions may be described as follows: an identity

$$1 + c_1 + c_2 + \cdots = (1 + c_1' + c_2' + \cdots) \cdot (1 + c_1'' + c_2'' + \cdots)$$

implies the identity

$$\sum_{n \geq 0} K_n(c_1, \ldots, c_n) = \sum_{i \geq 0} K_i(c_1', \ldots, c_i') \cdot \sum_{j \geq 0} K_j(c_1'', \ldots, c_j'').$$

A sequence $\mathcal{K} = \{K_i(c_1, \ldots, c_i), i \geq 0\}$ with $K_0 = 1$ satisfying the identities above is called a multiplicative Hirzebruch sequence.
Proposition

A multiplicative sequence $K$ is completely determined by the series

$$Q(x) = 1 + q_1 x + q_2 x^2 + \cdots \in R \otimes \mathbb{Q}[[x]],$$

where $x = c_1$, and $q_i = K_i(1,0,\ldots,0)$; moreover, every series $Q(x)$ as above determines a multiplicative sequence.

Proof.

Indeed, by considering the identity

$$1 + c_1 + \cdots + c_n = (1 + x_1) \cdots (1 + x_n)$$

we obtain from the multiplicative property that

$$Q(x_1) \cdots Q(x_n) = 1 + K_1(c_1) + K_2(c_1, c_2) + \cdots + K_n(c_1, \ldots, c_n) + K_{n+1}(c_1, \ldots, c_n, 0) + \cdots \qed$$
Along with $Q(x)$ it is convenient to consider the series $f(x) \in R \otimes \mathbb{Q}[[x]] = x + \cdots$ given by the identity $Q(x) = \frac{x}{f(x)}$.

Given a genus $\varphi: \Omega_U^* \otimes \mathbb{Q} \to R \otimes \mathbb{Q}$, the corresponding Hirzebruch sequence satisfies

$$K_n(c_1, \ldots, c_n) = \text{degree-2n part of } \prod_{i=1}^{n} \frac{x_i}{f(x_i)} \in R \otimes \mathbb{Q}[[c_1, \ldots, c_n]].$$

We regard $\prod_{i=1}^{n} \frac{x_i}{f(x_i)}$ as a universal characteristic class of complex $n$-plane bundles. Then the value of $\varphi$ on an $2n$-dimensional stably complex manifold $M$ is given by

$$\varphi[M] = \left( \prod_{i=1}^{n} \frac{x_i}{f(x_i)} (TM) \right) \langle M \rangle.$$

The Hirzebruch genus corresponding to a series $f(x) = x + \cdots \in R \otimes \mathbb{Q}[[x]]$ is the homomorphism $\varphi: \Omega_U^* \to R \otimes \mathbb{Q}$ given by the formula above.
Theorem

For every genus \( \varphi : \Omega^U \to R \), the exponential of the formal group law \( \varphi(F_U) \) is the series \( f(x) \in R \otimes \mathbb{Q}[\![x]\!] \) corresponding to \( \varphi \).

This can be proved either directly, by appealing to the construction of geometric cobordisms, or indirectly, by calculating the values of \( \varphi \) on projective spaces and comparing to the formula for the logarithm of the formal group law.

Example

The universal genus maps a stably complex manifold \( M \) to its bordism class \( [M] \in \Omega^U \) and therefore corresponds to the identity homomorphism \( \varphi_U : \Omega^U \to \Omega^U \).

Its corresponding series \( f_U(x) \) is the exponential of the universal formal group law of geometric cobordisms.
Example

We take $R = \mathbb{Z}$ in these examples.

1. The **top Chern genus** is given by $c[M] = c_n[M]$ for $[M] \in \Omega^U_{2n}$. We have $Q(x) = 1 + x$ and $f(x) = \frac{x}{1 + x}$. Note that $c[M]$ is the Euler characteristic of $M$ if $[M]$ is the cobordism class of an almost complex manifold $M$.

2. The **$L$-genus** $L[M]$ corresponds to the series $f(x) = \tanh(x)$. The $L$-genus coincides with the **signature** $\text{sign}(M)$ of the manifold by the classical result of Hirzebruch. This can be seen by observing that $\text{sign}(\mathbb{C}P^{2k}) = 1$ and $\text{sign}(\mathbb{C}P^{2k+1}) = 0$ and calculating the functional inverse series $g(u)$ (the logarithm).

