Polyhedral products, right-angled Coxeter groups, and hyperbolic manifolds

based on joint works with Victor Buchstaber, Nikolay Erokhovets, Mikiya Masuda, Seonjeong Park and Yakov Veryovkin

Taras Panov

Moscow State University, ITEP and IITP

Рождественские математические встречи
НМУ, Москва, 4–6 января 2017

1. Preliminaries

Polyhedral product

$(\boldsymbol{X}, \boldsymbol{A}) = \{(X_1, A_1), \ldots, (X_m, A_m)\}$ a sequence of pairs of spaces, $A_i \subset X_i$.

$\mathcal{K}$ a simplicial complex on $[m] = \{1, 2, \ldots, m\}$, $\varnothing \in \mathcal{K}$.

Given $I = \{i_1, \ldots, i_k\} \subset [m]$, set

$$(\boldsymbol{X}, \boldsymbol{A})^I = Y_1 \times \cdots \times Y_m \qquad \text{where } Y_i = \begin{cases} X_i & \text{if } i \in I, \\ A_i & \text{if } i \notin I. \end{cases}$$

The $\mathcal{K}$-polyhedral product of $(\boldsymbol{X}, \boldsymbol{A})$ is

$$(\boldsymbol{X}, \boldsymbol{A})^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (\boldsymbol{X}, \boldsymbol{A})^I = \bigcup_{I \in \mathcal{K}} \Bigl(\prod_{i \in I} X_i \times \prod_{j \notin I} A_j \Bigr).$$

Notation: $(X, A)^{\mathcal{K}} = (\boldsymbol{X}, \boldsymbol{A})^{\mathcal{K}}$ when all $(X_i, A_i) = (X, A)$;
$\boldsymbol{X}^{\mathcal{K}} = (\boldsymbol{X}, pt)^{\mathcal{K}}$, $X^{\mathcal{K}} = (X, pt)^{\mathcal{K}}$.

Categorical approach

Category of faces $\textsc{cat}(\mathcal{K})$.
Objects: simplices $I \in \mathcal{K}$. Morphisms: inclusions $I \subset J$.

$\textsc{top}$ the category of topological spaces.
Define the $\textsc{cat}(\mathcal{K})$-diagram

$$\begin{aligned} \mathcal{D}_{\mathcal{K}}(\boldsymbol{X}, \boldsymbol{A})\colon \textsc{cat}(\mathcal{K}) &\longrightarrow \textsc{top}, \\ I &\longmapsto (\boldsymbol{X}, \boldsymbol{A})^I, \end{aligned}$$

which maps the morphism $I \subset J$ of $\textsc{cat}(\mathcal{K})$ to the inclusion of spaces $(\boldsymbol{X}, \boldsymbol{A})^I \subset (\boldsymbol{X}, \boldsymbol{A})^J$.

Then we have

$$(\boldsymbol{X}, \boldsymbol{A})^{\mathcal{K}} = \operatorname{colim} \mathcal{D}_{\mathcal{K}}(\boldsymbol{X}, \boldsymbol{A}) = \operatorname*{colim}_{I \in \mathcal{K}} (\boldsymbol{X}, \boldsymbol{A})^I.$$

Example

Let $(X, A) = (S^1, pt)$, where S^1 is a circle. Then

$$(S^1)^{\mathcal{K}} = \bigcup_{I\in\mathcal{K}} (S^1)^I \subset (S^1)^m.$$

When $\mathcal{K} = \{\varnothing, \{1\}, \ldots, \{m\}\}$ (m disjoint points), the polyhedral product $(S^1)^{\mathcal{K}}$ is the wedge $(S^1)^{\vee m}$ of m circles.

When $\mathcal{K}$ consists of all proper subsets of $[m]$ (the boundary $\partial\Delta^{m-1}$ of an $(m-1)$-dimensional simplex), $(S^1)^{\mathcal{K}}$ is the fat wedge of m circles; it is obtained by removing the top-dimensional cell from the m-torus $(S^1)^m$.

For a general $\mathcal{K}$ on m vertices, $(S^1)^{\vee m} \subset (S^1)^{\mathcal{K}} \subset (S^1)^m$.

