Geometric structures on manifolds with torus action

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Moment-angle manifolds and complexes

A convex polyhedron in $\mathbb{R}^n$ obtained by intersecting m halfspaces:

$$P = \bigl\{\mathbf{x} \in \mathbb{R}^n \colon \langle \mathbf{a}_i, \mathbf{x}\rangle + b_i \geqslant 0 \quad \text{for } i = 1, \ldots, m\bigr\}.$$

Define an affine map

$$i_P \colon \mathbb{R}^n \to \mathbb{R}^m, \quad i_P(\mathbf{x}) = \bigl(\langle \mathbf{a}_1, \mathbf{x}\rangle + b_1, \ldots, \langle \mathbf{a}_m, \mathbf{x}\rangle + b_m\bigr).$$

If P has a vertex, then i_P is monomorphic, and $i_P(P)$ is the intersection of an n-plane with $\mathbb{R}^m_\geqslant = \{\mathbf{y} = (y_1, \ldots, y_m) \colon y_i \geqslant 0\}$.

Define the space $\mathcal{Z}_P$ from the diagram

$$\begin{array}{ccccc} \mathcal{Z}_P & \xrightarrow{i_Z} & \mathbb{C}^m & \qquad & (z_1, \ldots, z_m) \\ \downarrow & & \downarrow \mu & & \downarrow \\ P & \xrightarrow{i_P} & \mathbb{R}^m_\geqslant & & (|z_1|^2, \ldots, |z_m|^2) \end{array}$$

$\mathcal{Z}_P$ has a $\mathbb{T}^m$-action, $\mathcal{Z}_P/\mathbb{T}^m = P$, and i_Z is a $\mathbb{T}^m$-equivariant inclusion.

Prop 1. *If P is a simple polytope (more generally, if the presentation of P by inequalities is* generic*), then $\mathcal{Z}_P$ is a smooth manifold of dimension $m+n$.*

Proof. Write $i_P(\mathbb{R}^n)$ by $m-n$ linear equations in $(y_1,\ldots,y_m)\in\mathbb{R}^m$. Replace y_k by $|z_k|^2$ to obtain a presentation of $\mathcal{Z}_P$ by quadrics. $\square$

$\mathcal{Z}_P$: polytopal moment-angle manifold corresponding to P.

Similarly, by considering the projection $\mu\colon\mathbb{R}^m\to\mathbb{R}^m_\geqslant$ instead of $\mu\colon\mathbb{C}^m\to\mathbb{R}^m_\geqslant$ we obtain the real moment-angle manifold $\mathcal{R}_P\subset\mathbb{R}^m$.

Ex 1. $P=\{(x_1,x_2)\in\mathbb{R}^2\colon x_1\geqslant 0,\ x_2\geqslant 0,\ -\gamma_1x_1-\gamma_2x_2+1\geqslant 0\}$, $\gamma_1,\gamma_2>0$ (a 2-simplex). Then

$\mathcal{Z}_P=\{(z_1,z_2,z_3)\in\mathbb{C}^3\colon \gamma_1|z_1|^2+\gamma_2|z_2|^2+|z_3|^2)=1\}$ (a 5-sphere),

$\mathcal{R}_P=\{(u_1,u_2,u_3)\in\mathbb{R}^3\colon \gamma_1|u_1|^2+\gamma_2|u_2|^2+|u_3|^2)=1\}$ (a 2-sphere).

$\mathcal{K}$ an (abstract) simplicial complex on the set $[m] = \{1, \dots, m\}$.

$I = \{i_1, \dots, i_k\} \in \mathcal{K}$ a simplex. Always assume $\varnothing \in \mathcal{K}$.

Consider the unit polydisc in $\mathbb{C}^m$,

$$\mathbb{D}^m = \Big\{(z_1, \dots, z_m) \in \mathbb{C}^m \colon |z_i| \leqslant 1, \quad i = 1, \dots, m\Big\}.$$

Given $I \subset [m]$, set

$$B_I := \Big\{(z_1, \dots, z_m) \in \mathbb{D}^m \colon |z_j| = 1 \text{ for } j \notin I\Big\} \cong \prod_{i \in I} D^2 \times \prod_{i \notin I} S^1.$$

The moment-angle complex

$$\mathcal{Z}_{\mathcal{K}} := \bigcup_{I \in \mathcal{K}} B_I = \bigcup_{I \in \mathcal{K}} \Big(\prod_{i \in I} D^2 \times \prod_{i \notin I} S^1\Big) \subset \mathbb{D}^m$$

It is invariant under the coordinatewise action of the torus $\mathbb{T}^m$.

