Toric Topology and Geometry

based on joint works with Victor Buchstaber, Nikolay Erokhovets, Mikiya Masuda, Andrey Mironov, Yuri Ustinovsky, Misha Verbitsky

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Polytopes and moment-angle manifolds

A convex polytope in $\mathbb{R}^n$ is obtained by intersecting m halfspaces:

$$P = \bigl\{\boldsymbol{x} \in \mathbb{R}^n \colon \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle + b_i \geqslant 0 \quad \text{for } i = 1, \ldots, m\bigr\},$$

where $\boldsymbol{a}_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$.

Suppose each $F_i = P \cap \{\boldsymbol{x} \colon \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle + b_i = 0\}$ is a facet (m facets in total).

Define an affine map

$$i_P \colon \mathbb{R}^n \to \mathbb{R}^m, \quad i_P(\boldsymbol{x}) = \bigl(\langle \boldsymbol{a}_1, \boldsymbol{x}\rangle + b_1, \ldots, \langle \boldsymbol{a}_m, \boldsymbol{x}\rangle + b_m\bigr).$$

Then i_P is monomorphic, and $i_P(P) \subset \mathbb{R}^m$ is the intersection of an n-plane with $\mathbb{R}^m_{\geqslant} = \{\boldsymbol{y} = (y_1, \ldots, y_m) \colon y_i \geqslant 0\}$.

Define the space $\mathcal{Z}_P$ from the diagram

$$\begin{array}{ccccc}
\mathcal{Z}_P & \xrightarrow{i_Z} & \mathbb{C}^m & & (z_1,\ldots,z_m) \\
\downarrow & & \downarrow \mu & & \downarrow \\
P & \xrightarrow{i_P} & \mathbb{R}^m_{\geqslant} & & (|z_1|^2,\ldots,|z_m|^2)
\end{array}$$

$\mathcal{Z}_P$ has a $\mathbb{T}^m$-action, $\mathcal{Z}_P/\mathbb{T}^m = P$, and i_Z is a $\mathbb{T}^m$-equivariant inclusion.

A polytope P is simple if exactly $n = \dim P$ facets meet at each vertex.

Proposition

If P is simple, then $\mathcal{Z}_P$ is a smooth manifold of dimension $m + n$.

Proof.

Write $i_P(\mathbb{R}^n)$ by $m - n$ linear equations in $(y_1,\ldots,y_m) \in \mathbb{R}^m$. Replace y_k by $|z_k|^2$ to obtain a presentation of $\mathcal{Z}_P$ by quadrics. □

$\mathcal{Z}_P$ is the moment-angle manifold corresponding to P.

Similarly, by considering

$$\begin{array}{ccccc} \mathcal{R}_P & \longrightarrow & \mathbb{R}^m & & (u_1,\ldots,u_m) \\ \downarrow & & \downarrow\mu & & \downarrow \\ P & \xrightarrow{i_P} & \mathbb{R}^m_{\geqslant} & & (u_1^2,\ldots,u_m^2) \end{array}$$

we obtain the real moment-angle manifold $\mathcal{R}_P$.

Example

$P = \{(x_1,x_2)\in\mathbb{R}^2 \colon x_1 \geqslant 0,\ x_2\geqslant 0,\ -\gamma_1x_1-\gamma_2x_2+1\geqslant 0\}$, $\gamma_1,\gamma_2>0$ (a 2-simplex). Then
$\mathcal{Z}_P = \{(z_1,z_2,z_3)\in\mathbb{C}^3 \colon \gamma_1|z_1|^2+\gamma_2|z_2|^2+|z_3|^2=1\}$ (a 5-sphere),
$\mathcal{R}_P = \{(u_1,u_2,u_3)\in\mathbb{R}^3 \colon \gamma_1|u_1|^2+\gamma_2|u_2|^2+|u_3|^2=1\}$ (a 2-sphere).

Right-angled polytopes and hyperbolic manifolds

Let P be a polytope in n-dimensional Lobachevsky space $\mathbb{L}^n$ with right angles between adjacent facets (a right-angled n-polytope).

