New constructions of Hamiltonian-minimal Lagrangian submanifolds

based on joint works with Andrey E. Mironov

Taras Panov

Moscow State University

Integrable Systems
CSF Ascona, 19-24 June 2016

Basics

Let M be a Kähler manifold with symplectic form ω, $\dim_{\mathbb{R}} M = 2n$.

An immersion $i\colon N \looparrowright M$ of an n-manifold N is Lagrangian if $i^*(\omega) = 0$. If i is an embedding, then $i(N)$ is a Lagrangian submanifold of M.

A vector field ξ on M is Hamiltonian if the 1-form $\omega(\,\cdot\,,\xi)$ is exact.

A Lagrangian immersion $i\colon N \looparrowright M$ is Hamiltonian minimal (H-minimal) if the variations of the volume of $i(N)$ along all Hamiltonian vector fields with compact support are zero, i.e.

$$\frac{d}{dt}\,\mathrm{vol}(i_t(N))\big|_{t=0} = 0,$$

where $i_t(N)$ is a Hamiltonian deformation of $i(N) = i_0(N)$.

Overview

Explicit examples of H-minimal Lagrangian submanifolds in $\mathbb{C}^m$ and $\mathbb{C}P^m$ were constructed in the work of Yong-Geun Oh, Castro–Urbano, Hélein–Romon, Amarzaya–Ohnita, among others.

In 2003 Mironov suggested a universal construction providing an H-minimal Lagrangian immersion in $\mathbb{C}^m$ from an intersection of special real quadrics.

The same intersections of real quadrics are known to toric geometers and topologists as (real) moment-angle manifolds. They appear, for instance, as the level sets of the moment map in the construction of Hamiltonian toric manifolds via symplectic reduction.

Here we combine Mironov's construction with the methods of toric topology to produce new examples of H-minimal Lagrangian **embeddings** with interesting and complicated topology.

Polytopes and moment-angle manifolds

A convex polytope in $\mathbb{R}^n$ is obtained by intersecting m halfspaces:

$$P = \{\boldsymbol{x} \in \mathbb{R}^n \colon \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle + b_i \geqslant 0 \quad \text{for } i = 1, \dots, m\}.$$

Suppose each $F_i = P \cap \{\boldsymbol{x} \colon \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle + b_i = 0\}$ is a facet (m facets in total).

Define an affine map

$$i_P \colon \mathbb{R}^n \to \mathbb{R}^m, \quad i_P(\boldsymbol{x}) = \big(\langle \boldsymbol{a}_1, \boldsymbol{x} \rangle + b_1, \dots, \langle \boldsymbol{a}_m, \boldsymbol{x} \rangle + b_m\big).$$

Then i_P is monomorphic, and $i_P(P) \subset \mathbb{R}^m$ is the intersection of an n-plane with $\mathbb{R}^m_{\geqslant} = \{\boldsymbol{y} = (y_1, \dots, y_m) \colon y_i \geqslant 0\}$.

Define the space $\mathcal{Z}_P$ from the diagram

$$
\begin{array}{ccccc}
\mathcal{Z}_P & \xrightarrow{i_Z} & \mathbb{C}^m & \qquad & (z_1, \ldots, z_m) \\
\downarrow & & \downarrow \mu & & \downarrow \\
P & \xrightarrow{i_P} & \mathbb{R}^m_{\geqslant} & & (|z_1|^2, \ldots, |z_m|^2)
\end{array}
$$

$\mathcal{Z}_P$ has a $\mathbb{T}^m$-action, $\mathcal{Z}_P/\mathbb{T}^m = P$, and i_Z is a $\mathbb{T}^m$-equivariant inclusion.

A polytope P is simple if exactly $n = \dim P$ facets meet at each vertex.

Proposition

If P is simple, then $\mathcal{Z}_P$ is a smooth manifold of dimension $m + n$.

Proof.

Write $i_P(\mathbb{R}^n)$ by $m - n$ linear equations in $(y_1, \ldots, y_m) \in \mathbb{R}^m$. Replace y_k by $|z_k|^2$ to obtain a presentation of $\mathcal{Z}_P$ by quadrics. □

$\mathcal{Z}_P$ is the moment-angle manifold corresponding to P.

