On the cohomology of partial quotients of moment-angle manifolds.

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1. Basics

$\mathcal{K}$ a simplicial complex on $[m] = \{1, \dots, m\}$.
For each simplex $I = \{i_1, \dots, i_k\} \in \mathcal{K}$, set

$$(D^2, S^1)^I = \{(x_1, \dots, x_m) \in (D^2)^m \colon x_i \in S^1 = \partial D^2 \text{ when } i \notin I\}.$$

The moment-angle complex is the polyhedral product

$$\mathcal{Z}_{\mathcal{K}} = (D^2, S^1)^{\mathcal{K}} = \bigcup\nolimits_{I \in \mathcal{K}} (D^2, S^1)^I \subset (D^2)^m.$$

$\mathcal{Z}_{\mathcal{K}}$ is a manifold whenever $\mathcal{K}$ is a triangulated sphere, and can be smoothed when $\mathcal{K}$ is a boundary of a polytope or is a starshaped sphere (comes from a complete simplicial fan).

Also define

$$BT^{\mathcal{K}} = (\mathbb{C}P^\infty, pt)^{\mathcal{K}} = \bigcup_{I\in\mathcal{K}} BT^I \subset BT^m = (\mathbb{C}P^\infty)^m.$$

The cohomology of $BT^{\mathcal{K}}$ (with coefficients in a commutative ring R) is the face ring of $\mathcal{K}$:

$$H^*(BT^{\mathcal{K}}) \cong R[\mathcal{K}] = R[v_1,\ldots,v_m]/(v_{i_1}\cdots v_{i_k}\colon \{i_1,\ldots,i_k\} \notin \mathcal{K}),$$

where $\deg v_i = 2$.

There is a homotopy fibration

$$\mathcal{Z}_{\mathcal{K}} \longrightarrow BT^{\mathcal{K}} \longrightarrow BT^m$$

2. Partial quotients

The torus T^m acts on $\mathcal{Z}_\mathcal{K}$ coordinatewise.
We consider freely acting subtori $H \subset T^m$ and partial quotients $\mathcal{Z}_\mathcal{K}/H$.
The manifolds $\mathcal{Z}_\mathcal{K}/H$ have recently attracted attention as they support complex-analytic structures, usually non-Kähler, with interesting geometry.

We turn the face ring $R[\mathcal{K}]$ into a module over the polynomial ring $H^*(B(T^m/H))$ via the homomorphism

$$H^*(B(T^m/H)) \to H^*(BT^m) = R[v_1, \ldots, v_m] \to R[\mathcal{K}]$$

Theorem

For any commutative ring R, there is an isomorphism of graded algebras

$$H^*(\mathcal{Z}_\mathcal{K}/H; R) \cong \mathrm{Tor}_{H^*(B(T^m/H);R)}(R[\mathcal{K}], R).$$

3. Proof of the main theorem

The Eilenberg–Moore spectral sequence of the homotopy fibration $\mathcal{Z}_\mathcal{K}/H \to BT^\mathcal{K} \to B(T^m/H)$ has

$$E_2 = \mathrm{Tor}_{H^*(B(T^m/H))}(R[\mathcal{K}], R)$$

and converges to

$$H^*(\mathcal{Z}_\mathcal{K}/H) \cong \mathrm{Tor}_{C^*(B(T^m/H))}(C^*(BT^\mathcal{K}), R).$$

We shall establish a multiplicative isomorphism

$$\mathrm{Tor}_{H^*(B(T^m/H))}(R[\mathcal{K}], R) \to \mathrm{Tor}_{C^*(B(T^m/H))}(C^*(BT^\mathcal{K}), R);$$

it would also imply the collapse of the Eilenberg–Moore spectral sequence.

For any torus T^k we consider the map of R-modules

$$\varphi\colon H^*(BT^k) = (H^*(BT^1))^{\otimes k} \xrightarrow{i} (C^*(BT^1))^{\otimes k} \xrightarrow{\times} C^*(BT^k),$$

where C^* denotes the normalised singular cochain functor with coefficients in R, the map i is the k-fold tensor product of the map $H^*(BT^1) = R[v] \to C^*(BT^1)$ sending v to any representing cochain, and $\times$ is the k-fold cross-product.
The map φ induces an isomorphism in cohomology.

We have

$$R[\mathcal{K}] = H^*(BT^{\mathcal{K}}) = \lim_{I \in \mathcal{K}} H^*(BT^I)$$

where each $H^*(BT^I)$ is a polynomial ring on $|I|$ generators, the (inverse) limit is taken in the category of graded algebras for the diagram consisting of projections $H^*(BT^I) \to H^*(BT^J)$ corresponding to $J \subset I \in \mathcal{K}$.