Denote by $G(P)$ the group generated by reflections in the facets of P. It is a right-angled Coxeter group given by the presentation

$$G(P) = \langle g_1, \ldots, g_m \mid g_i^2 = 1,\ g_i g_j = g_j g_i \text{ if } F_i \cap F_j \neq \varnothing \rangle,$$

where g_i denotes the reflection in the facet F_i.

The group $G(P)$ acts on $\mathbb{L}^n$ discretely with finite isotropy subgroups and with fundamental domain P.

Lemma

Consider an epimorphism $\varphi\colon G(P)\to\mathbb{Z}_2^k$. *The subgroup* $\operatorname{Ker}\varphi\subset G(P)$ *does not contain elements of finite order if and only if the images of the reflections in any* $\leqslant k$ *facets of* P *that have a common vertex are linearly independent in* $\mathbb{Z}_2^k$.
In this case the group $\operatorname{Ker}\varphi$ *acts freely on* $\mathbb{L}^n$.

The quotient $N=\mathbb{L}^n/\operatorname{Ker}\varphi$ is a hyperbolic n-manifold. It is composed of $|\mathbb{Z}_2^k|=2^k$ copies of P and has a Riemannian metric of constant negative curvature. Furthermore, the manifold N is aspherical (the Eilenberg–Mac Lane space $K(\operatorname{Ker}\varphi,1)$), as its universal cover $\mathbb{L}^n$ is contractible.

Which combinatorial n-polytopes have right-angled realisations in $\mathbb{L}^n$?
In dim 3, there is a nice criterion going back to Pogorelov's work of 1967:

Theorem (Pogorelov, Andreev)

A combinatorial 3-polytope $P \neq \Delta^3$ can be realised as a right-angled polytope in $\mathbb{L}^3$ if and only if it is simple, and does not have 3- and 4-belts of facets. Furthermore, such a realisation is unique up to isometry.

We refer to the above class of 3-polytopes as the Pogorelov class $\mathcal{P}$. A polytope from the class $\mathcal{P}$ does not have triangular or quadrangular facets. The Pogorelov class contains all fullerenes (simple 3-polytopes with only pentagonal and hexagonal facets).

There is no classification of right-angled polytopes in $\mathbb{L}^4$. For $n \geqslant 5$, right-angled polytopes in $\mathbb{L}^n$ do not exist (by a theorem of Vinberg).

Given a right-angled polytope P, how to find an epimorphism $\varphi\colon G(P) \to \mathbb{Z}_2^k$ with $\operatorname{Ker}\varphi$ acting freely on $\mathbb{L}^n$?

One can consider the abelianisation: $G(P) \xrightarrow{\mathrm{ab}} \mathbb{Z}_2^m$, with $\operatorname{Ker}\mathrm{ab} = G'(P)$, the commutator subgroup.
The corresponding n-manifold $\mathbb{L}^n/G'(P)$ is the real moment-angle manifold $\mathcal{R}_P$, described as an intersection of quadrics in the beginning of this talk.

Corollary

If P is a right-angled polytope in $\mathbb{L}^n$, then the real moment-angle manifold $\mathcal{R}_P$ admits a hyperbolic structure as $\mathbb{L}^n/G'(P)$, where $G'(P)$ is the commutator subgroup of the corresponding right-angled Coxeter group. The manifold $\mathcal{R}_P$ is composed of 2^m copies of P.

A more econimical way to obtain a hyperbolic manifold is to consider $\varphi\colon G(P) \to \mathbb{Z}_2^n$. Such an epimorphism factors as $G(P) \xrightarrow{\mathrm{ab}} \mathbb{Z}_2^m \xrightarrow{\Lambda} \mathbb{Z}_2^n$, where Λ is a linear map.

The subgroup $\operatorname{Ker}\varphi$ acts freely on $\mathbb{L}^n$ if and only the Λ-images of any n facets of P that meet at a vertex form a basis of $\mathbb{Z}_2^n$.
Such Λ is called a $\mathbb{Z}_2$-characteristic function.

Proposition

Any simple 3-polytope admits a characteristic function.

Proof.