Example

Let $(X, A) = (\mathbb{R}, \mathbb{Z})$. Then

$$\mathcal{L}_{\mathcal{K}} := (\mathbb{R}, \mathbb{Z})^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (\mathbb{R}, \mathbb{Z})^I \subset \mathbb{R}^m.$$

When $\mathcal{K}$ consists of m disjoint points, $\mathcal{L}_{\mathcal{K}}$ is a grid in $\mathbb{R}^m$ consisting of all lines parallel to one of the coordinate axis and passing though integer points.

When $\mathcal{K} = \partial\Delta^{m-1}$, the complex $\mathcal{L}_{\mathcal{K}}$ is the union of all integer hyperplanes parallel to coordinate hyperplanes.

Example

Let $(X, A) = (\mathbb{R}P^\infty, pt)$, where $\mathbb{R}P^\infty = B\mathbb{Z}_2$. Then

$$(\mathbb{R}P^\infty)^\mathcal{K} = \bigcup_{I\in\mathcal{K}} (\mathbb{R}P^\infty)^I \subset (\mathbb{R}P^\infty)^m.$$

Example

Let $(X, A) = (D^1, S^0)$, where $D^1 = [-1, 1]$ and $S^0 = \{1, -1\}$. The real moment-angle complex is

$$\mathcal{R}_\mathcal{K} := (D^1, S^0)^\mathcal{K} = \bigcup_{I\in\mathcal{K}} (D^1, S^0)^I.$$

It is a cubic subcomplex in the m-cube $(D^1)^m = [-1, 1]^m$.

When $\mathcal{K}$ consists of m disjoint points, $\mathcal{R}_\mathcal{K}$ is the 1-dimensional skeleton of the cube $[-1, 1]^m$. When $\mathcal{K} = \partial\Delta^{m-1}$, $\mathcal{R}_\mathcal{K}$ is the boundary of the cube $[-1, 1]^m$. Also, $\mathcal{R}_\mathcal{K}$ is a topological manifold when $|\mathcal{K}|$ is a sphere.

The four polyhedral products above are related by the two homotopy fibrations

$$(\mathbb{R},\mathbb{Z})^{\mathcal{K}}=\mathcal{L}_{\mathcal{K}}\longrightarrow(S^1)^{\mathcal{K}}\longrightarrow(S^1)^m,$$

$$(D^1,S^0)^{\mathcal{K}}=\mathcal{R}_{\mathcal{K}}\longrightarrow(\mathbb{R}P^\infty)^{\mathcal{K}}\longrightarrow(\mathbb{R}P^\infty)^m.$$

By analogy with the polyhedral product of spaces $\boldsymbol{X}^{\mathcal{K}} = \operatorname{colim}_{I\in\mathcal{K}} \boldsymbol{X}^I$, we may consider the following more general construction of a discrete group.

Graph product

$\boldsymbol{G} = (G_1, \ldots, G_m)$ a sequence of m discrete groups, $G_i \neq \{1\}$.

Given $I = \{i_1, \ldots, i_k\} \subset [m]$, set

$$\boldsymbol{G}^I = \Bigl\{(g_1, \ldots, g_m) \in \prod_{k=1}^{m} G_k \colon \; g_k = 1 \quad \text{for } k \notin I\Bigr\}.$$

Then consider the following $\mathrm{CAT}(\mathcal{K})$-diagram of groups:

$$\mathcal{D}_{\mathcal{K}}(\boldsymbol{G}) \colon \mathrm{CAT}(\mathcal{K}) \longrightarrow \mathrm{GRP}, \qquad I \longmapsto \boldsymbol{G}^I,$$

which maps a morphism $I \subset J$ to the canonical monomorphism $\boldsymbol{G}^I \to \boldsymbol{G}^J$.