Now consider the diagram

$$\begin{array}{ccccccc} R & \longleftarrow & H^*(B(T^m/H)) & \longrightarrow & \lim_{I\in\mathcal K} H^*(BT^I) & =R[\mathcal K]=H^*(BT^{\mathcal K}) \\ \| & & \Downarrow & & \Downarrow & \\ R & \longleftarrow & C^*(B(T^m/H)) & \longrightarrow & \lim_{I\in\mathcal K} C^*(BT^I) & \\ \| & & \| & & \uparrow & \\ R & \longleftarrow & C^*(B(T^m/H)) & \longrightarrow & C^*(\mathrm{colim}_{I\in\mathcal K} BT^I) & = C^*(BT^{\mathcal K}) \end{array}$$

where the double arrows denote derivatives of φ and the horizontal arrows on the right are induced by the maps $BT^I \to BT^m \to BT^m/H$.

All vertical arrows above induce isomorphisms in cohomology (for the bottom right arrow this follows from excision).
If the diagram was commutative in the category DA of differential graded algebras (i.e. consisted of multiplicative maps), then the standard functoriality of Tor would have implied the required isomorphism

$$\mathrm{Tor}_{H^*(B(T^m/H))}(R[\mathcal K], R) \cong \mathrm{Tor}_{C^*(B(T^m/H))}(C^*(BT^{\mathcal K}), R) \cong H^*(\mathcal Z_{\mathcal K}/H).$$

$$\begin{array}{ccccccc}
R & \longleftarrow & H^*(B(T^m/H)) & \longrightarrow & \lim_{I\in\mathcal{K}} H^*(BT^I) & =R[\mathcal{K}]=H^*(BT^{\mathcal{K}}) \\
\| & & \Downarrow & & \Downarrow & \\
R & \longleftarrow & C^*(B(T^m/H)) & \longrightarrow & \lim_{I\in\mathcal{K}} C^*(BT^I) & \\
\| & & \| & & \uparrow & \\
R & \longleftarrow & C^*(B(T^m/H)) & \longrightarrow & C^*(\mathrm{colim}_{I\in\mathcal{K}} BT^I) & = C^*(BT^{\mathcal{K}})
\end{array}$$

The lower part of the diagram is indeed a commutative diagram in DA. The upper part is not commutative though, and the double arrow maps are not morphisms in DA as φ is not multiplicative.

Nevertheless, Tor enjoys extended functoriality with respect to morphisms in the category DASH, provided that the diagram above is homotopy commutative in DASH, by [Munkholm74, 5.4].

The objects of DASH are the same as in DA, while morphisms $A \Rightarrow A'$ are coalgebra maps $BA \to BA'$ of the bar constructions.

The map φ and the double arrows above are morphisms in DASH by [Munkholm74, 7.3].

$$\begin{array}{ccccccc}
R & \longleftarrow & H^*(B(T^m/H)) & \longrightarrow & \lim_{I\in\mathcal K} H^*(BT^I) & =R[\mathcal K]=H^*(BT^{\mathcal K}) \\
\| & & \Downarrow & & \Downarrow & \\
R & \longleftarrow & C^*(B(T^m/H)) & \longrightarrow & \lim_{I\in\mathcal K} C^*(BT^I) & \\
\| & & \| & & \uparrow & \\
R & \longleftarrow & C^*(B(T^m/H)) & \longrightarrow & C^*(\mathrm{colim}_{I\in\mathcal K} BT^I) & = C^*(BT^{\mathcal K})
\end{array}$$

To see that the upper right square is homotopy commutative, it is enough to establish the homotopy commutativity of the diagram

$$\begin{array}{ccccc}
H^*(B(T^m/H)) & \longrightarrow & H^*(BT^I) & \longrightarrow & H^*(BT^J) \\
\Downarrow & & \Downarrow & & \Downarrow \\
C^*(B(T^m/H)) & \longrightarrow & C^*(BT^I) & \longrightarrow & C^*(BT^J)
\end{array}$$

for any $J \subset I \in \mathcal{K}$.

The right square is commutative in the standard sense by the construction of φ (note that we are using normalised cochains), while the left square is homotopy commutative by [Munkholm74, 7.3].

It remains to prove that the isomorphism

$$\mathrm{Tor}_{H^*(B(T^m/H))}(R[\mathcal{K}], R) \to \mathrm{Tor}_{C^*(B(T^m/H))}(C^*(BT^{\mathcal{K}}), R)$$

is multiplicative.

Let us take a closer look on how the product structure is defined on both sides.