Given a 4-colouring of the facets of P, we assign to a facet of ith colour the ith basis vector $\boldsymbol{e}_i \in \mathbb{Z}^3$ for $i = 1, 2, 3$ and the vector $\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3$ for $i = 4$. The resulting map $\Lambda\colon \mathbb{Z}_2^m \to \mathbb{Z}_2^3$ satisfies the required condition, as any three of the four vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3$ form a basis of $\mathbb{Z}^3$. □

Manifolds $N(P, \Lambda) = \mathbb{L}^3 / \operatorname{Ker} \varphi$ obtained from right-angled 3-polytopes $P \in \mathcal{P}$ and characteristic functions $\Lambda\colon \mathbb{Z}_2^m \to \mathbb{Z}_2^3$ are called hyperbolic 3-manifolds of Löbell type. They were introduced and studied by A. Vesnin in 1987. Each $N(P, \Lambda)$ is composed of $|\mathbb{Z}_2^3| = 8$ copies of P.

In particular, one obtains a hyperbolic 3-manifold from any 4-colouring of a right-angled 3-polytope P. Löbell was first to consider a hyperbolic 3-manifold coming from a (unique) 4-colouring of the dodecahedron.

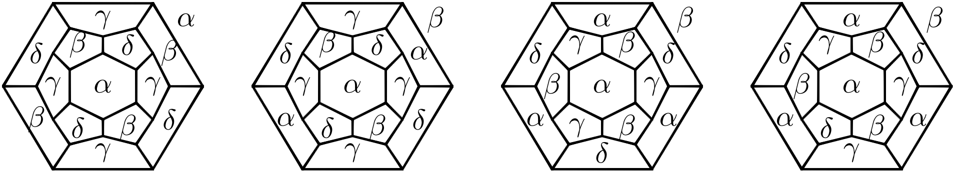

Figure: Four non-equivalent 4-colouring of the 'barell' fullerene with 14 facets.

Pairs (P, Λ) and (P', Λ') are equivalent if P and P' are combinatorially equivalent, and $\Lambda, \Lambda' \colon \mathbb{Z}_2^m \to \mathbb{Z}_2^n$ differ by an automorphism of $\mathbb{Z}_2^n$.

Theorem (Buchstaber-Erokhovets-Masuda-P-Park)

Let $N = N(P, \Lambda)$ and $N' = N(P', \Lambda')$ be two hyperbolic 3-manifolds of Löbell type corresponding to right-angled 3-polytopes P and P'. Then the following conditions are equivalent:

(a) *there is a cohomology ring isomorphism* $\varphi \colon H^*(N; \mathbb{Z}_2) \xrightarrow{\cong} H^*(N'; \mathbb{Z}_2)$;

(b) *there is a diffeomorphism* $N \cong N'$;

(c) *there is an equivalence of $\mathbb{Z}_2$-characteristic pairs* $(P, \Lambda) \sim (P', \Lambda')$.

In particular, hyperbolic 3-manifolds corresponding to non-equivalent 4-colourings of P are not diffeomorphic.

The difficult implication is (a)⇒(c). Its proof builds upon the wealth of cohomological techniques of toric topology.

Moment-angle complexes and polyhedral products

$\mathcal{K}$ an (abstract) simplicial complex on the set $[m] = \{1, \ldots, m\}$.
$I = \{i_1, \ldots, i_k\} \in \mathcal{K}$ a simplex. Always assume $\varnothing \in \mathcal{K}$.

Consider the unit polydisc in $\mathbb{C}^m$,

$$\mathbb{D}^m = \big\{(z_1, \ldots, z_m) \in \mathbb{C}^m \colon |z_i| \leqslant 1, \quad i = 1, \ldots, m\big\}.$$

Given $I \subset [m]$, set

$$B_I := \big\{(z_1, \ldots, z_m) \in \mathbb{D}^m \colon |z_j| = 1 \text{ for } j \notin I\big\} \cong \prod_{i \in I} D^2 \times \prod_{i \notin I} S^1.$$

The moment-angle complex

$$\mathcal{Z}_{\mathcal{K}} := \bigcup_{I \in \mathcal{K}} B_I = \bigcup_{I \in \mathcal{K}} \Big(\prod_{i \in I} D^2 \times \prod_{i \notin I} S^1 \Big) \subset \mathbb{D}^m$$

It is invariant under the coordinatewise action of the torus $\mathbb{T}^m$.