Ex 2. $\mathcal{K} = 2$ points, then $\mathcal{Z}_{\mathcal{K}} = D^2 \times S^1 \cup S^1 \times D^2 \cong S^3$.

$\mathcal{K} = \Delta$, then $\mathcal{Z}_{\mathcal{K}} = (D^2 \times D^2 \times S^1) \cup (D^2 \times S^1 \times D^2) \cup (S^1 \times D^2 \times D^2) \cong S^5$.

More generally, let X a space, and $A \subset X$. Given $I \subset [m]$, set

$$(X,A)^I = \Big\{(x_1,\ldots,x_m) \in \prod_{i=1}^m X \colon x_j \in A \text{ for } j \notin I\Big\} \cong \prod_{i\in I} X \times \prod_{i\notin I} A.$$

The $\mathcal{K}$-polyhedral product of (X,A) is

$$\mathcal{Z}_{\mathcal{K}}(X,A) = \bigcup_{I\in\mathcal{K}} (X,A)^I \subset X^m.$$

Another important example is the complement of the coordinate subspace arrangement corresponding to $\mathcal{K}$:

$$U(\mathcal{K}) = \mathbb{C}^m \setminus \bigcup_{\{i_1,\ldots,i_k\}\notin\mathcal{K}} \{\mathbf{z} \in \mathbb{C}^m \colon z_{i_1} = \ldots = z_{i_k} = 0\},$$

namely,

$$U(\mathcal{K}) = \mathcal{Z}_{\mathcal{K}}(\mathbb{C},\mathbb{C}^\times),$$

where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.

Thm 1. *$\mathcal{Z}_{\mathcal{K}} \subset U(\mathcal{K})$ is a $\mathbb{T}^m$-deformation retract of $U(\mathcal{K})$.*

Thm 2. *If P is a simple polytope, $\mathcal{K}_P = \partial(P^*)$ (the dual triangulation), then $\mathcal{Z}_{\mathcal{K}_P} \cong \mathcal{Z}_P$ ($\mathbb{T}^m$-equivariantly homeomorphic).*

In particular, $\mathcal{Z}_{\mathcal{K}_P}$ is a manifold. More generally,

Prop 2. *Assume $|\mathcal{K}| \cong S^{n-1}$ (a sphere triangulation with m vertices). Then $\mathcal{Z}_{\mathcal{K}}$ is a closed manifold of dimension $m+n$.*

Geometric structures I. Non-Kähler complex structures

Recall: if $\mathcal{K} = \mathcal{K}_P$ is the dual triangulation of a simple convex polytope P, then $\mathcal{Z}_P = \mathcal{Z}_{\mathcal{K}_P}$ has a canonical smooth structure (e.g. as a nondegenerate intersection of Hermitian quadrics in $\mathbb{C}^m$).

Let $\mathcal{K}$ be a sphere triangulation, i.e. $|\mathcal{K}| \cong S^{n-1}$.
A realisation $|\mathcal{K}| \subset \mathbb{R}^n$ is starshaped if there is a point $\mathbf{x} \notin |\mathcal{K}|$ such that any ray from $\mathbf{x}$ intersects $|\mathcal{K}|$ in exactly one point.

A convex triangulation $\mathcal{K}_P$ is starshaped, but not vice versa!

$\mathcal{K}$ has a starshaped realisation if and only if it is the underlying complexes of a complete simplicial fan Σ.

Also recall $U(\mathcal{K}) = \mathbb{C}^m \setminus \bigcup_{\{i_1,\ldots,i_k\}\notin\mathcal{K}} \{\mathbf{z} \in \mathbb{C}^m \colon z_{i_1} = \ldots = z_{i_k} = 0\}$.

$\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}^n$ the generators of the 1-dim cones of Σ. Define a map

$$A\colon \mathbb{R}^m \to \mathbb{R}^n, \quad \mathbf{e}_i \mapsto \mathbf{a}_i,$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_m$ is the standard basis of $\mathbb{R}^m$. Set

$$\mathbb{R}^m_> = \{(y_1, \ldots, y_m) \in \mathbb{R}^m \colon y_i > 0\},$$

and define

$$R := \exp(\operatorname{Ker} A) = \Bigl\{(y_1, \ldots, y_m) \in \mathbb{R}^m_> \colon \prod_{i=1}^m y_i^{\langle \mathbf{a}_i, \mathbf{u}\rangle} = 1 \text{ for all } \mathbf{u} \in \mathbb{R}^n\Bigr\},$$

$R \subset \mathbb{R}^m_>$ acts on $U(\mathcal{K}) \subset \mathbb{C}^m$ by coordinatewise multiplications.

