Right-angled polytopes, hyperbolic manifolds and torus actions

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Polytopes and moment-angle manifolds

A convex polytope in $\mathbb{R}^n$ is a bounded intersection of m halfspaces:

$$P = \big\{\boldsymbol{x} \in \mathbb{R}^n \colon \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle + b_i \geqslant 0 \quad \text{for } i = 1, \ldots, m\big\},$$

where $\boldsymbol{a}_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$.

Assume that $F_i = P \cap \{\boldsymbol{x} \colon \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle + b_i = 0\}$ is a facet for each i.
$\mathcal{F} = \{F_1, \ldots, F_m\}$ the set of facets of P.

Define an affine map

$$i_P \colon \mathbb{R}^n \to \mathbb{R}^m, \quad i_P(\boldsymbol{x}) = \big(\langle \boldsymbol{a}_1, \boldsymbol{x}\rangle + b_1, \ldots, \langle \boldsymbol{a}_m, \boldsymbol{x}\rangle + b_m\big).$$

Then i_P is injective, and $i_P(P) \subset \mathbb{R}^m$ is the intersection of an n-dimensional plane with $\mathbb{R}^m_\geqslant = \{\boldsymbol{y} = (y_1, \ldots, y_m) \colon y_i \geqslant 0\}$.

Define the space $\mathcal{Z}_P$ from the diagram

$$\begin{array}{ccccc} \mathcal{Z}_P & \xrightarrow{i_Z} & \mathbb{C}^m & \qquad & (z_1,\ldots,z_m) \\ \big\downarrow & & \big\downarrow \mu & & \big\downarrow \\ P & \xrightarrow{i_P} & \mathbb{R}^m_{\geqslant} & & (|z_1|^2,\ldots,|z_m|^2) \end{array}$$

Explicitly, $\mathcal{Z}_P = \mu^{-1}(i_P(P))$. It has a T^m-action with the quotient $\mathcal{Z}_P/T^m = P$.

P is simple if there are $n = \dim P$ facets meeting at each vertex.

Proposition

If P is a simple polytope, then $\mathcal{Z}_P$ is a smooth $(m+n)$-dim manifold.

Proof.

Write $i_P(\mathbb{R}^n)$ by $(m-n)$ linear equations in $(y_1,\ldots,y_m) \in \mathbb{R}^m$. Replacing each y_k by $|z_k|^2$ we obtain a presentation of $\mathcal{Z}_P$ by Hermitian quadrics. □

$\mathcal{Z}_P$ is the moment-angle manifold (corresponding to P).

Similarly, considering

$$\begin{array}{ccccc} \mathcal{R}_P & \longrightarrow & \mathbb{R}^m & \qquad & (u_1,\ldots,u_m) \\ \big\downarrow & & \big\downarrow \mu & & \big\downarrow \\ P & \xrightarrow{i_P} & \mathbb{R}^m_{\geqslant} & & (u_1^2,\ldots,u_m^2) \end{array}$$

we obtain a real moment-angle manifold $\mathcal{R}_P$.

Example

$P = \{(x_1, x_2) \in \mathbb{R}^2\colon x_1 \geqslant 0,\ x_2 \geqslant 0,\ -\gamma_1 x_1 - \gamma_2 x_2 + 1 \geqslant 0\}$, $\gamma_1, \gamma_2 > 0$ (a 2-simplex). Then

$\mathcal{Z}_P = \{(z_1, z_2, z_3) \in \mathbb{C}^3\colon \gamma_1|z_1|^2 + \gamma_2|z_2|^2 + |z_3|^2 = 1\}$ (a 5-sphere),

$\mathcal{R}_P = \{(u_1, u_2, u_3) \in \mathbb{R}^3\colon \gamma_1|u_1|^2 + \gamma_2|u_2|^2 + |u_3|^2 = 1\}$ (a 2-sphere).

Right-angled polytopes and hyperbolic manifolds

Let P be a polytope in n-dimensional Lobachevsky space $\mathbb{L}^n$ with right angles between adjacent facets (a right-angled n-polytope).

Denote by $G(P)$ the group generated by reflections in the facets of P. It is a right-angled Coxeter group given by the presentation

$$G(P) = \langle g_1, \ldots, g_m \mid g_i^2 = 1,\ g_i g_j = g_j g_i \text{ if } F_i \cap F_j \neq \varnothing \rangle,$$

where g_i denotes the reflection in the facet F_i.