Example

$\mathcal{K} = \bullet\ \bullet$ (2 points), then $\mathcal{Z}_\mathcal{K} = D^2 \times S^1 \cup S^1 \times D^2 \cong S^3$.

$\mathcal{K} = \triangle$ (the boundary of a triangle), then
$\mathcal{Z}_\mathcal{K} = (D^2 \times D^2 \times S^1) \cup (D^2 \times S^1 \times D^2) \cup (S^1 \times D^2 \times D^2) \cong S^5$.

More generally, let X a space, and $A \subset X$. Given $I \subset [m]$, set

$$(X, A)^I = \Big\{(x_1, \ldots, x_m) \in \prod_{i=1}^m X \colon x_j \in A \text{ for } j \notin I\Big\} \cong \prod_{i \in I} X \times \prod_{i \notin I} A.$$

The $\mathcal{K}$-polyhedral product of (X, A) is

$$(X, A)^\mathcal{K} = \bigcup_{I \in \mathcal{K}} (X, A)^I \subset X^m.$$

Have $\mathcal{Z}_\mathcal{K} = (D^2, S^1)^\mathcal{K}$.
$\mathcal{R}_\mathcal{K} = (D^1, S^0)^\mathcal{K}$ the real moment-angle complex.

Theorem

If P is a simple polytope, $\mathcal{K}_P = \partial(P^*)$ (the dual simplicial complex), then $\mathcal{Z}_{\mathcal{K}_P} \cong \mathcal{Z}_P$ and $\mathcal{R}_{\mathcal{K}_P} \cong \mathcal{R}_P$.

In particular, $\mathcal{Z}_{\mathcal{K}_P} = (D^2, S^1)^{\mathcal{K}}$ and $\mathcal{R}_{\mathcal{K}_P} = (D^1, S^0)^{\mathcal{K}}$ are manifolds. More generally,

Proposition

Assume $|\mathcal{K}| \cong S^{n-1}$ (a sphere triangulation with m vertices). Then $\mathcal{Z}_{\mathcal{K}}$ is a closed manifold of dimension $m + n$.

Cohomology of moment-angle complexes

The face ring (the Stanley–Reisner ring) of $\mathcal{K}$ is given by

$$\mathbb{Z}[\mathcal{K}] := \mathbb{Z}[v_1, \ldots, v_m] \big/ \big(v_{i_1} \cdots v_{i_k} = 0 \quad \text{if } \{i_1, \ldots, i_k\} \notin \mathcal{K}\big)$$

where $\deg v_i = 2$.

Theorem (Buchstaber-P)

There are isomorphisms of rings

$$\begin{aligned} H^*(\mathcal{Z}_\mathcal{K}) &\cong \operatorname{Tor}_{\mathbb{Z}[v_1,\ldots,v_m]}(\mathbb{Z}[\mathcal{K}], \mathbb{Z}) \\ &\cong H\big[\Lambda[u_1, \ldots, u_m] \otimes \mathbb{Z}[\mathcal{K}], d\big], \qquad du_i = v_i, dv_i = 0 \\ &\cong \bigoplus_{I \notin \mathcal{K}} \widetilde{H}^{*-|I|-1}(\mathcal{K}_I) \qquad \mathcal{K}_I = \mathcal{K}|_I \end{aligned}$$

(Quasi)toric manifolds and small covers

Let P be a simple n-polytope with the set of facets $\mathcal{F} = \{F_1, \ldots, F_m\}$.

A characteristic function is a map $\Lambda\colon \mathcal{F} \to \mathbb{Z}^n$ such that $\Lambda(F_{i_1}), \ldots, \Lambda(F_{i_n})$ is a basis of $\mathbb{Z}^n$ whenever the facets $F_{i_1}, \ldots, F_{i_n}$ meet at a vertex.
A characteristic function defines a linear map $\Lambda\colon \mathbb{Z}^{\mathcal{F}} = \mathbb{Z}^m \to \mathbb{Z}^n$ and the corresponding homomorphism of tori $\Lambda\colon T^m \to T^n$.