Thm 3. *Let Σ be a complete simplicial fan in $\mathbb{R}^n$ with m one-dimensional cones, and let $\mathcal{K} = \mathcal{K}_\Sigma$ be its underlying simplicial complex. Then*

(a) the group $R \cong \mathbb{R}^{m-n}$ acts on $U(\mathcal{K})$ freely and properly, so the quotient $U(\mathcal{K})/R$ is a smooth $(m+n)$-dimensional manifold;

(b) $U(\mathcal{K})/R$ is $\mathbb{T}^m$-equivariantly homeomorphic to $\mathcal{Z}_\mathcal{K}$.

Therefore, $\mathcal{Z}_\mathcal{K}$ can be smoothed canonically.

Assume $m - n$ is even and set $\ell = \frac{m-n}{2}$.

Choose a linear map $\Psi \colon \mathbb{C}^\ell \to \mathbb{C}^m$ satisfying the two conditions:

(a) $\mathrm{Re} \circ \Psi \colon \mathbb{C}^\ell \to \mathbb{R}^m$ is a monomorphism.

(b) $A \circ \mathrm{Re} \circ \Psi = 0$.

The composite map of the top line in the following diagram is zero:

$$\begin{array}{ccccccc}
\mathbb{C}^\ell & \xrightarrow{\Psi} & \mathbb{C}^m & \xrightarrow{\mathrm{Re}} & \mathbb{R}^m & \xrightarrow{A} & \mathbb{R}^n \\
 & & \big\downarrow \exp & & \big\downarrow \exp & & \big\downarrow \exp \\
 & & (\mathbb{C}^\times)^m & \xrightarrow{|\cdot|} & \mathbb{R}^m_> & \xrightarrow{\exp A} & \mathbb{R}^n_>
\end{array}$$

where $|\cdot|$ denotes the map $(z_1, \dots, z_m) \mapsto (|z_1|, \dots, |z_m|)$.

Now set

$$C = \exp \Psi(\mathbb{C}^\ell) = \left\{ \left(e^{\langle \psi_1, \mathbf{w} \rangle}, \dots, e^{\langle \psi_m, \mathbf{w} \rangle} \right) \in (\mathbb{C}^\times)^m \right\}$$

Then $C \cong \mathbb{C}^\ell$ is a complex-analytic (but not algebraic) subgroup in $(\mathbb{C}^\times)^m$. It acts on $U(\mathcal{K})$ by holomorphic transformations.

Ex 3. Let $\mathcal{K}$ be empty on 2 elements (that is, $\mathcal{K}$ has two ghost vertices). We therefore have $n = 0$, $m = 2$, $\ell = 1$, and $A\colon \mathbb{R}^2 \to 0$ is a zero map. Let $\Psi\colon \mathbb{C} \to \mathbb{C}^2$ be given by $z \mapsto (z, \alpha z)$ for some $\alpha \in \mathbb{C}$, so that

$$C = \left\{(e^z, e^{\alpha z})\right\} \subset (\mathbb{C}^\times)^2.$$

Condition (b) above is void, while (a) is equivalent to that $\alpha \notin \mathbb{R}$. Then $\exp\Psi\colon \mathbb{C} \to (\mathbb{C}^\times)^2$ is an embedding, and the quotient $(\mathbb{C}^\times)^2/C$ with the natural complex structure is a complex torus $T^2_{\mathbb{C}}$ with parameter $\alpha \in \mathbb{C}$:

$$(\mathbb{C}^\times)^2/C \cong \mathbb{C}/(\mathbb{Z} \oplus \alpha\mathbb{Z}) = T^2_{\mathbb{C}}(\alpha).$$

Similarly, if $\mathcal{K}$ is empty on 2ℓ elements (so that $n = 0$, $m = 2\ell$), we may obtain any complex torus $T^{2\ell}_{\mathbb{C}}$ as the quotient $(\mathbb{C}^\times)^{2\ell}/C$.

Thm 4. *Let* Σ *be a complete simplicial fan in* $\mathbb{R}^n$ *with* m *one-dimensional cones, and let* $\mathcal{K} = \mathcal{K}_\Sigma$ *be its underlying simplicial complex. Assume that* $m - n = 2\ell$. *Then*

(a) *the holomorphic action of the group* $C \cong \mathbb{C}^\ell$ *on* $U(\mathcal{K})$ *is free and proper, so the quotient* $U(\mathcal{K})/C$ *is a compact complex* $(m-\ell)$*-manifold;*

(b) *there is a* $\mathbb{T}^m$*-equivariant diffeomorphism* $U(\mathcal{K})/C \cong \mathcal{Z}_\mathcal{K}$ *defining a complex structure on* $\mathcal{Z}_\mathcal{K}$ *in which* $\mathbb{T}^m$ *acts holomorphically.*

Ex 4 (Hopf manifold)**.** Let Σ be the complete fan in $\mathbb{R}^n$ whose cones are generated by all proper subsets of $n+1$ vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n, -\mathbf{e}_1 - \ldots - \mathbf{e}_n$.