The group $G(P)$ acts on $\mathbb{L}^n$ discretely with finite isotropy subgroups and with fundamental domain P.

Lemma (A. Vesnin, 1987)

Consider an epimorphism $\varphi\colon G(P) \to \mathbb{Z}_2^k$, $k \geqslant n$. *The subgroup* $\operatorname{Ker}\varphi \subset G(P)$ *does not contain elements of finite order if and only if the images of the reflections in any* n *facets of* P *that have a common vertex are linearly independent in* $\mathbb{Z}_2^k$.
In this case the group $\operatorname{Ker}\varphi$ *acts freely on* $\mathbb{L}^n$.

The quotient $N = \mathbb{L}^n / \operatorname{Ker}\varphi$ is a hyperbolic n-manifold. It is composed of $|\mathbb{Z}_2^k| = 2^k$ copies of P and has a Riemannian metric of constant negative curvature. Furthermore, the manifold N is aspherical (the Eilenberg–Mac Lane space $K(\operatorname{Ker}\varphi, 1)$), as its universal cover $\mathbb{L}^n$ is contractible.

Which combinatorial n-polytopes have right-angled realisations in $\mathbb{L}^n$?
In dim 3, there is a nice criterion going back to Pogorelov's work of 1967:

Theorem (Pogorelov, Andreev)

A combinatorial 3-polytope $P \neq \Delta^3$ can be realised as a right-angled polytope in $\mathbb{L}^3$ if and only if it is simple, and does not have 3- and 4-belts of facets. Furthermore, such a realisation is unique up to isometry.

We refer to the above class of 3-polytopes as the Pogorelov class $\mathcal{P}$.
A Pogorelov polytope does not have triangular or quadrangular facets.
The Pogorelov class contains all fullerenes (simple 3-polytopes with only pentagonal and hexagonal facets).
The conditions specifying Pogorelov polytopes also feature as the no-$\triangle$ and no-$\square$ condition in Gromov's theory of hyperbolic groups.

There is no classification of right-angled polytopes in $\mathbb{L}^4$. For $n \geqslant 5$, right-angled polytopes in $\mathbb{L}^n$ do not exist [Vinberg].

Given a right-angled polytope P, how to find an epimorphism $\varphi\colon G(P) \to \mathbb{Z}_2^k$ with $\operatorname{Ker}\varphi$ acting freely on $\mathbb{L}^n$?

One can consider the abelianisation: $G(P) \xrightarrow{\mathrm{ab}} \mathbb{Z}_2^m$, with $\operatorname{Ker}\mathrm{ab} = G'(P)$, the commutator subgroup.
The corresponding n-manifold $\mathbb{L}^n/G'(P)$ is the real moment-angle manifold $\mathcal{R}_P$, described as an intersection of quadrics in the beginning of this talk.

Corollary

If P is a right-angled polytope in $\mathbb{L}^n$, then the real moment-angle manifold $\mathcal{R}_P$ admits a hyperbolic structure as $\mathbb{L}^n/G'(P)$, where $G'(P)$ is the commutator subgroup of the corresponding right-angled Coxeter group. The manifold $\mathcal{R}_P$ is composed of 2^m copies of P.

A more econimical way to obtain a hyperbolic manifold is to consider $\varphi\colon G(P) \to \mathbb{Z}_2^n$. Such an epimorphism factors as $G(P) \xrightarrow{\mathrm{ab}} \mathbb{Z}_2^m \xrightarrow{\varLambda} \mathbb{Z}_2^n$, where $\varLambda$ is a linear map.

The subgroup $\mathrm{Ker}\,\varphi$ acts freely on $\mathbb{L}^n$ if and only the $\varLambda$-images of any n facets of P that meet at a vertex form a basis of $\mathbb{Z}_2^n$.
Such $\varLambda$ is called a $\mathbb{Z}_2$-characteristic function.

Proposition

Any simple 3-polytope admits a characteristic function.

Proof.