Proposition

The subgroup $\operatorname{Ker} \Lambda \cong T^{m-n}$ *acts freely on* $\mathcal{Z}_P$.

The quotient $M(P, \Lambda) = \mathcal{Z}_P / \operatorname{Ker} \Lambda$ is called a quasitoric manifold.
It is a smooth $2n$-dimensional manifold with an action of the n-torus $T^m / \operatorname{Ker} \Lambda \cong T^n$ with quotient P.

Similarly, by considering a $\mathbb{Z}_2$-characteristic function $\varLambda\colon \mathbb{Z}_2^m \to \mathbb{Z}_2^n$ one obtains a small cover of P as the quotient $\mathcal{R}_P/\operatorname{Ker}\varLambda$.
A small cover $N(P,\varLambda)$ is a smooth n-dimensional manifold with an action of $\mathbb{Z}_2^n$ with quotient P.

Hyperbolic manifolds of Löbell type are small covers over 3-dimensional polytopes from the Pogorelov class $\mathcal{P}$. (Recall that $P\in\mathcal{P}$ admits a right-angled realisation in Lobachevsky 3-space $\mathbb{L}^3$.)

How to produce characteristic functions $\varLambda\colon \mathbb{Z}^m \to \mathbb{Z}^n$?

(a) A simple n-polytope P is Delzant if the normals $\boldsymbol{a}_{i_1},\ldots,\boldsymbol{a}_{i_n}$ to the facets $F_{i_1},\ldots,F_{i_n}$ form a basis of $\mathbb{Z}^n$ whenever $F_{i_1},\ldots,F_{i_n}$ meet at a vertex.
A Delzant polytope defines a characteristic function $\varLambda\colon \mathbb{Z}^m \to \mathbb{Z}^n$, $F_i \mapsto \boldsymbol{a}_i$.
The quotient $M = \mathcal{Z}_P/\operatorname{Ker}\varLambda$ is a (symplectic) toric manifold.

(b) For 3-dimensional polytopes P, any regular 4-colouring of facets gives a characteristic function, like in the $\mathbb{Z}_2$ case.

Theorem (Danilov–Jurkiewicz, Davis–Januszkiewicz)

Let $M = M(P, \Lambda)$ be a quasitoric manifold over a simple n-polytope P. The cohomology ring $H^*(M;\mathbb{Z})$ is generated by the degree-two classes $[v_i]$ dual to the oriented characteristic submanifolds M_i, and is given by

$$H^*(M;\mathbb{Z}) \cong \mathbb{Z}[v_1, \ldots, v_m]/\mathcal{I}, \qquad \deg v_i = 2,$$

where $\mathcal{I}$ is the ideal generated by elements of the following two types:

(a) $v_{i_1} \cdots v_{i_k}$ *such that* $F_{i_1} \cap \cdots \cap F_{i_k} = \varnothing$ *in* P;

(b) $\sum_{i=1}^{m} \langle \Lambda(F_i), \boldsymbol{x} \rangle v_i$, *for any vector* $\boldsymbol{x} \in \mathbb{Z}^n$.

The $\mathbb{Z}_2$-cohomology ring $H^*(N;\mathbb{Z}_2)$ of a small cover $N = N(P,\Lambda)$ has a similar description, but with generators v_i of degree 1.

Theorem (Buchstaber-Erokhovets-Masuda-P-Park)

Let $M = M(P, \Lambda)$ and $M' = M(P', \Lambda')$ be two quasitoric 6-manifolds, where P is a 3-polytope from the Pogorelov class $\mathcal{P}$. The following conditions are equivalent:

(a) *there is a cohomology ring isomorphism $\varphi\colon H^*(M;\mathbb{Z}) \xrightarrow{\cong} H^*(M';\mathbb{Z})$;*

(b) *there is a diffeomorphism $M \cong M'$;*

(c) *there is an equivalence of characteristic pairs $(P, \Lambda) \sim (P', \Lambda')$.*

Idea of proof (for both theorems).