Similarly, by considering

$$\begin{array}{ccccc} \mathcal{R}_P & \longrightarrow & \mathbb{R}^m & & (u_1,\ldots,u_m) \\ \big\downarrow & & \big\downarrow \mu & & \big\downarrow \\ P & \xrightarrow{i_P} & \mathbb{R}^m_\geqslant & & (u_1^2,\ldots,u_m^2) \end{array}$$

we obtain the real moment-angle manifold $\mathcal{R}_P$.

Example

$P = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1 \geqslant 0,\ x_2 \geqslant 0,\ -\gamma_1 x_1 - \gamma_2 x_2 + 1 \geqslant 0\}$, $\gamma_1, \gamma_2 > 0$ (a 2-simplex). Then
$\mathcal{Z}_P = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \colon \gamma_1|z_1|^2 + \gamma_2|z_2|^2 + |z_3|^2 = 1\}$ (a 5-sphere),
$\mathcal{R}_P = \{(u_1, u_2, u_3) \in \mathbb{R}^3 \colon \gamma_1|u_1|^2 + \gamma_2|u_2|^2 + |u_3|^2 = 1\}$ (a 2-sphere).

Torus actions

Have intersections of quadrics

$$\mathcal{Z}_P = \{\boldsymbol{z} = (z_1, \ldots, z_m) \in \mathbb{C}^m \colon \gamma_1 |z_1|^2 + \cdots + \gamma_m |z_m|^2 = c\},$$
$$\mathcal{R}_P = \{\boldsymbol{u} = (u_1, \ldots, u_m) \in \mathbb{R}^m \colon \gamma_1 u_1^2 + \cdots + \gamma_m u_m^2 = c\}$$

where $\gamma_1, \ldots, \gamma_m$ and c are vectors in $\mathbb{R}^{m-n}$.

Assume that the polytope P is rational. Then have two lattices:
$\Lambda = \mathbb{Z}\langle \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \rangle \subset \mathbb{R}^n$ and $L = \mathbb{Z}\langle \gamma_1, \ldots, \gamma_m \rangle \subset \mathbb{R}^{m-n}$.

Consider the $(m-n)$-torus $T_P = \left\{ \left(e^{2\pi i \langle \gamma_1, \varphi \rangle}, \ldots, e^{2\pi i \langle \gamma_m, \varphi \rangle} \right) \in \mathbb{T}^m \right\}$,
i.e. $T_P = \mathbb{R}^{m-n}/L^*$, and set $D_P = \frac{1}{2} L^*/L^* \cong (\mathbb{Z}_2)^{m-n}$.

Proposition

The $(m-n)$-torus T_P acts on $\mathcal{Z}_P$ almost freely.

Main construction

Consider the map

$$f\colon \mathcal{R}_P \times T_P \longrightarrow \mathbb{C}^m,$$
$$(\boldsymbol{u},\varphi) \mapsto \boldsymbol{u}\cdot\varphi = \left(u_1 e^{2\pi i\langle\gamma_1,\varphi\rangle},\ldots,u_m e^{2\pi i\langle\gamma_m,\varphi\rangle}\right).$$

Note $f(\mathcal{R}_P \times T_P) \subset \mathcal{Z}_P$ is the set of T_P-orbits through $\mathcal{R}_P \subset \mathbb{C}^m$.
Have an m-dimensional manifold

$$N_P = \mathcal{R}_P \times_{D_P} T_P.$$

Lemma

$f\colon \mathcal{R}_P \times T_P \to \mathbb{C}^m$ induces an immersion $j\colon N_P \looparrowright \mathbb{C}^m$.

Theorem (Mironov)

The immersion $j\colon N_P \looparrowright \mathbb{C}^m$ is H-minimal Lagrangian.

Question

When $j\colon N_P \looparrowright \mathbb{C}^m$ is an embedding?