3. The **Todd genus** $td[M]$ corresponds to the series $f(x) = 1 - e^{-x}$. The associated formal group law is given by $F(u, v) = u + v - uv$, so the Todd genus is integral on any complex bordism class.

The logarithm is given by $-\ln(1 - u) = \sum_{k \geq 1} \frac{u^k}{k}$, which implies $td[\mathbb{C}P^k] = 1$ for any $k$. The **$Q$-series** is

$$Q(x) = \frac{x}{1 - e^{-x}} = \sum_{k \geq 0} (-1)^k \frac{B_k}{k!} x^k.$$
Example

4. Another important example from the original work of Hirzebruch is given by the $\chi_y$-genus. It corresponds to the series

$$f(x) = \frac{1 - e^{-x(1+y)}}{1 + ye^{-x(1+y)}},$$

where $y$ is a parameter. Setting $y = -1$, $y = 0$ and $y = 1$ we get $c_n[M]$, the Todd genus $td[M]$ and the $L$-genus $L[M] = \text{sign}(M)$ respectively.

When working with graded rings, it is convenient to consider the 2-parameter homogeneous genus corresponding to

$$f(x) = \frac{e^{ax} - e^{bx}}{ae^{bx} - be^{ax}}, \quad \text{deg } a = \text{deg } b = -2.$$

It is called the $\chi_{a,b}$-genus.

One gets the original $\chi_y$-genus by setting $a = y$, $b = -1$. 
A multiplicative generalised cohomology theory $X \mapsto h^*(X)$ is complex oriented if it has a choice of Euler class for every complex vector bundle. Such a choice is determined by a choice of an element $c_1^h \in \tilde{h}^2(\mathbb{C}P\infty)$ which restricts to 1 under the composite map

$$\tilde{h}^2(\mathbb{C}P\infty) \to \tilde{h}^2(\mathbb{C}P^1) \cong h^0(pt).$$

$c_1^h$ is called the universal first Chern class in the theory $h^*$. For a complex line bundle $\xi$ over $X$ classified by a map $f : X \to BU(1)$, the first Chern class is defined by $c_1^h(\xi) = f^*(c_1^h) \in \tilde{h}^2(X)$.

Examples of complex oriented theories include ordinary cohomology, complex $K$-theory and complex cobordism.
Given two complex line bundles $\xi, \eta$ over $X$ with $u = c_1^U(\xi)$ and $v = c_1^U(\eta)$,

$$F_h(u, v) = c_1^h(\xi \otimes \eta)$$

is a formal group law over $h^*(pt)$, as in the case of complex cobordism. The f.g.l. $F_h$ is classified by a ring map $\Omega_U = U^*(pt) \to h^*(pt)$ (a genus), which extends to a transformation of cohomology theories $U^*(X) \to h^*(X)$.

Therefore, a complex oriented cohomology theory $h^*$ defines a formal group law $F_h$ and the corresponding genus $\Omega_U \to h^*(pt)$.

On the other hand, given a genus $\varphi : \Omega_U \to R$, one may try to define a cohomology theory by setting $h^*_\varphi(X) = U^*(X) \otimes_{\Omega_U} R$.

The functor $X \mapsto h^*_\varphi(X)$ is homotopy invariant and has the excision property. However, tensoring with $R$ may fail to preserve exact sequences. A criterion for $h^*_\varphi(X)$ to be a cohomology theory is given next.
Define the \( n \)-th power in \( F_U \) as \([n](u) = F_U([n-1](u), u)\) and \([0](u) = 0\).

For each prime \( p \), write

\[
[p](u) = pu + \cdots + t_1 u^p + \cdots + t_n u^{p^n} + \cdots,
\]

where \( t_i \in \Omega_U^{-2(p^n-1)} \).

**Theorem (Landweber Exact Functor Theorem)\)**

In order that \( U_*(X) \otimes_{\Omega_U} R \) be a homology theory, it suffices that for each prime \( p \), the sequence \( p, t_1, \ldots, t_n, \ldots \) of elements in \( \Omega_U \) be \( R \)-regular. That is, it is required that the multiplication by \( p \) on \( R \), and by \( t_n \) on \( R/(pR + \cdots + t_{n-1}R) \) for \( n \geq 1 \), be injective.