To make $m-n$ even we add one 'empty' 1-cone. We have $m = n+2$, $\ell = 1$. Then $A\colon \mathbb{R}^{n+2} \to \mathbb{R}^n$ is given by the matrix $(\mathbf{0}\ I\ -\mathbf{1})$, where I is the unit $n \times n$ matrix, and $\mathbf{0}$, $\mathbf{1}$ are the n-columns of zeros and units respectively.

We have that $\mathcal{K}$ is the boundary of an n-dim simplex with $n+1$ vertices and 1 ghost vertex, $\mathcal{Z}_{\mathcal{K}} \cong S^1 \times S^{2n+1}$, and $U(\mathcal{K}) = \mathbb{C}^\times \times (\mathbb{C}^{n+1} \setminus \{0\})$.

Take $\Psi\colon \mathbb{C} \to \mathbb{C}^{n+2}$, $z \mapsto (z, \alpha z, \ldots, \alpha z)$ for some $\alpha \in \mathbb{C}$, $\alpha \notin \mathbb{R}$. Then

$$C = \Big\{(e^z, e^{\alpha z}, \ldots, e^{\alpha z})\colon z \in \mathbb{C}\Big\} \subset (\mathbb{C}^\times)^{n+2},$$

and $\mathcal{Z}_{\mathcal{K}}$ acquires a complex structure as the quotient $U(\mathcal{K})/C$:

$$\mathbb{C}^\times \times \Big(\mathbb{C}^{n+1} \setminus \{0\}\Big) \Big/ \{(t, \mathbf{w}) \sim (e^z t, e^{\alpha z}\mathbf{w})\} \cong \Big(\mathbb{C}^{n+1} \setminus \{0\}\Big) \Big/ \{\mathbf{w} \sim e^{2\pi i \alpha}\mathbf{w}\},$$

where $t \in \mathbb{C}^\times$, $\mathbf{w} \in \mathbb{C}^{n+1} \setminus \{0\}$. The latter quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ is known as the Hopf manifold.

Holomorphic bundles over toric varieties.

Manifolds $\mathcal{Z}_\mathcal{K}$ corresponding to complete *regular* (in particular, *rational*) simplicial fans are total spaces of holomorphic principal bundles over toric varieties with fibre a complex torus. This allows us to calculate invariants of the complex structures on $\mathcal{Z}_\mathcal{K}$, such as Hodge numbers and Dolbeault cohomology.

A toric variety is a normal algebraic variety X on which an algebraic torus $(\mathbb{C}^\times)^n$ acts with a dense (Zariski open) orbit.

Toric varieties are classified by *rational* fans. Under this correspondence,

$$\begin{aligned} \text{complete fans} &\longleftrightarrow \text{compact varieties}\\ \text{normal fans of polytopes} &\longleftrightarrow \text{projective varieties}\\ \text{regular fans} &\longleftrightarrow \text{nonsingular varieties}\\ \text{simplicial fans} &\longleftrightarrow \text{orbifolds}\end{aligned}$$

Σ complete, simplicial, *rational*;
$\mathbf{a}_1, \ldots, \mathbf{a}_m$ *primitive integral* generators of 1-cones;
$\mathbf{a}_i = (a_{i1}, \ldots, a_{in}) \in \mathbb{Z}^n$.

Constr 1 ('Cox construction'). Let $A_{\mathbb{C}} \colon \mathbb{C}^m \to \mathbb{C}^n$, $\mathbf{e}_i \mapsto \mathbf{a}_i$,

$$\begin{aligned} \exp A_{\mathbb{C}} \colon (\mathbb{C}^\times)^m &\to (\mathbb{C}^\times)^n, \\ (z_1, \ldots, z_m) &\mapsto \Bigl(\prod_{i=1}^m z_i^{a_{i1}}, \ldots, \prod_{i=1}^m z_i^{a_{in}} \Bigr) \end{aligned}$$

Set $G = \operatorname{Ker} \exp A_{\mathbb{C}}$.
This is an $(m-n)$-dimensional algebraic subgroup in $(\mathbb{C}^\times)^m$.
It acts almost freely (with finite isotropy subgroups) on $U(\mathcal{K}_\Sigma)$.
If Σ is regular, then $G \cong (\mathbb{C}^\times)^{m-n}$ and the action is free.