The graph product of the groups $G_1, \ldots, G_m$ is

$$\boldsymbol{G}^{\mathcal{K}} = \operatorname{colim}^{\mathrm{GRP}} \mathcal{D}_{\mathcal{K}}(\boldsymbol{G}) = \operatorname{colim}^{\mathrm{GRP}}_{I\in\mathcal{K}} \boldsymbol{G}^I.$$

The graph product $\boldsymbol{G}^{\mathcal{K}}$ depends only on the 1-skeleton (graph) of $\mathcal{K}$. Namely,

Proposition

The is an isomorphism of groups

$$\boldsymbol{G}^{\mathcal{K}} \cong \mathop{\bigstar}_{k=1}^{m} G_k \big/ (g_i g_j = g_j g_i \ \text{for } g_i \in G_i,\, g_j \in G_j,\, \{i,j\} \in \mathcal{K}),$$

where $\bigstar_{k=1}^{m} G_k$ *denotes the free product of the groups* G_k*.*

Example

Let $G_i = \mathbb{Z}$. Then $\boldsymbol{G}^{\mathcal{K}}$ is the right-angled Artin group

$$RA_{\mathcal{K}} = F(g_1, \ldots, g_m)\big/(g_i g_j = g_j g_i \text{ for } \{i, j\} \in \mathcal{K}),$$

where $F(g_1, \ldots, g_m)$ is a free group with m generators.

When $\mathcal{K}$ is a full simplex, we have $RA_{\mathcal{K}} = \mathbb{Z}^m$. When $\mathcal{K}$ is m points, we obtain a free group of rank m.

Example

Let $G_i = \mathbb{Z}_2$. Then $\boldsymbol{G}^{\mathcal{K}}$ is the right-angled Coxeter group

$$RC_{\mathcal{K}} = F(g_1, \ldots, g_m)\big/(g_i^2 = 1,\ g_i g_j = g_j g_i \text{ for } \{i, j\} \in \mathcal{K}).$$

2. Classifying spaces

The homotopy fibrations $\mathcal{L}_\mathcal{K} \to (S^1)^\mathcal{K} \to (S^1)^m$ and $\mathcal{R}_\mathcal{K} \to (\mathbb{R}P^\infty)^\mathcal{K} \to (\mathbb{R}P^\infty)^m$ are generalised as follows.

Proposition

There is a homotopy fibration

$$(E\mathbf{G}, \mathbf{G})^\mathcal{K} \longrightarrow (B\mathbf{G})^\mathcal{K} \longrightarrow \prod_{k=1}^m BG_k.$$

A missing face (a minimal non-face) of $\mathcal{K}$ is a subset $I \subset [m]$ such that $I \notin \mathcal{K}$, but $J \in \mathcal{K}$ for each $J \subsetneq I$.

$\mathcal{K}$ a flag complex if each of its missing faces consists of two vertices. Equivalently, $\mathcal{K}$ is flag if any set of vertices of $\mathcal{K}$ which are pairwise connected by edges spans a simplex.

Every flag complex $\mathcal{K}$ is determined by its 1-skeleton $\mathcal{K}^1$.

Theorem

Let $\boldsymbol{G}^{\mathcal{K}}$ be a graph product group.

1. $\pi_1((B\boldsymbol{G})^{\mathcal{K}}) \cong \boldsymbol{G}^{\mathcal{K}}$.
2. *Both spaces $(B\boldsymbol{G})^{\mathcal{K}}$ and $(E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}}$ are aspherical if and only if $\mathcal{K}$ is flag. Hence, $B(\boldsymbol{G}^{\mathcal{K}}) = (B\boldsymbol{G})^{\mathcal{K}}$ whenever $\mathcal{K}$ is flag.*
3. $\pi_i((B\boldsymbol{G})^{\mathcal{K}}) \cong \pi_i((E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}})$ *for* $i \geqslant 2$.
4. $\pi_1((E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}})$ *is isomorphic to the kernel of the canonical projection* $\boldsymbol{G}^{\mathcal{K}} \to \prod_{k=1}^{m} G_k$.

Proof

(1) Proceed inductively by adding simplices to $\mathcal{K}$ one by one and use van Kampen's Theorem. The base of the induction is $\mathcal{K}$ consisting of m disjoint points. Then $(B\boldsymbol{G})^{\mathcal{K}}$ is the wedge $BG_1 \vee \cdots \vee BG_m$, and $\pi_1((B\boldsymbol{G})^{\mathcal{K}})$ is the free product $G_1 \star \cdots \star G_m$.