Given a 4-colouring of the facets of P, we assign to a facet of ith colour the ith basis vector $\boldsymbol{e}_i \in \mathbb{Z}^3$ for $i = 1, 2, 3$ and the vector $\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3$ for $i = 4$. The resulting map $\varLambda\colon \mathbb{Z}_2^m \to \mathbb{Z}_2^3$ satisfies the required condition, as any three of the four vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3$ form a basis of $\mathbb{Z}^3$. □

Definition (A. Vesnin, 1987)

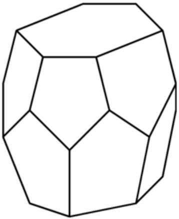
$N(P,\Lambda) = \mathbb{L}^3 / \operatorname{Ker}\varphi$ a hyperbolic 3-manifold of Löbell type.
It is composed of $|\mathbb{Z}_2^3| = 8$ copies of a right-angled 3-polytope $P \in \mathcal{P}$ glued along their facets;
the gluing is prescribed by the characteristic function $\Lambda\colon \mathbb{Z}_2^m \to \mathbb{Z}_2^3$.

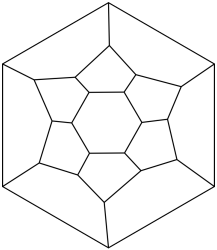
In particular, one obtains a hyperbolic 3-manifold $N(P,\chi)$ from any regular 4-colouring $\chi\colon \mathcal{F} \to \{1,2,3,4\}$ of a right-angled 3-polytope P.

Löbell (1931) was first to consider hyperbolic 3-manifolds coming from 4-colourings of a family of right-angled polytopes starting from the dodecahedron.

Example: Löbell polytopes Q_k ("barrel" fullerenes)

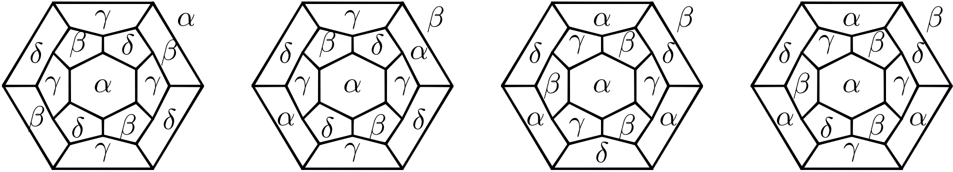
For $k \geqslant 5$, let Q_k be a simple 3-polytope with two "top" and "bottom" k-gonal facets and $2k$ pentagonal facets forming two k-belts around the top and bottom, so that Q_k has $2k+2$ facets in total.
Note that Q_5 is a combinatorial dodecahedron, while Q_6 is a fullerene.





It is easy to see that $Q_k \in \mathcal{P}$, so it admits a right-angled realisation in $\mathbb{L}^3$.

Consider hyperbolic 3-manifolds $N(Q_k, \chi)$ corresponding to 4-colourings χ of Q_k. For example, a dodecahedron Q_5 has a unique 4-colouring up to equivalence, while Q_6 has four non-equivalent regular 4-colourings (4-colourings are equivalent if they differ by a permutation of colours).



Conjecture (Vesnin, 1991)

Hyperbolic 3-manifolds $N(Q_k, \chi_1)$ and $N(Q_k, \chi_2)$ are diffeomorphic (isometric) if and only if the 4-colourings χ_1 and χ_2 are equivalent.

By 2009 the conjecture was verified for all k except $6, 8$ using deep results on arithmetic groups (Margulis commensurator theorem).
However, it remained open for Q_6 and Q_8.

Pairs (P, Λ) and (P', Λ') are equivalent if P and P' are combinatorially equivalent, and $\Lambda, \Lambda' \colon \mathbb{Z}_2^m \to \mathbb{Z}_2^n$ differ by an automorphism of $\mathbb{Z}_2^n$.

Theorem (Buchstaber–Erokhovets–Masuda–P–Park)

Let $N = N(P, \Lambda)$ and $N' = N(P', \Lambda')$ be two hyperbolic 3-manifolds of Löbell type corresponding to right-angled 3-polytopes P and P'. Then the following conditions are equivalent:

(a) *there is a cohomology ring isomorphism* $\varphi\colon H^*(N;\mathbb{Z}_2) \xrightarrow{\cong} H^*(N';\mathbb{Z}_2)$;

(b) *there is a diffeomorphism $N \cong N'$;*

(c) *there is an equivalence of $\mathbb{Z}_2$-characteristic pairs $(P, \Lambda) \sim (P', \Lambda')$.*

The difficult implication is (a)$\Rightarrow$(c). Its proof builds upon the wealth of cohomological techniques of toric topology.

Specifying to $\mathbb{Z}_2$-characteristic functions Λ coming from colourings χ we obtain:

Theorem (Buchstaber–P)

Hyperbolic 3-manifolds $N(P, \chi_1)$ and $N(P', \chi_2)$ corresponding to right-angled polytopes P and P' are diffeomorphic (isometric) if and only if the 4-colourings χ_1 and χ_2 are equivalent.