A simple rational polytope P is Delzant if for any vertex $v \in P$ the set of vectors $\boldsymbol{a}_{i_1}, \ldots, \boldsymbol{a}_{i_n}$ normal to the facets meeting at v forms a basis of the lattice $\Lambda = \mathbb{Z}\langle \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \rangle$:

$$\mathbb{Z}\langle \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \rangle = \mathbb{Z}\langle \boldsymbol{a}_{i_1}, \ldots, \boldsymbol{a}_{i_n} \rangle \qquad \text{for any } v = F_{i_1} \cap \cdots \cap F_{i_n}.$$

Theorem

The following conditions are equivalent:

1. *$j\colon N_P \to \mathbb{C}^m$ is an embedding of an H-minimal Lagrangian submanifold;*
2. *the $(m-n)$-torus T_P acts on $\mathcal{Z}_P$ freely.*
3. *P is a Delzant polytope.*

Get an H-minimal Lagrangian submanifold N_P in $\mathbb{C}^m$ for any Delzant n-polytope P with m facets!

Explicit constructions of families of Delzant polytopes are known in toric geometry and topology:

- simplices and cubes in all dimensions;
- products and face truncations;
- associahedra (Stasheff polytopes), permutahedra, and generalisations.

Examples

Example (one quadric)

Let $P = \Delta^{m-1}$ (a simplex), i.e. $m - n = 1$.
$\mathcal{R}_{\Delta^{m-1}}$ is given by a single quadric

$$\gamma_1 u_1^2 + \cdots + \gamma_m u_m^2 = c$$

with $\gamma_i > 0$, i.e. $\mathcal{R}_{\Delta^{m-1}} \cong S^{m-1}$.
Then

$$N \cong S^{m-1} \times_{\mathbb{Z}_2} S^1 \cong \begin{cases} S^{m-1} \times S^1 & \text{if } \tau \text{ preserves the orient. of } S^{m-1}, \\ \mathcal{K}^m & \text{if } \tau \text{ reverses the orient. of } S^{m-1}, \end{cases}$$

where τ is the involution and $\mathcal{K}^m$ is an m-dimensional Klein bottle.

Proposition (one quadric)

We obtain an H-minimal Lagrangian embedding of $N_{\Delta^{m-1}} \cong S^{n-1} \times_{\mathbb{Z}_2} S^1$ in $\mathbb{C}^m$ whenever $\gamma_1 = \cdots = \gamma_m$ in $\gamma_1 u_1^2 + \cdots + \gamma_m u_m^2 = c$.
The topology of $N_{\Delta^{m-1}} = N(m)$ depends on the parity of m:

$$N(m) \cong S^{m-1} \times S^1 \qquad \text{if } m \text{ is even},$$
$$N(m) \cong \mathcal{K}^m \qquad \text{if } m \text{ is odd}.$$

The Klein bottle $\mathcal{K}^m$ with even m does *not* admit Lagrangian embeddings in $\mathbb{C}^m$ [Nemirovsky, Shevchishin].

Theorem (two quadrics)

Let $m - n = 2$, *i.e.* $P \simeq \Delta^{p-1} \times \Delta^{q-1}$.

- $\mathcal{R}_P$ *is diffeomorphic to* $\mathcal{R}(p,q) \cong S^{p-1} \times S^{q-1}$ *given by*

$$\begin{aligned} u_1^2 + \ldots + u_k^2 + u_{k+1}^2 + \cdots + u_p^2 \qquad\qquad\qquad\qquad &= 1, \\ u_1^2 + \ldots + u_k^2 \qquad\qquad\qquad\qquad + u_{p+1}^2 + \cdots + u_m^2 &= 2, \end{aligned}$$

where $p + q = m$, $0 < p < m$ *and* $0 \leqslant k \leqslant p$.

- *If* $N_P \to \mathbb{C}^m$ *is an embedding, then* N_P *is diffeomorphic to*

$$N_k(p,q) = \mathcal{R}(p,q) \times_{\mathbb{Z}_2 \times \mathbb{Z}_2} (S^1 \times S^1),$$

where the two involutions act on $\mathcal{R}(p,q)$ *by*

$\psi_1 \colon (u_1, \ldots, u_m) \mapsto (-u_1, \ldots, -u_k, -u_{k+1}, \ldots, -u_p, u_{p+1}, \ldots, u_m)$,

$\psi_2 \colon (u_1, \ldots, u_m) \mapsto (-u_1, \ldots, -u_k, u_{k+1}, \ldots, u_p, -u_{p+1}, \ldots, -u_m)$.

There is a fibration $N_k(p,q) \to S^{q-1} \times_{\mathbb{Z}_2} S^1 = N(q)$ *with fibre* $N(p)$.