Proof

(2) To see that $B(\boldsymbol{G}^{\mathcal{K}}) = (B\boldsymbol{G})^{\mathcal{K}}$ when $\mathcal{K}$ is flag, consider the map

$$\operatorname{colim}_{I\in\mathcal{K}} B\boldsymbol{G}^{I} = (B\boldsymbol{G})^{\mathcal{K}} \to B(\boldsymbol{G}^{\mathcal{K}}). \tag{1}$$

According to [PRV], the homotopy fibre of (1) is $\operatorname{hocolim}_{I\in\mathcal{K}} \boldsymbol{G}^{\mathcal{K}}/\boldsymbol{G}^{I}$, which is homeomorphic to the identification space

$$\bigl(B\mathrm{CAT}(\mathcal{K}) \times \boldsymbol{G}^{\mathcal{K}}\bigr)/\sim . \tag{2}$$

Here $B\mathrm{CAT}(\mathcal{K})$ is homeomorphic to the cone on $|\mathcal{K}|$. The equivalence relation $\sim$ is defined as follows: $(x,gh)\sim(x,g)$ whenever $h \in \boldsymbol{G}^{I}$ and $x \in B(I\downarrow\mathrm{CAT}(\mathcal{K}))$, where $I\downarrow\mathrm{CAT}(\mathcal{K})$ is the *undercategory*, and $B(I\downarrow\mathrm{CAT}(\mathcal{K}))$ is homeomorphic to the star of I in $\mathcal{K}$.
When $\mathcal{K}$ is a flag complex, the identification space (2) is contractible by [PRV]. Therefore, the map (1) is a homotopy equivalence, which implies that $(B\boldsymbol{G})^{\mathcal{K}}$ is aspherical when $\mathcal{K}$ is flag.

Proof

Assume now that $\mathcal{K}$ is not flag. Choose a missing face $J = \{j_1, \ldots, j_k\} \subset [m]$ with $k \geqslant 3$ vertices. Let $\mathcal{K}_J = \{I \in \mathcal{K} \colon I \subset J\}$.

Then $(B\boldsymbol{G})^{\mathcal{K}_J}$ is the fat wedge of the spaces $\{BG_j, j \in J\}$, and it is a retract of $(B\boldsymbol{G})^{\mathcal{K}}$.

The homotopy fibre of the inclusion $(B\boldsymbol{G})^{\mathcal{K}_J} \to \prod_{j \in J} BG_j$ is $\Sigma^{k-1} G_{j_1} \wedge \cdots \wedge G_{j_k}$, a wedge of $(k-1)$-dimensional spheres.
Hence, $\pi_{k-1}((B\boldsymbol{G})^{\mathcal{K}_J}) \neq 0$ where $k \geqslant 3$.
Thus, $(B\boldsymbol{G})^{\mathcal{K}_J}$ and $(B\boldsymbol{G})^{\mathcal{K}}$ are non-aspherical.

The rest of the proof (the asphericity of $(E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}}$ and statements (3) and (4)) follow from the homotopy exact sequence of the fibration $(E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}} \to (B\boldsymbol{G})^{\mathcal{K}} \to \prod_{k=1}^{m} BG_k$.

Specialising to the cases $G_k = \mathbb{Z}$ and $G_k = \mathbb{Z}_2$ respectively we obtain:

Corollary

Let $RA_{\mathcal{K}}$ be a right-angled Artin group.

1. $\pi_1((S^1)^{\mathcal{K}}) \cong RA_{\mathcal{K}}$.
2. *Both* $(S^1)^{\mathcal{K}}$ *and* $\mathcal{L}_{\mathcal{K}} = (\mathbb{R}, \mathbb{Z})^{\mathcal{K}}$ *are aspherical iff* $\mathcal{K}$ *is flag.*
3. $\pi_i((S^1)^{\mathcal{K}}) \cong \pi_i(\mathcal{L}_{\mathcal{K}})$ *for* $i \geqslant 2$.
4. $\pi_1(\mathcal{L}_{\mathcal{K}})$ *is isomorphic to the commutator subgroup* $RA'_{\mathcal{K}}$.

Corollary

Let $RC_{\mathcal{K}}$ be a right-angled Coxeter group.

1. $\pi_1((\mathbb{R}P^\infty)^{\mathcal{K}}) \cong RC_{\mathcal{K}}$.
2. *Both* $(\mathbb{R}P^\infty)^{\mathcal{K}}$ *and* $\mathcal{R}_{\mathcal{K}} = (D^1, S^0)^{\mathcal{K}}$ *are aspherical iff* $\mathcal{K}$ *is flag.*
3. $\pi_i((\mathbb{R}P^\infty)^{\mathcal{K}}) \cong \pi_i(\mathcal{R}_{\mathcal{K}})$ *for* $i \geqslant 2$.
4. $\pi_1(\mathcal{R}_{\mathcal{K}})$ *is isomorphic to the commutator subgroup* $RC'_{\mathcal{K}}$.