$V_\Sigma = U(\mathcal{K}_\Sigma)/G$ the toric variety associated to Σ.
The quotient torus $(\mathbb{C}^\times)^m/G \cong (\mathbb{C}^\times)^n$ acts on V_Σ with a dense orbit.

Observe that $\mathbb{C}^\ell \cong C \subset G \cong (\mathbb{C}^\times)^{m-n}$ as a complex subgroup.

Prop 3.

(a) The toric variety V_Σ is homeomorphic to the quotient of $\mathcal{Z}_{\mathcal{K}_\Sigma}$ by the holomorphic action of G/C.

(b) If Σ is regular, then there is a holomorphic principal bundle $\mathcal{Z}_{\mathcal{K}_\Sigma} \to V_\Sigma$ with fibre the compact complex torus G/C of dimension ℓ.

Rem 1. For singular varieties V_Σ the quotient projection $\mathcal{Z}_{\mathcal{K}_\Sigma} \to V_\Sigma$ is a holomorphic principal Seifert bundle for an appropriate orbifold structure on V_Σ.

Submanifolds and analytic subsets.

The complex structure on $\mathcal{Z}_\mathcal{K}$ is determined by two pieces of data:
– the complete simplicial fan Σ with generators $\mathbf{a}_1, \ldots, \mathbf{a}_m$;
– the ℓ-dimensional holomorphic subgroup $C \subset (\mathbb{C}^\times)^m$.

If this data is *generic* (in particular, the fan Σ is not rational), then there is no holomorphic principal torus fibration $\mathcal{Z}_\mathcal{K} \to V_\Sigma$ over a toric variety V_Σ.

However, there still exists a holomorphic ℓ-dimensional *foliation* $\mathcal{F}$ with a transverse Kähler form $\omega_\mathcal{F}$. This form can be used to describe submanifolds and analytic subsets in $\mathcal{Z}_\mathcal{K}$.

Consider the complexified map $A_{\mathbb{C}}\colon \mathbb{C}^m \to \mathbb{C}^n$, $\mathbf{e}_i \mapsto \mathbf{a}_i$, and the following complex $(m-n)$-dimensional subgroup in $(\mathbb{C}^\times)^m$:

$$G = \exp(\operatorname{Ker} A_{\mathbb{C}}) = \Big\{\Big(e^{z_1}, \ldots, e^{z_m}\Big) \in (\mathbb{C}^\times)^m\colon (z_1, \ldots, z_m) \in \operatorname{Ker} A_{\mathbb{C}}\Big\}.$$

Note $C \subset G$.

The group G acts on $U(\mathcal{K})$, and its orbits define a holomorphic foliation on $U(\mathcal{K})$. Since $G \subset (\mathbb{C}^\times)^m$, this action is free on open subset $(\mathbb{C}^\times)^m \subset U(\mathcal{K})$, so that the generic leaf of the foliation has complex dimension $m - n = 2\ell$.

The ℓ-dimensional closed subgroup $C \subset G$ acts on $U(\mathcal{K})$ freely and properly by Theorem 4, so that $U(\mathcal{K})/C$ carries a holomorphic action of the quotient group $D = G/C$.

$\mathcal{F}$: the holomorphic foliation on $U(\mathcal{K})/C \cong \mathcal{Z}_{\mathcal{K}}$ by the orbits of D.

The subgroup $G \subset (\mathbb{C}^\times)^m$ is closed if and only if it is isomorphic to $(\mathbb{C}^\times)^{2\ell}$; in this case the subspace $\operatorname{Ker} A \subset \mathbb{R}^m$ is rational. Then Σ is a rational fan and V_Σ is the quotient $U(\mathcal{K})/G$. The foliation $\mathcal{F}$ gives rise to a holomorphic principal Seifert fibration $\pi\colon \mathcal{Z}_\mathcal{K} \to V_\Sigma$ with fibres compact complex tori G/C.

For a generic configuration of nonzero vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$, G is biholomorphic to $\mathbb{C}^{2\ell}$ and $D = G/C$ is biholomorphic to $\mathbb{C}^\ell$.

A $(1,1)$-form $\omega_{\mathcal{F}}$ on the complex manifold $\mathcal{Z}_{\mathcal{K}}$ is called transverse Kähler with respect to the foliation $\mathcal{F}$ if
(a) $\omega_{\mathcal{F}}$ is closed, i.e. $d\omega_{\mathcal{F}} = 0$;
(b) $\omega_{\mathcal{F}}$ is nonnegative and the zero space of $\omega_{\mathcal{F}}$ is the tangent space of $\mathcal{F}$.