If the condition above is satisfied for the homomorphism \( \Omega^U \rightarrow h_*(pt) \) coming from a complex oriented homology theory \( h_* \), the theory \( h_* \) is called Landweber exact. In this case, the canonical transformation

\[
U_*(X) \otimes_{\Omega_U} h_*(pt) \rightarrow h_*(X)
\]

is an equivalence of homology theories.
Example

1. The Thom homomorphism $U^* \to H^*$ gives rise to the augmentation genus $\varepsilon: \Omega_U \to \mathbb{Z}$ sending each element of nonzero degree in $\Omega_U$ to zero. It corresponds to the series $f(x) = x$.

The ordinary cohomology theory $H^*$ is not Landweber exact, because $\varepsilon(t_1) = 0$ and hence the multiplication by $t_1$ is zero on $\mathbb{Z}/p\mathbb{Z}$. Indeed, it is known that the identity $U_*(X) \otimes_{\Omega_U} \mathbb{Z} = H_*(X)$ does not hold in general.

On the other hand, the rational cohomology theory $H^*(X; \mathbb{Q})$ is Landweber exact; we have $U_*(X) \otimes_{\Omega_U} \mathbb{Q} = H_*(X; \mathbb{Q})$. The reason is that $\mathbb{Q}/p\mathbb{Q} = 0$.

2. The Todd genus $td: \Omega_U \to \mathbb{Z}$ defines a $\Omega_U$-module structure on $\mathbb{Z}$, which we denote by $\mathbb{Z}_{td}$ for emphasis.

The $p$-th power in the corresponding formal group law is given by

$$[p]_{td}(u) = 1 - (1 - u)^p = pu + \cdots + u^p,$$

so $t_1$ acts identically on $\mathbb{Z}_{td}/p\mathbb{Z}_{td}$. Hence, Landweber’s Theorem applies, and we get a cohomology theory $U^*(X) \otimes_{\Omega_U} \mathbb{Z}_{td}$. 
Example

On the other hand, there is a natural transformation

$$\mu_c : U^*(X) \to K^*(X)$$

from complex cobordism to complex $K$-theory (graded mod 2), due to Conner and Floyd. Since $\mu_c : \Omega U \to K^*(pt)$ is the same as the Todd genus, the above transformation factors through a transformation

$$\tilde{\mu}_c : U^*(X) \otimes_{\Omega U} \mathbb{Z}_{td} \to K^*(X)$$

which is an equivalence by the uniqueness theorem for cohomology theories.

We therefore obtain the celebrated result of Conner and Floyd which states that complex cobordism determines complex $K$-theory.
Example

We can also obtain \( \mathbb{Z} \)-graded \( K \)-theory (which remembers the dimension of complex line bundles) by a similar procedure.

Then we have \( K^*(pt) = \mathbb{Z}[\beta, \beta^{-1}] \) where \( \beta = 1 - \bar{\eta} \) as the Bott element in \( \tilde{K}^0(\mathbb{C}P^1) = K^{-2}(pt) \), \( \deg \beta = -2 \).

We view \( \mathbb{Z}[\beta, \beta^{-1}] \) as a graded \( \Omega_U \)-module via the homomorphism \( [M^2]n \mapsto \text{td}[M^2] \beta^n \). The corresponding formal group law has the \( p \)-th power is given by

\[
[p]_{\beta}(u) = pu + \cdots + \beta^{p-1}u^p.
\]

Landweber’s Theorem applies because the multiplication by \( \beta^{p-1} \) is an isomorphism \( \mathbb{Z}_p[\beta, \beta^{-1}] \rightarrow \mathbb{Z}_p[\beta, \beta^{-1}] \), and \( \mathbb{Z}_p[\beta, \beta^{-1}] / (\beta^{p-1}) = 0 \). We therefore obtain an equivalence of cohomology theories

\[
U^*(X) \otimes_{\Omega_U} \mathbb{Z}[\beta, \beta^{-1}] \xrightarrow{\cong} K^*(X).
\]

The conclusion is that both \( \mathbb{Z}_2 \)- and \( \mathbb{Z} \)-graded versions of complex \( K \)-theory are Landweber exact.