Example

Let $\mathcal{K}$ be an m-cycle (the boundary of an m-gon).
A simple argument with Euler characteristic shows that $\mathcal{R}_{\mathcal{K}}$ is homeomorphic to a closed orientable surface of genus $(m-4)2^{m-3}+1$.
(This observation goes back to a 1938 work of Coxeter.)
Therefore, the commutator subgroup of the corresponding right-angled Coxeter group $RC_{\mathcal{K}}$ is a surface group.

Similarly, when $|\mathcal{K}| \cong S^2$ (which is equivalent to $\mathcal{K}$ being the boundary of a 3-dimensional simplicial polytope), $\mathcal{R}_{\mathcal{K}}$ is a 3-dimensional manifold.
Therefore, the commutator subgroup of the corresponding $RC_{\mathcal{K}}$ is a 3-manifold group.

3. The structure of the commutator subgroups

We have

$$\operatorname{Ker}\Bigl(\boldsymbol{G}^{\mathcal{K}} \to \prod_{k=1}^{m} G_k\Bigr) = \pi_1((E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}}).$$

In the case of right-angled Artin or Coxeter groups (or when each G_k is abelian), the group above is the commutator subgroup $(\boldsymbol{G}^{\mathcal{K}})'$.

The next goal is to study the group $\pi_1((E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}})$, identify the class of $\mathcal{K}$ for which this group is free, and describe a generator set.

A graph Γ is called chordal (in other terminology, triangulated) if each of its cycles with $\geqslant 4$ vertices has a chord.

By a result of Fulkerson–Gross, a graph is chordal if and only if its vertices can be ordered in such a way that, for each vertex i, the lesser neighbours of i form a complete subgraph. (A perfect elimination order.)

Theorem (P–Veryovkin)

The following conditions are equivalent:

1. $\mathrm{Ker}(\boldsymbol{G}^{\mathcal{K}} \to \prod_{k=1}^{m} G_k)$ *is a free group;*
2. $(E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}}$ *is homotopy equivalent to a wedge of circles;*
3. $\mathcal{K}^1$ *is a chordal graph.*

Proof

(2)⇒(1) Because $\mathrm{Ker}\Big(\boldsymbol{G}^{\mathcal{K}} \to \prod_{k=1}^{m} G_k\Big) = \pi_1((E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}})$.

(3)⇒(2) Use induction and perfect elimination order.

(1)⇒(3) Assume that $\mathcal{K}^1$ is not chordal. Then, for each chordless cycle of length $\geqslant 4$, one can find a subgroup in $\mathrm{Ker}(\boldsymbol{G}^{\mathcal{K}} \to \prod_{k=1}^{m} G_k)$ which is a surface group. Hence, $\mathrm{Ker}(\boldsymbol{G}^{\mathcal{K}} \to \prod_{k=1}^{m} G_k)$ is not a free group.

Corollary

Let $RA_{\mathcal{K}}$ and $RC_{\mathcal{K}}$ be the right-angled Artin and Coxeter groups corresponding to a simplicial complex $\mathcal{K}$.

(a) *The commutator subgroup $RA'_{\mathcal{K}}$ is free if and only if $\mathcal{K}^1$ is a chordal graph.*

(b) *The commutator subgroup $RC'_{\mathcal{K}}$ is free if and only if $\mathcal{K}^1$ is a chordal graph.*

Part (a) is the result of Servatius, Droms and Servatius.

The difference between (a) and (b) is that the commutator subgroup $RA'_{\mathcal{K}}$ is infinitely generated, unless $RA_{\mathcal{K}} = \mathbb{Z}^m$, while the commutator subgroup $RC'_{\mathcal{K}}$ is finitely generated. We elaborate on this in the next theorem.

Let $(g,h)=g^{-1}h^{-1}gh$ denote the group commutator of g,h.