Example (three quadrics)

In the case $m - n = 3$ the topology of compact manifolds $\mathcal{R}_P$ and $\mathcal{Z}_P$ was fully described by [Lopez de Medrano]. Each manifold is diffeomorphic to a product of three spheres, or to a connected sum of products of spheres, with two spheres in each product.

The simplest case is $n = 2$ and $m = 5$: a Delzant pentagon.

In this case $\mathcal{R}_P$ is an oriented surface of genus 5, and $\mathcal{Z}_P$ is diffeomorphic to a connected sum of 5 copies of $S^3 \times S^4$.

Get an H-minimal Lagrangian submanifold $N_P \subset \mathbb{C}^5$ which is the total space of a bundle over T^3 with fibre a surface of genus 5.

Proposition

Let P be an m-gon. Then $\mathcal{R}_P$ is an orientable surface S_g of genus $g = 1 + 2^{m-3}(m-4)$.

Get an H-minimal Lagrangian submanifold $N_P \subset \mathbb{C}^m$ which is the total space of a bundle over T^{m-2} with fibre S_g. It is an aspherical manifold (for $m \geqslant 4$) whose fundamental group enters into the short exact sequence

$$1 \longrightarrow \pi_1(S_g) \longrightarrow \pi_1(N) \longrightarrow \mathbb{Z}^{m-2} \longrightarrow 1.$$

For $n > 2$ and $m - n > 3$ the topology of $\mathcal{R}_P$ and $\mathcal{Z}_P$ is even more complicated.

Generalisation to toric manifolds

Consider 2 sets of quadrics:

$$\mathcal{Z}_\Gamma = \{z \in \mathbb{C}^m \colon \sum_{k=1}^m \gamma_k |z_k|^2 = \boldsymbol{c}\}, \quad \gamma_k, \boldsymbol{c} \in \mathbb{R}^{m-n};$$
$$\mathcal{Z}_\Delta = \left\{z \in \mathbb{C}^m \colon \sum_{k=1}^m \delta_k |z_k|^2 = \boldsymbol{d}\right\}, \quad \delta_k, \boldsymbol{d} \in \mathbb{R}^{m-\ell};$$

s. t. the polytopes corresponding to $\mathcal{Z}_\Gamma$, $\mathcal{Z}_\Delta$ *and* $\mathcal{Z}_\Gamma \cap \mathcal{Z}_\Delta$ are Delzant.

Define $\mathcal{R}_\Gamma$, $T_\Gamma \cong \mathbb{T}^{m-n}$, $D_\Gamma \cong \mathbb{Z}_2^{m-n}$, $\mathcal{R}_\Delta$, $T_\Delta \cong \mathbb{T}^{m-\ell}$, $D_\Delta \cong \mathbb{Z}_2^{m-\ell}$ as before.

The idea is to use the first set of quadrics to produce a toric manifold M via symplectic reduction, and then use the second set of quadrics to define an H-minimal Lagrangian submanifold in M.

$M := \mathbb{C}^m /\!\!/ T_\Gamma = \mathcal{Z}_\Gamma / T_\Gamma$ a toric manifold, $\dim M = 2n$.

Real points $\mathcal{R}_\Gamma / D_\Gamma \subset \mathcal{Z}_\Gamma / T_\Gamma = M$.
$R := (\mathcal{R}_\Gamma \cap \mathcal{R}_\Delta)/D_\Gamma$ subset of real points of M, $\dim R = n + \ell - m$.

Define $N := R \times_{D_\Delta} T_\Delta \subset M, \quad \dim N = n$.

Theorem

N is an H-minimal Lagrangian submanifold in M.

Idea of proof.