In particular, Vesnin's conjecture holds for *all* Löbell polytopes Q_k.

Cohomology of moment-angle manifolds

The face ring (the Stanley–Reisner ring) of a simple polytope P

$$\mathbb{Z}[P] := \mathbb{Z}[v_1, \ldots, v_m] \big/ \big(v_{i_1} \cdots v_{i_k} = 0 \quad \text{for } F_{i_1} \cap \cdots \cap F{i_k} = \varnothing \big)$$

where $\deg v_i = 2$.

Theorem

There are ring isomorphisms

$$\begin{aligned} H^*(\mathcal{Z}_P) &\cong \operatorname{Tor}_{\mathbb{Z}[v_1,\ldots,v_m]}(\mathbb{Z}[P], \mathbb{Z}) \\ &\cong H\big(\Lambda[u_1, \ldots, u_m] \otimes \mathbb{Z}[P], d\big) && du_i = v_i, dv_i = 0 \\ &\cong \bigoplus_{I \subset \{1,\ldots,m\}} \widetilde{H}^{*-|I|-1}(P_I) && P_I = \bigcup_{i \in I} F_i \end{aligned}$$

(Quasi)toric manifolds and small covers

P a simple n-polytope, $\mathcal{F} = \{F_1, \ldots, F_m\}$ the set of facets.

A characteristic function is map $\Lambda\colon \mathcal{F} \to \mathbb{Z}^n$ such that $\Lambda(F_{i_1}), \ldots, \Lambda(F_{i_n})$ is a basis of $\mathbb{Z}^n$ whenever $F_{i_1}, \ldots, F_{i_n}$ intersect in a vertex.
A characteristic function defines a linear map $\Lambda\colon \mathbb{Z}^{\mathcal{F}} = \mathbb{Z}^m \to \mathbb{Z}^n$ and a homomorphism of tori $\Lambda\colon T^m \to T^n$.

Proposition

The subgroup $\operatorname{Ker} \Lambda \cong T^{m-n}$ *acts freely on* $\mathcal{Z}_P$.

$M(P, \Lambda) = \mathcal{Z}_P / \operatorname{Ker} \Lambda$ is a quasitoric manifold.
It is a smooth $2n$-dimensional manifold with an action of the n-torus $T^m / \operatorname{Ker} \Lambda \cong T^n$ and the quotient P.

By considering $\mathbb{Z}_2$-characteristic functions $\Lambda\colon \mathbb{Z}_2^m \to \mathbb{Z}_2^n$, we obtain small covers of P as the quotients $\mathcal{R}_P/\operatorname{Ker}\Lambda$.
A small cover $N(P,\Lambda)$ is a smooth n-dimensional manifold with an action of $\mathbb{Z}_2^n$ and the quotient P.

Proposition

In dimension 3, a real moment-angle manifold $\mathcal{R}_P$ and a small cover $N(P,\Lambda)$ admit a hyperbolic structure if and only if P is a Pogorelov polytope.

Proof.

P is a Pogorelov polytope $\quad\Leftrightarrow\quad$ dual $\mathcal{K}$ has no $\triangle$ and $\square$

If P is a Pogorelov polytope, then $\mathcal{R}_P$ and $N(P,\Lambda)$ are hyperbolic.

If $\mathcal{K}$ has a $\triangle$, then $R_\triangle \cong S^2$ retracts off $\mathcal{R}_P$, so $\mathcal{R}_P$ cannot be hyperbolic (as it has $\pi_2 \neq 0$).

If $\mathcal{K}$ has a $\square$, then $R_\square \cong T^2$ retracts off $\mathcal{R}_P$, which is impossible for a hyperbolic manifold. $\square$

Theorem (Danilov–Jurkiewicz, Davis–Januszkiewicz)

Let $M = M(P, \Lambda)$ be a quasitoric manifold over a simple n-polytope P. The cohomology ring $H^*(M; \mathbb{Z})$ is generated by the 2-dimensional classes $[v_i]$ dual to the characteristic submanifolds M_i, $i = 1, \ldots, m$, and is given by

$$H^*(M; \mathbb{Z}) \cong \mathbb{Z}[v_1, \ldots, v_m]/\mathcal{I}, \qquad \deg v_i = 2,$$

where $\mathcal{I}$ is the ideal generated by elements of two kinds:

(a) $v_{i_1} \cdots v_{i_k}$, *where* $F_{i_1} \cap \cdots \cap F_{i_k} = \varnothing$ *in* P;

(b) $\displaystyle\sum_{i=1}^{m} \langle \Lambda(F_i), \boldsymbol{x} \rangle v_i$ *for any* $\boldsymbol{x} \in \mathbb{Z}^n$.