Theorem (P–Veryovkin)

The commutator subgroup $RC'_{\mathcal K}$ has a finite minimal generator set consisting of $\sum_{J\subset[m]}\operatorname{rank}\widetilde H_0(\mathcal K_J)$ iterated commutators

$$(g_j,g_i),\quad (g_{k_1},(g_j,g_i)),\quad \ldots,\quad (g_{k_1},(g_{k_2},\cdots(g_{k_{m-2}},(g_j,g_i))\cdots)),$$

where $k_1<k_2<\cdots<k_{\ell-2}<j>i$, $k_s\ne i$ for any s, and i is the smallest vertex in a connected component not containing j of the subcomplex $\mathcal K_{\{k_1,\ldots,k_{\ell-2},j,i\}}$.

Idea of proof

First consider the case $\mathcal K=m$ points. Then $\mathcal R_{\mathcal K}$ is the 1-skeleton of an m-cube and $RC'_{\mathcal K}=\pi_1(\mathcal R_{\mathcal K})$ is a free group of rank $\sum_{\ell=2}^m(\ell-1)\binom{m}{\ell}$. It agrees with the total number of nested commutators in the list.

Then eliminate the extra nested commutators using the commutation relations $(g_i,g_j)=1$ for $\{i,j\}\in\mathcal K$.

Idea of proof

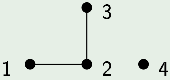
To see that the given generating set is minimal, argue as follows. The first homology group $H_1(\mathcal{R}_\mathcal{K})$ is $RC'_\mathcal{K}/RC''_\mathcal{K}$. On the other hand,

$$H_1(\mathcal{R}_\mathcal{K}) \cong \sum_{J\subset[m]} \widetilde{H}_0(\mathcal{K}_J).$$

Hence, the number of generators in the abelian group $H_1(\mathcal{R}_\mathcal{K}) \cong RC'_\mathcal{K}/RC''_\mathcal{K}$ is $\sum_{J\subset[m]} \mathrm{rank}\, \widetilde{H}_0(\mathcal{K}_J)$, and the latter number agrees with the number of iterated commutators in the in generator set for $RC'_\mathcal{K}$ constructed above.

Example

Let $\mathcal{K} =$



Then the commutator subgroup $RC'_{\mathcal{K}}$ is free with the following basis:

$$(g_3,g_1),\ (g_4,g_1),\ (g_4,g_2),\ (g_4,g_3),$$
$$(g_2,(g_4,g_1)),\ (g_3,(g_4,g_1)),\ (g_1,(g_4,g_3)),\ (g_3,(g_4,g_2)),$$
$$(g_2,(g_3,(g_4,g_1))).$$

Example

Let $\mathcal{K}$ be an m-cycle with $m \geqslant 4$ vertices.
Then $\mathcal{K}^1$ is not a chordal graph, so the group $RC'_{\mathcal{K}}$ is not free.

In fact, $\mathcal{R}_{\mathcal{K}}$ is an orientable surface of genus $(m-4)2^{m-3}+1$, so $RC'_{\mathcal{K}} \cong \pi_1(\mathcal{R}_{\mathcal{K}})$ is a one-relator group.

The are similar results of Grbic, P., Theriault and Wu describing the commutator subalgebra of the graded Lie algebra given by

$$L_{\mathcal{K}} = FL\langle u_1, \ldots, u_m\rangle \big/ \big([u_i, u_i] = 0,\ [u_i, u_j] = 0 \text{ for } \{i, j\} \in \mathcal{K}\big),$$

where $FL\langle u_1, \ldots, u_m\rangle$ is the free graded Lie algebra on generators u_i of degree one, and $[a, b] = -(-1)^{|a||b|}[b, a]$ denotes the graded Lie bracket.

The commutator subalgebra is the kernel of the Lie algebra homomorphism $L_{\mathcal{K}} \to CL\langle u_1, \ldots, u_m\rangle$ to the commutative (trivial) Lie algebra.

The graded Lie algebra $L_{\mathcal{K}}$ is a graph product similar to the right-angled Coxeter group $RC_{\mathcal{K}}$.

It has a similar colimit decomposition, with each $G_i = \mathbb{Z}_2$ replaced by the trivial Lie algebra $CL\langle u\rangle = FL\langle u\rangle/([u, u] = 0)$ and the colimit taken in the category of graded Lie algebras.