A complete simplicial fan Σ in $\mathbb{R}^n$ is called weakly normal if there exists a (not necessarily simple) n-dimensional polytope P such that Σ is a simplicial subdivision of the normal fan Σ_P.

Thm 5. *Assume that Σ is a weakly normal fan. Then there exists an exact $(1,1)$-form $\omega_{\mathcal{F}}$ on $\mathcal{Z}_{\mathcal{K}} = U(\mathcal{K})/C$ which is transverse Kähler for the foliation $\mathcal{F}$ on the dense open subset $(\mathbb{C}^\times)^m/C \subset U(\mathcal{K})/C$.*

For each $J \subset [m]$, define the corresponding coordinate submanifold in $\mathcal{Z}_\mathcal{K}$ by

$$\mathcal{Z}_{\mathcal{K}_J} = \{(z_1, \ldots, z_m) \in \mathcal{Z}_\mathcal{K} \colon z_i = 0 \quad \text{for } i \notin J\}.$$

Obviously, $\mathcal{Z}_{\mathcal{K}_J}$ is identified with the quotient of

$$U(\mathcal{K}_J) = \{(z_1, \ldots, z_m) \in U(\mathcal{K}) \colon z_i = 0 \quad \text{for } i \notin J\}$$

by $C \cong \mathbb{C}^\ell$. In particular, $U(\mathcal{K}_J)/C$ is a complex submanifold in $\mathcal{Z}_\mathcal{K} = U(\mathcal{K})/C$.

Observe that the closure of any $(\mathbb{C}^\times)^m$-orbit of $U(\mathcal{K})$ has the form $U(\mathcal{K}_J)$ for some $J \subset [m]$ (in particular, the dense orbit corresponds to $J = [m]$). Similarly, the closure of any $(\mathbb{C}^\times)^m/C$-orbit of $\mathcal{Z}_\mathcal{K} \cong U(\mathcal{K})/C$ has the form $\mathcal{Z}_{\mathcal{K}_J}$.

Thm 6. *Assume that the data defining a complex structure on* $\mathcal{Z}_{\mathcal{K}} = U(\mathcal{K})/C$ *is generic. Then any divisor of* $\mathcal{Z}_{\mathcal{K}}$ *is a union of coordinate divisors.*

Furthermore, if Σ *is a weakly normal fan, then any compact irreducible analytic subset* $Y \subset \mathcal{Z}_{\mathcal{K}}$ *of positive dimension is a coordinate submanifold.*

Cor 1. *Under generic assumptions, there are no non-constant meromorphic functions on* $\mathcal{Z}_{\mathcal{K}}$.

Geometric structures II. Lagrangian submanifolds

M a Kähler manifold with symplectic form ω, $\dim_{\mathbb{R}} M = 2n$.

An immersion $i\colon N \looparrowright M$ of an n-manifold N is Lagrangian if $i^*(\omega) = 0$. If i is an embedding, then $i(N)$ is a Lagrangian submanifold of M.

A vector field ξ on M is Hamiltonian if the 1-form $\omega(\,\cdot\,,\xi)$ is exact.

A Lagrangian immersion $i\colon N \looparrowright M$ is Hamiltonian minimal (H-minimal) if the variations of the volume of $i(N)$ along all Hamiltonian vector fields with compact support are zero, i.e.

$$\frac{d}{dt}\mathrm{vol}(i_t(N))\Big|_{t=0} = 0,$$

where $i_0(N) = i(N)$, $i_t(N)$ is a Hamiltonian deformation of $i(N)$, and $\mathrm{vol}(i_t(N))$ is the volume of the deformed part of $i_t(N)$.

Recall: P a simple polytope

$$P = \Big\{\mathbf{x} \in \mathbb{R}^n \colon \langle \mathbf{a}_i, \mathbf{x}\rangle + b_i \geqslant 0 \quad \text{for } i = 1, \ldots, m\Big\}.$$

The polytopal moment-angle manifold $\mathcal{Z}_P$,

$$\begin{array}{ccccc}
\mathcal{Z}_P & \xrightarrow{i_Z} & \mathbb{C}^m & & (z_1, \ldots, z_m) \\
\big\downarrow & & \big\downarrow \mu & & \big\downarrow \\
P & \xrightarrow{i_P} & \mathbb{R}^m_{\geqslant} & & (|z_1|^2, \ldots, |z_m|^2)
\end{array}$$

can be written as the intersection of $m - n$ real quadrics,

$$\mathcal{Z}_P = \Big\{\mathbf{z} = (z_1, \ldots, z_m) \in \mathbb{C}^m \colon \sum_{k=1}^{m} \gamma_{jk}|z_k|^2 = c_j, \quad \text{for } 1 \leqslant j \leqslant m - n\Big\}.$$