The $\mathbb{Z}_2$-cohomology ring $H^*(N; \mathbb{Z}_2)$ of a small cover $N = N(P, \Lambda)$ has the same description, with generators v_i of degree 1.

Theorem (Buchstaber-Erokhovets-Masuda-P-Park)

Let $M = M(P, \Lambda)$ and $M' = M(P', \Lambda')$ be quasitoric 6-manifolds, where P is a Pogorelov 3-polytope. The following conditions are equivalent:

(a) *There is a ring isomorphism $\varphi\colon H^*(M;\mathbb{Z}) \xrightarrow{\cong} H^*(M';\mathbb{Z})$;*

(b) *There is a diffeomorphism $M \cong M'$;*

(c) *There is an equivalence of characteristic pairs $(P, \Lambda) \sim (P', \Lambda')$.*

Idea of proof (for both theorems).

We need to prove that a ring iso $\varphi\colon H^*(N;\mathbb{Z}) \xrightarrow{\cong} H^*(N';\mathbb{Z})$ implies an equivalence $(P,\Lambda) \sim (P',\Lambda')$.

An iso $\varphi\colon H^*(N;\mathbb{Z}_2) \xrightarrow{\cong} H^*(N';\mathbb{Z}_2)$ implies an iso $\varphi\colon H^*(M;\mathbb{Z}_2) \xrightarrow{\cong} H^*(M';\mathbb{Z}_2)$ (as every $\mathbb{Z}_2$-characteristic function of a 3-polytope lifts to a $\mathbb{Z}$-characteristic function!)

An iso $\varphi\colon H^*(M;\mathbb{Z}_2) = \mathbb{Z}_2[P]/\mathcal{J}_\Lambda \xrightarrow{\cong} \mathbb{Z}_2[P']/\mathcal{J}_{\Lambda'} = H^*(M';\mathbb{Z}_2)$ implies an iso

$$\psi\colon H^*(\mathcal{Z}_P;\mathbb{Z}_2) = \operatorname{Tor}_{\mathbb{Z}_2[v_1,\ldots,v_m]/\mathcal{J}_\Lambda}(\mathbb{Z}_2[P]/\mathcal{J}_\Lambda,\mathbb{Z}_2)$$
$$\xrightarrow{\cong} \operatorname{Tor}_{\mathbb{Z}_2[v_1,\ldots,v_m]/\mathcal{J}_{\Lambda'}}(\mathbb{Z}_2[P']/\mathcal{J}_{\Lambda'},\mathbb{Z}_2) = H^*(\mathcal{Z}_{P'};\mathbb{Z}_2)$$

The results of Fan, Ma and Wang imply that ψ maps the set of canonical generators $\{[u_iv_j] \in H^3(\mathcal{Z}_P)\colon F_i \cap F_j = \varnothing\}$ bijectively to the corresponding set for $\mathcal{Z}_{P'}$.
This implies that φ maps the set $\{[v_i] \in H^2(M)\}$ bijectively to the corresponding set for M', giving an equivalence $(P,\Lambda) \sim (P',\Lambda')$. □

Cohomological rigidity

Problem

Let $M = M(P, \Lambda)$ and $M' = M(P', \Lambda')$ be quasitoric manifolds with isomorphic integer cohomology rings. Are they homeomorphic?

Our result gives a positive answer in the case of quasitoric 6-manifolds over Pogorelov polytopes.

Proposition

6-dimensional quasitoric manifolds M and M' are diffeomorphic if there is an isomorphism $\varphi\colon H^*(M;\mathbb{Z}) \xrightarrow{\cong} H^*(M';\mathbb{Z})$ preserving the first Pontryagin class, i. e. $\varphi(p_1(M)) = p_1(M')$.

We have $p_1(M) = v_1^2 + \cdots + v_m^2 \in H^4(M)$. However, we were not able to establish the invariance of $p_1(M)$ directly...

References

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[2] V. Buchstaber and T. Panov. *On manifolds defined by 4-colourings of simple 3-polytopes.* Russian Math. Surveys 71 (2016), no. 6, 1137–1139; arXiv:1703.06801.