4. Right-angled polytopes and hyperbolic manifolds

Let P be a polytope in n-dimensional Lobachevsky space $\mathbb{L}^n$ with right angles between adjacent facets (a right-angled n-polytope).

Denote by $G(P)$ the group generated by reflections in the facets of P. It is a right-angled Coxeter group given by the presentation

$$G(P) = \langle g_1, \ldots, g_m \mid g_i^2 = 1,\; g_i g_j = g_j g_i \text{ if } F_i \cap F_j \neq \varnothing \rangle,$$

where g_i denotes the reflection in the facet F_i.

The group $G(P)$ acts on $\mathbb{L}^n$ discretely with finite isotropy subgroups and with fundamental domain P.

Lemma

Consider an epimorphism $\varphi\colon G(P) \to \mathbb{Z}_2^k$. *The subgroup* $\operatorname{Ker}\varphi \subset G(P)$ *does not contain elements of finite order if and only if the images of the reflections in any* $\leqslant k$ *facets of* P *that have a common vertex are linearly independent in* $\mathbb{Z}_2^k$.
In this case the group $\operatorname{Ker}\varphi$ *acts freely on* $\mathbb{L}^n$.

The quotient $N = \mathbb{L}^n/\operatorname{Ker}\varphi$ is a hyperbolic n-manifold. It is composed of $|\mathbb{Z}_2^k| = 2^k$ copies of P and has a Riemannian metric of constant negative curvature. Furthermore, the manifold N is aspherical (the Eilenberg–Mac Lane space $K(\operatorname{Ker}\varphi, 1)$), as its universal cover $\mathbb{L}^n$ is contractible.

Which combinatorial n-polytopes have right-angled realisations in $\mathbb{L}^n$?
In dim 3, there is a nice criterion going back to Pogorelov's work of 1967:

Theorem (Pogorelov, Andreev)

A combinatorial 3-polytope $P \neq \Delta^3$ can be realised as a right-angled polytope in $\mathbb{L}^3$ if and only if it is simple, and does not have 3- and 4-belts of facets. Furthermore, such a realisation is unique up to isometry.

We refer to the above class of 3-polytopes as the Pogorelov class $\mathcal{P}$. A polytope from the class $\mathcal{P}$ does not have triangular or quadrangular facets. The Pogorelov class contains all fullerenes (simple 3-polytopes with only pentagonal and hexagonal facets).

There is no classification of right-angled polytopes in $\mathbb{L}^4$. For $n \geqslant 5$, right-angled polytopes in $\mathbb{L}^n$ do not exist [Vinberg].

Given a right-angled polytope P, how to find an epimorphism $\varphi\colon G(P) \to \mathbb{Z}_2^k$ with $\operatorname{Ker}\varphi$ acting freely on $\mathbb{L}^n$?

One can consider the abelianisation: $G(P) \xrightarrow{\mathrm{ab}} \mathbb{Z}_2^m$, with $\operatorname{Ker}\mathrm{ab} = G'(P)$, the commutator subgroup.

The corresponding n-manifold $\mathbb{L}^n/G'(P)$ is the real moment-angle manifold $\mathcal{R}_P$, described as an intersection of quadrics in the beginning of this talk.

Corollary

If P is a right-angled polytope in $\mathbb{L}^n$, then the real moment-angle manifold $\mathcal{R}_P$ admits a hyperbolic structure as $\mathbb{L}^n/G'(P)$, where $G'(P)$ is the commutator subgroup of the corresponding right-angled Coxeter group. The manifold $\mathcal{R}_P$ is composed of 2^m copies of P.

A more econimical way to obtain a hyperbolic manifold is to consider $\varphi\colon G(P) \to \mathbb{Z}_2^n$. Such an epimorphism factors as $G(P) \xrightarrow{\mathrm{ab}} \mathbb{Z}_2^m \xrightarrow{\Lambda} \mathbb{Z}_2^n$, where Λ is a linear map.

The subgroup $\operatorname{Ker}\varphi$ acts freely on $\mathbb{L}^n$ if and only the Λ-images of any n facets of P that meet at a vertex form a basis of $\mathbb{Z}_2^n$.
Such Λ is called a $\mathbb{Z}_2$-characteristic function.