Also have the real moment-angle manifold,

$$\mathcal{R}_P = \Big\{\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{R}^m \colon \sum_{k=1}^{m} \gamma_{jk} u_k^2 = c_j, \quad \text{for } 1 \leqslant j \leqslant m-n\Big\}.$$

Set $\gamma_k = (\gamma_{1k}, \ldots, \gamma_{m-n,k}) \in \mathbb{R}^{m-n}$ for $1 \leqslant k \leqslant m$.

Assume that the polytope P is rational. Then have two lattices:

$$\Lambda = \mathbb{Z}\langle \mathbf{a}_1, \ldots, \mathbf{a}_m \rangle \subset \mathbb{R}^n \quad \text{and} \quad L = \mathbb{Z}\langle \gamma_1, \ldots, \gamma_m \rangle \subset \mathbb{R}^{m-n}.$$

Consider the $(m-n)$-torus

$$T_P = \Big\{\Big(e^{2\pi i \langle \gamma_1, \varphi \rangle}, \ldots, e^{2\pi i \langle \gamma_m, \varphi \rangle}\Big) \in \mathbb{T}^m\Big\},$$

i.e. $T_P = \mathbb{R}^{m-n}/L^*$, and set

$$D_P = \frac{1}{2} L^*/L^* \cong (\mathbb{Z}/2)^{m-n}.$$

Prop 4. *The $(m-n)$-torus T_P acts on $\mathcal{Z}_P$ almost freely.*

Consider the map

$$f\colon \mathcal{R}_P \times T_P \longrightarrow \mathbb{C}^m,$$
$$(\mathbf{u}, \varphi) \mapsto \mathbf{u}\cdot\varphi = (u_1 e^{2\pi i\langle\gamma_1,\varphi\rangle}, \ldots, u_m e^{2\pi i\langle\gamma_m,\varphi\rangle}).$$

Note $f(\mathcal{R}_P \times T_P) \subset \mathcal{Z}_P$ is the set of T_P-orbits through $\mathcal{R}_P \subset \mathbb{C}^m$.

Have an m-dimensional manifold

$$N_P = \mathcal{R}_P \times_{D_P} T_P.$$

Lemma 1. $f\colon \mathcal{R}_P \times T_P \to \mathbb{C}^m$ *induces an immersion* $j\colon N_P \looparrowright \mathbb{C}^m$.

Thm 7 (Mironov). *The immersion* $i_\Gamma\colon N_\Gamma \looparrowright \mathbb{C}^m$ *is H-minimal Lagrangian.*

When it is an embedding?

A simple rational polytope P is Delzant if for any vertex $v \in P$ the set of vectors $\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_n}$ normal to the facets meeting at v forms a basis of the lattice $\Lambda = \mathbb{Z}\langle \mathbf{a}_1, \dots, \mathbf{a}_m \rangle$:

$$\mathbb{Z}\langle \mathbf{a}_1, \dots, \mathbf{a}_m \rangle = \mathbb{Z}\langle \mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_n} \rangle \qquad \text{for any } v = F_{i_1} \cap \dots \cap F_{i_n}.$$

Thm 8. *The following conditions are equivalent:*

1) $j\colon N_P \to \mathbb{C}^m$ *is an* embedding *of an H-minimal Lagrangian submanifold;*

2) the $(m-n)$*-torus* T_P *acts on* $\mathcal{Z}_P$ freely*.*

3) P *is a* Delzant *polytope.*

Explicit constructions of families of Delzant polytopes are known in toric geometry and topology:

- simplices and cubes in all dimensions;
- products and face cuts;
- associahedra (Stasheff ptopes), permutahedra, and generalisations.

Ex 5 (one quadric). Let $P = \Delta^{m-1}$ (a simplex), i.e. $m - n = 1$ and $\mathcal{R}_{\Delta^{m-1}}$ is given by a single quadric

$$\gamma_1 u_1^2 + \cdots + \gamma_m u_m^2 = c \qquad (1)$$

with $\gamma_i > 0$, i.e. $\mathcal{R}_{\Delta^{m-1}} \cong S^{m-1}$. Then

$$N \cong S^{m-1} \times_{\mathbb{Z}/2} S^1 \cong \begin{cases} S^{m-1} \times S^1 & \text{if } \tau \text{ preserves the orient. of } S^{m-1}, \\ \mathcal{K}^m & \text{if } \tau \text{ reverses the orient. of } S^{m-1}, \end{cases}$$

where τ is the involution and $\mathcal{K}^m$ is an m-dimensional Klein bottle.