We have a commutative diagram

$$\begin{array}{ccccc} R\otimes R & \longleftarrow & C^*(B(T^m/H))\otimes C^*(B(T^m/H)) & \longrightarrow & C^*(BT^{\mathcal K})\otimes C^*(BT^{\mathcal K}) \\ \downarrow & & \Downarrow & & \Downarrow \\ R & \longleftarrow & C^*(B(T^m/H)) & \longrightarrow & C^*(BT^{\mathcal K}) \end{array}$$

Using the functoriality of Tor in DASH we get a natural map

$$\mathrm{Tor}_{C^*(B(T^m/H))\otimes C^*(B(T^m/H))}(C^*(BT^{\mathcal K})\otimes C^*(BT^{\mathcal K}), R\otimes R) \\ \to \mathrm{Tor}_{C^*(B(T^m/H))}(C^*(BT^{\mathcal K}), R)$$

which, composed with the classical Künneth-like map

$$\mathrm{Tor}_{C^*(B(T^m/H))}(C^*(BT^{\mathcal K}), R)\otimes \mathrm{Tor}_{C^*(B(T^m/H))}(C^*(BT^{\mathcal K}), R) \\ \to \mathrm{Tor}_{C^*(B(T^m/H))\otimes C^*(B(T^m/H))}(C^*(BT^{\mathcal K})\otimes C^*(BT^{\mathcal K}), R\otimes R),$$

gives the multiplicative structure in $\mathrm{Tor}_{C^*(B(T^m/H))}(C^*(BT^{\mathcal K}), R)$.

It can be checked that this multiplicative structure is the same as the one defined via the Eilenberg–Zilber theorem and used in the Eilenberg–Moore isomorphism $\mathrm{Tor}_{C^*(B(T^m/H))}(C^*(BT^{\mathcal K}), R)\cong H^*(\mathcal Z_{\mathcal K}/H)$.

The product in $\mathrm{Tor}_{H^*(B(T^m/H))}(H^*(BT^{\mathcal{K}}), R)$ is defined similarly.

Denote $B = C^*(B(T^m/H))$ and $M = C^*(BT^{\mathcal{K}})$.
The diagram

$$\begin{array}{ccc} \mathrm{Tor}_{HB}(HM, R) \otimes \mathrm{Tor}_{HB}(HM, R) & \longrightarrow & \mathrm{Tor}_{HB\otimes HB}(HM \otimes HM, R \otimes R) \\ \downarrow & & \downarrow \\ \mathrm{Tor}_{B}(M, R) \otimes \mathrm{Tor}_{B}(M, R) & \longrightarrow & \mathrm{Tor}_{B\otimes B}(M \otimes M, R \otimes R) \end{array}$$

in which the vertical arrows are isomorphisms of R-modules, is commutative, because the corresponding 3-dimensional diagram in which each $\mathrm{Tor}_B(M, R)$ is replaced by $R \leftarrow B \rightarrow M$ is homotopy commutative in DASH.

Therefore, the R-module isomorphism $\mathrm{Tor}_{HB}(HM, R) \to \mathrm{Tor}_B(M, R)$ is multiplicative with respect to the multiplicative structure given.
The proof is complete.

4. Remarks and examples

When R is a field of zero characteristic, one can avoid appealing to the category DASH by using a commutative cochain model in the argument above.
One can also avoid using DASH when H is a trivial subgroup, as in [Buchstaber-Panov15, Ex. 8.1.12].

Examples of quotients $\mathcal{Z}_\mathcal{K}/H$ include compact toric manifolds M (when H has maximal possible dimension).
In this case $R[\mathcal{K}]$ is a free $H^*(B(T^m/H))$-module, and Theorem 1 reduces to the well-known description of the cohomology:

$$H^*(M;R) \cong \mathrm{Tor}_{H^*(B(T^m/H);R)}(R[\mathcal{K}],R) = R[\mathcal{K}]/p^*H^*(B(T^m/H)).$$

Another series of examples are 'projective' moment-angle manifolds $\mathcal{Z}_\mathcal{K}/S^1_d$ corresponding to the diagonal subcircle $H = S^1_d \subset T^m$. When $\mathcal{K}$ is the boundary of a polytope, $\mathcal{Z}_\mathcal{K}/S^1_d$ admits a complex-analytic structure as an LVM-manifold.

In this case Theorem 1 together with the Koszul resolution gives the following isomorphism:

$$H^*(\mathcal{Z}_\mathcal{K}/S^1_d) \cong H(\Lambda[t_1, \ldots, t_{m-1}] \otimes R[\mathcal{K}], d)$$

where the cohomology of the differential graded algebra on the right hand side is taken with respect to $dt_i = v_i - v_m$, $dv_j = 0$, $\deg t_i = 1$.

There is also a similar description of the Dolbeault cohomology of the complex quotients $\mathcal{Z}_\mathcal{K}/H$ [Panov-Ustinovsky12].

References

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