Consider $\widetilde{M} := M /\!\!/ T_\Delta = (\mathcal{Z}_\Gamma \cap \mathcal{Z}_\Delta)/(T_\Gamma \times T_\Delta)$. Then

$$\widetilde{N} := N/T_\Delta = (\mathcal{R}_\Gamma \cap \mathcal{R}_\Delta)/(D_\Gamma \times D_\Delta) \hookrightarrow (\mathcal{Z}_\Gamma \cap \mathcal{Z}_\Delta)/(T_\Gamma \times T_\Delta) = \widetilde{M}$$

is a minimal (totally geodesic) submanifold. □

Example

1. 1. If $m - n = 0$, then the set of quadrics defining $\mathcal{Z}_\Gamma$ is void, so $\mathcal{Z}_\Gamma = M = \mathbb{C}^m$ and we get the original construction of H-minimal Lagrangian submanifolds N in $\mathbb{C}^m$.
2. If $m - \ell = 0$, then the set of quadrics defining $\mathcal{Z}_\Delta$ is void, so $N = R = \mathcal{R}_\Gamma / D_\Gamma$ is set of real points of $M = \mathcal{Z}_\Gamma / T_\Gamma$. The submanifold N is minimal (totally geodesic) in M.
3. If $m - n = 1$ and $\mathcal{Z}_\Gamma$ is compact, then $\mathcal{Z}_\Gamma \cong S^{2m-1}$, and we obtain H-minimal Lagrangian submanifolds in $M = \mathcal{Z}_\Gamma / T_\Gamma = \mathbb{C}P^{m-1}$.

Integrable systems and dynamics

General fact: the above constructed H-minimal Lagrangian submanifolds $N \subset \mathbb{C}^m$ are 'special' degenerations of Liouville tori for certain integrable systems.

Have $\mathcal{R} \subset N \subset \mathcal{Z}$, where $\mathcal{Z}$ is the level set for $m-n$ Hermitian quadrics

$$H_1 = \cdots = H_{m-n} = 0,$$

and $\mathcal{R}$ is its real part

$$H_1^{\mathbb{R}} = \cdots = H_{m-n}^{\mathbb{R}} = 0,$$

Then N can be written by

$$H_1 = \cdots = H_{m-n} = 0, \quad F_1 = \ldots = F_n = 0,$$

where $\{H_i, H_j\} = \{H_i, F_k\} = \{F_k, F_l\} = 0$ and $F_1, \ldots, F_n$ are *polynomial* integrals.

Example

Consider the quadric $x_1^2 + x_2^2 = 1$ giving rise to an embedded H-minimal Lagrangian 2-torus $N = T^2 \subset \mathbb{C}^2$.
It is given by two equations

$$H = x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1, \quad F = x_1 y_2 - x_2 y_1 = 0,$$

with $\{H, F\} = 0$.
This torus is included to the family of Liouville tori given by

$$H = x_1^2 + y_1^2 + x_2^2 + y_2^2 = C, \quad F = x_1 y_2 - x_2 y_1 = D,$$

which degenerates to a circle at $D = C/2$.

Example

Now consider the quadric $x_1^2 + 2x_2^2 = 1$ giving rise to an immersed H-minimal Klein bottle $K \looparrowright \mathbb{C}^2$.
It is given by two equations

$$H = x_1^2 + y_1^2 + 2x_2^2 + 2y_2^2 = 1, \quad F = x_1^2 y_2 - y_1^2 y_2 - 2x_1 x_2 y_1 = 0,$$

with $\{H, F\} = 0$.
Have a family of Liouville tori

$$H = x_1^2 + y_1^2 + x_2^2 + y_2^2 = C, \quad F = x_1^2 y_2 - y_1^2 y_2 - 2x_1 x_2 y_1 = D,$$

which degenerates to a Klein bottle at $D = 0$.

References

- Victor Buchstaber and Taras Panov. *Toric Topology*. Math. Surv. and Monogr., 204, Amer. Math. Soc., Providence, RI, 2015.
- Andrey E. Mironov. *New examples of Hamilton-minimal and minimal Lagrangian submanifolds in $\mathbb{C}^n$ and $\mathbb{C}P^n$.* Sbornik Math. **195** (2004), no. 1–2, 85–96.
- Andrey E. Mironov and Taras Panov. *Intersections of quadrics, moment-angle manifolds, and Hamiltonian-minimal Lagrangian embeddings*. Funct. Anal. and Appl. **47** (2013), no. 1, 38–49.
- Andrey E. Mironov and Taras Panov. *Hamiltonian minimal Lagrangian submanifolds in toric varieties*. Russian Math. Surveys **68** (2013), no. 2, 392–394.
- Taras Panov. *Geometric structures on moment-angle manifolds.* Russian Math. Surveys **68** (2013), no. 3, 503–568.