Prop 5. *We obtain an H-minimal Lagrangian embedding of* $N_{\Delta^{m-1}} \cong S^{n-1} \times_{\mathbb{Z}/2} S^1$ *in* $\mathbb{C}^m$ *if and only if* $\gamma_1 = \cdots = \gamma_m$ *in (1). The topological type of* $N_{\Delta^{m-1}} = N(m)$ *depends only on the parity of* m*:*

$$N(m) \cong S^{m-1} \times S^1 \qquad \textit{if } m \textit{ is even},$$
$$N(m) \cong \mathcal{K}^m \qquad \textit{if } m \textit{ is odd}.$$

The Klein bottle $\mathcal{K}^m$ with even m does *not* admit Lagrangian embeddings in $\mathbb{C}^m$ [Nemirovsky, Shevchishin].

Ex 6 (two quadrics).

Thm 9. *Let $m-n=2$, i.e. $P \simeq \Delta^{p-1} \times \Delta^{q-1}$.*

(a) $\mathcal{R}_P$ is diffeomorphic to $\mathcal{R}(p,q) \cong S^{p-1} \times S^{q-1}$ given by

$$\begin{array}{ll} u_1^2 + \ldots + u_k^2 + u_{k+1}^2 + \cdots + u_p^2 & = 1, \\ u_1^2 + \ldots + u_k^2 \qquad\qquad\qquad\qquad + u_{p+1}^2 + \cdots + u_m^2 & = 2, \end{array}$$

where $p+q=m$, $0<p<m$ and $0 \leqslant k \leqslant p$.

(b) If $N_P \to \mathbb{C}^m$ is an embedding, then N_P is diffeomorphic to

$$N_k(p,q) = \mathcal{R}(p,q) \times_{\mathbb{Z}/2 \times \mathbb{Z}/2} (S^1 \times S^1),$$

where the two involutions act on $\mathcal{R}(p,q)$ by

$$\begin{aligned} \psi_1 &\colon (u_1,\ldots,u_m) \mapsto (-u_1,\ldots,-u_k,-u_{k+1},\ldots,-u_p,u_{p+1},\ldots,u_m), \\ \psi_2 &\colon (u_1,\ldots,u_m) \mapsto (-u_1,\ldots,-u_k,u_{k+1},\ldots,u_p,-u_{p+1},\ldots,-u_m). \end{aligned} \tag{2}$$

There is a fibration $N_k(p,q) \to S^{q-1} \times_{\mathbb{Z}/2} S^1 = N(q)$ with fibre $N(p)$ (the manifold from the previous example), which is trivial for $k=0$.

Ex 7 (three quadrics)**.**

In the case $m-n=3$ the topology of compact manifolds $\mathcal{R}_P$ and $\mathcal{Z}_P$ was fully described by [Lopez de Medrano]. Each manifold is diffeomorphic to a product of three spheres, or to a connected sum of products of spheres, with two spheres in each product.

The simplest P with $m-n=3$ is a (Delzant) pentagon, e.g.

$$P=\Big\{(x_1,x_2)\in\mathbb{R}^2\colon x_1\geqslant 0,\ x_2\geqslant 0,\ -x_1+2\geqslant 0,\ -x_2+2\geqslant 0,\ -x_1-x_2+3\geqslant 0\Big\}.$$

In this case $\mathcal{R}_P$ is an oriented surface of genus 5, and $\mathcal{Z}_P$ is diffeomorphic to a connected sum of 5 copies of $S^3\times S^4$.

Get an H-minimal Lagrangian submanifold $N_P\subset\mathbb{C}^5$ which is the total space of a bundle over T^3 with fibre a surface of genus 5.

Prop 6. *Let P be an m-gon. Then $\mathcal{R}_P$ is an orientable surface S_g of genus $g = 1 + 2^{m-3}(m-4)$.*

Get an H-minimal Lagrangian submanifold $N_P \subset \mathbb{C}^m$ which is the total space of a bundle over T^{m-2} with fibre S_g. It is an aspherical manifold (for $m \geqslant 4$) whose fundamental group enters into the short exact sequence

$$1 \longrightarrow \pi_1(S_g) \longrightarrow \pi_1(N) \longrightarrow \mathbb{Z}^{m-2} \longrightarrow 1.$$

For $n > 2$ and $m - n > 3$ the topology of $\mathcal{R}_P$ and $\mathcal{Z}_P$ is even more complicated.

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