We need to prove (a)⇒(c). A ring isomorphism $\varphi\colon H^*(N;\mathbb{Z}_2) \xrightarrow{\cong} H^*(N';\mathbb{Z}_2)$ implies an isomorphism $\varphi\colon H^*(M;\mathbb{Z}_2) \xrightarrow{\cong} H^*(M';\mathbb{Z}_2)$, which in turn implies an isomorphism $\psi\colon H^*(\mathcal{Z}_P;\mathbb{Z}_2) \xrightarrow{\cong} H^*(\mathcal{Z}_{P'};\mathbb{Z}_2)$. Using the specifics of the combinatorics of $P \in \mathcal{P}$ one can conclude that P is combinatorially equivalent to P', and Λ is equivalent to Λ'. □

Complex geometry of moment-angle manifolds

Moment-angle manifolds provide a wide class of examples of non-Kähler compact complex manifolds. A complex moment-angle manifold $\mathcal{Z}$ is constructed via certain combinatorial data, called a complete simplicial fan. In the case of rational fans, the manifold $\mathcal{Z}$ is the total space of a holomorphic bundle over a toric variety with fibres compact complex tori. In general, a complex moment-angle manifold $\mathcal{Z}$ is equipped with a canonical holomorphic foliation $\mathcal{F}$ which is equivariant with respect to the $(\mathbb{C}^\times)^m$-action. Examples of moment-angle manifolds include Hopf manifolds, Calabi–Eckmann manifolds, and their deformations.
We construct transversely Kähler metrics on moment-angle manifolds, under some restriction on the combinatorial data. We prove that any Kähler submanifold in a moment-angle manifold is contained in a leaf of the foliation $\mathcal{F}$. For a generic moment-angle manifold $\mathcal{Z}$, we prove that all subvarieties are moment-angle manifolds of smaller dimension and there are only finitely many of them. This implies, in particular, that $\mathcal{Z}$ does not support non-constant meromorphic functions.

Recall: if $\mathcal{K} = \mathcal{K}_P$ is the dual triangulation of a simple convex polytope P, then $\mathcal{Z}_P = \mathcal{Z}_{\mathcal{K}_P}$ has a canonical smooth structure as a nondegenerate intersection of Hermitian quadrics in $\mathbb{C}^m$.

Let $\mathcal{K}$ be a sphere triangulation, i.e. $|\mathcal{K}| \cong S^{n-1}$.

A realisation $|\mathcal{K}| \subset \mathbb{R}^n$ is starshaped if there is a point $\boldsymbol{x} \notin |\mathcal{K}|$ such that any ray from $\boldsymbol{x}$ intersects $|\mathcal{K}|$ in exactly one point.

A convex triangulation $\mathcal{K}_P$ is starshaped, but not vice versa!

$\mathcal{K}$ has a starshaped realisation if and only if it is the underlying complexes of a complete simplicial fan Σ.

The complement of the coordinate subspace arrangement defined by $\mathcal{K}$

$$
\begin{aligned}
U(\mathcal{K}) &= (\mathbb{C}, \mathbb{C} \setminus \{0\})^{\mathcal{K}} \\
&= \mathbb{C}^m \setminus \bigcup_{\{i_1,\dots,i_k\} \notin \mathcal{K}} \{z \in \mathbb{C}^m \colon z_{i_1} = \dots = z_{i_k} = 0\}.
\end{aligned}
$$

Theorem (P-Ustinovsky)

Let Σ be a complete simplicial fan in $\mathbb{R}^n$ with m one-dimensional cones, and let $\mathcal{K}$ be its underlying simplicial complex. Assume that $m - n = 2\ell$. Then

(a) *the holomorphic action of the group $C \cong \mathbb{C}^\ell$ on $U(\mathcal{K})$ is free and proper, so the quotient $U(\mathcal{K})/C$ is a compact complex manifold of dimension $(m - \ell)$;*

(b) *there is a $\mathbb{T}^m$-equivariant diffeomorphism $U(\mathcal{K})/C \cong \mathcal{Z}_{\mathcal{K}}$ defining a complex structure on $\mathcal{Z}_{\mathcal{K}}$ in which $\mathbb{T}^m$ acts holomorphically.*