Proposition

Any simple 3-polytope admits a characteristic function.

Proof.

Given a 4-colouring of the facets of P, we assign to a facet of ith colour the ith basis vector $\boldsymbol{e}_i \in \mathbb{Z}^3$ for $i = 1, 2, 3$ and the vector $\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3$ for $i = 4$. The resulting map $\Lambda\colon \mathbb{Z}_2^m \to \mathbb{Z}_2^3$ satisfies the required condition, as any three of the four vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3$ form a basis of $\mathbb{Z}^3$. ☐

Manifolds $N(P,\Lambda) = \mathbb{L}^3 / \operatorname{Ker}\varphi$ obtained from right-angled 3-polytopes $P \in \mathcal{P}$ and characteristic functions $\Lambda\colon \mathbb{Z}_2^m \to \mathbb{Z}_2^3$ are called hyperbolic 3-manifolds of Löbell type. They were introduced and studied by A. Vesnin in 1987. Each $N(P,\Lambda)$ is composed of $|\mathbb{Z}_2^3| = 8$ copies of P.

In particular, one obtains a hyperbolic 3-manifold from any 4-colouring of a right-angled 3-polytope P. Löbell was first to consider a hyperbolic 3-manifold coming from a (unique) 4-colouring of the dodecahedron.

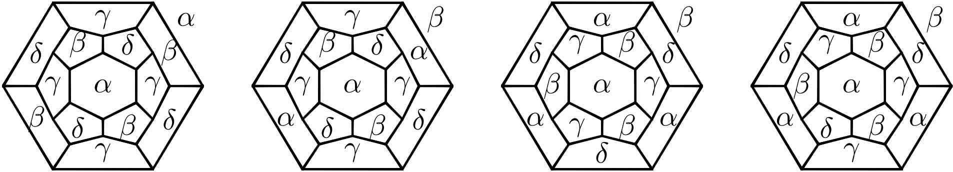


Figure: Four non-equivalent 4-colouring of the 'barell' fullerene with 14 facets.

Pairs (P, Λ) and (P', Λ') are equivalent if P and P' are combinatorially equivalent, and $\Lambda, \Lambda' \colon \mathbb{Z}_2^m \to \mathbb{Z}_2^n$ differ by an automorphism of $\mathbb{Z}_2^n$.

Theorem (Buchstaber–Erokhovets–Masuda–P–Park)

Let $N = N(P, \Lambda)$ and $N' = N(P', \Lambda')$ be two hyperbolic 3-manifolds of Löbell type corresponding to right-angled 3-polytopes P and P'. Then the following conditions are equivalent:

(a) *there is a cohomology ring isomorphism* $\varphi \colon H^*(N; \mathbb{Z}_2) \xrightarrow{\cong} H^*(N'; \mathbb{Z}_2)$;

(b) *there is a diffeomorphism* $N \cong N'$;

(c) *there is an equivalence of $\mathbb{Z}_2$-characteristic pairs* $(P, \Lambda) \sim (P', \Lambda')$.

In particular, hyperbolic 3-manifolds corresponding to non-equivalent 4-colourings of P are not diffeomorphic.

The difficult implication is (a)$\Rightarrow$(c). Its proof builds upon the wealth of cohomological techniques of toric topology.

References

[1] T. Panov, N. Ray and R. Vogt. *Colimits, Stanley-Reiner algebras, and loop spaces*, in: "Categorical Decomposition Techniques in Algebraic Topology" (G. Arone et al eds.), Progress in Mathematics, vol. 215, Birkhauser, Basel, 2004, pp. 261–291.

[2] J. Grbic, T. Panov, S. Theriault and J. Wu. *The homotopy types of moment-angle complexes for flag complexes.* Trans. of the Amer. Math. Soc. **368** (2016), no. 9, 6663–6682.

[3] T. Panov and Ya. Veryovkin. *Polyhedral products and commutator subgroups of right-angled Artin and Coxeter groups*. Sbornik Math. 207 (2016), no. 11; arXiv:1603.06902.

[4] V. Buchstaber, N. Erokhovets, M. Masuda, T. Panov and S. Park. *Cohomological rigidity of manifolds defined by right-angled 3-dimensional polytopes.* Russian Math. Surveys 72 (2017), to appear; arXiv:1610.07575.