Example (complex tori)

Let $\mathcal{K}$ be empty complex on 2 elements. Then $n = 0$, $m = 2$, $\ell = 1$. Set

$$C = \{(e^z, e^{\alpha z})\} \subset (\mathbb{C}^\times)^2$$

with $\alpha \notin \mathbb{R}$. Then $\mathbb{C} \to (\mathbb{C}^\times)^2$ is an embedding, and the quotient $(\mathbb{C}^\times)^2/C$ is a complex torus $T^2_{\mathbb{C}}$ with parameter $\alpha \in \mathbb{C}$:

$$(\mathbb{C}^\times)^2/C \cong \mathbb{C}/(\mathbb{Z} \oplus \alpha\mathbb{Z}) = T^2_{\mathbb{C}}(\alpha).$$

Similarly, if $\mathcal{K}$ is empty on 2ℓ elements (so that $n = 0$, $m = 2\ell$), we obtain any complex torus $T^{2\ell}_{\mathbb{C}}$ as the quotient $(\mathbb{C}^\times)^{2\ell}/C$.

Example (Hopf manifold)

Let $\mathcal{K}$ be the boundary of an n-dim simplex with $n+1$ vertices and 1 ghost vertex. Then

$$\mathcal{Z}_{\mathcal{K}} \cong S^1 \times S^{2n+1}, \quad U(\mathcal{K}) = \mathbb{C}^\times \times (\mathbb{C}^{n+1} \setminus \{0\}).$$

Set

$$C = \{(e^z, e^{\alpha z}, \ldots, e^{\alpha z})\colon z \in \mathbb{C}\} \subset (\mathbb{C}^\times)^{n+2}$$

with $\alpha \in \mathbb{C}$, $\alpha \notin \mathbb{R}$.

Then and $\mathcal{Z}_{\mathcal{K}}$ acquires a complex structure as the quotient $U(\mathcal{K})/C$:

$$\mathbb{C}^\times \times (\mathbb{C}^{n+1} \setminus \{0\}) / \{(t, \boldsymbol{w}) \sim (e^z t, e^{\alpha z} \boldsymbol{w})\} \cong (\mathbb{C}^{n+1} \setminus \{0\}) / \{\boldsymbol{w} \sim e^{2\pi i \alpha} \boldsymbol{w}\},$$

where $t \in \mathbb{C}^\times$, $\boldsymbol{w} \in \mathbb{C}^{n+1} \setminus \{0\}$. This is known as the Hopf manifold.

Assume that Σ is a rational fan, i. e. the generators $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ span a lattice. It defines an algebraic subgroup $G \subset (\mathbb{C}^\times)^m$, $G \cong (\mathbb{C}^\times)^{m-n}$.

The corresponding toric variety V_Σ is the quotient $U(\mathcal{K})/G$.

Proposition

(a) *The toric variety V_Σ is homeomorphic to the quotient of $\mathcal{Z}_{\mathcal{K}_\Sigma}$ by the holomorphic action of G/C.*

(b) *If Σ is a nonsingular fan, then there is a holomorphic principal bundle $\mathcal{Z}_{\mathcal{K}_\Sigma} \to V_\Sigma$ with fibre the compact complex torus G/C of dimension ℓ.*

For singular varieties V_Σ the quotient projection $\mathcal{Z}_{\mathcal{K}_\Sigma} \to V_\Sigma$ is a holomorphic principal Seifert bundle for an appropriate orbifold structure on V_Σ.

Theorem (P-Ustinovsky-Verbitsky)

Assume that the data defining a complex structure on $\mathcal{Z}_{\mathcal{K}} = U(\mathcal{K})/C$ is generic. Then any divisor of $\mathcal{Z}_{\mathcal{K}}$ is a union of coordinate divisors.

Furthermore, if Σ is a weakly normal fan, then any compact irreducible analytic subset $Y \subset \mathcal{Z}_{\mathcal{K}}$ of positive dimension is a coordinate submanifold.

Corollary

Under generic assumptions, there are no non-constant meromorphic functions on $\mathcal{Z}_{\mathcal{K}}$, i. e. its algebraic dimension is zero.

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