On toric generators in the unitary and special unitary bordism rings

based on joint work with Zhi Lu (Fudan University)

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1. Unitary bordism

Elements of the unitary bordism ring Ω^U are the complex bordism classes of stably complex manifolds. A stably complex manifold is a pair $(M, c_{\mathcal T})$ consisting of a smooth manifold M and a stably complex structure $c_{\mathcal T}$, where the latter is determined by a choice of an isomorphism

$$c_{\mathcal T}\colon \mathcal{T}M \oplus \underline{\mathbb{R}}^N \xrightarrow{\cong} \xi$$

between the stable tangent bundle of M and a complex vector bundle ξ.

Theorem (Milnor–Novikov)

- *Two stably complex manifold M and N represent the same bordism classes in Ω^U iff their sets of Chern characteristic numbers coincide.*
- *Ω^U is a polynomial ring on generators in every even degree:*

$$\Omega^U \cong \mathbb{Z}[a_i, i > 0], \quad \deg a_i = 2i.$$

Polynomial generators of Ω^U can be detected using a special characteristic class s_n. It is the polynomial in the universal Chern classes $c_1, \ldots, c_n$ obtained by expressing the symmetric polynomial $x_1^n + \cdots + x_n^n$ via the elementary symmetric functions $\sigma_i(x_1, \ldots, x_n)$ and replacing each σ_i by c_i.

$s_n[M] = s_n(\mathcal{T}M)\langle M\rangle$: the corresponding characteristic number.

Theorem

The bordism class of a stably complex manifold M^{2i} may be taken to be the polynomial generator $a_i \in \Omega^U_{2i}$ iff

$$s_i[M^{2i}] = \begin{cases} \pm 1 & \text{if} \quad i+1 \neq p^s \quad \text{for any prime } p, \\ \pm p & \text{if} \quad i+1 = p^s \quad \text{for some prime } p \text{ and integer } s > 0. \end{cases}$$

Problem

Find nice geometric representatives in (unitary) bordism classes; e.g., smooth algebraic varieties and/or manifolds with large symmetry groups.

2. Toric manifolds

A toric variety is a normal complex algebraic variety V containing an algebraic torus $(\mathbb{C}^\times)^n$ as a Zariski open subset in such a way that the natural action of $(\mathbb{C}^\times)^n$ on itself extends to an action on V.
We only consider nonsingular complete (compact in the usual topology) toric varieties, also known as toric manifolds.

Projective toric manifolds V are determined by simple n-dimensional lattice polytopes P. Irreducible torus-invariant divisors on V (or connected torus-invariant submanifolds of codimension 2) are the toric subvarieties of complex codimension 1 corresponding to the facets of P.

We assume that there are m facets of P, denote the corresponding inward-pointing normal primitive vectors by $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$, and denote the corresponding codimension-1 subvarieties by $V_1, \ldots, V_m$.
Each $\boldsymbol{a}_i$ defines a one-dimensional subgroup of the torus, which fixes pointwise the corresponding subvariety V_i.

Theorem (Danilov–Jurkiewicz)

Let V be a toric manifold of complex dimension n. The cohomology ring $H^*(V;\mathbb{Z})$ is generated by the degree-two classes v_i dual to the invariant submanifolds V_i, and is given by

$$H^*(V;\mathbb{Z}) \cong \mathbb{Z}[v_1,\ldots,v_m]/\mathcal{I}, \qquad \deg v_i = 2,$$

where $\mathcal{I}$ is the ideal generated by elements of the following two types:

- $v_{i_1}\cdots v_{i_k}$ *such that the facets $i_1,\ldots,i_k$ do not intersect in P;*
- $\displaystyle\sum_{i=1}^{m}\langle \boldsymbol{a}_i,\boldsymbol{x}\rangle v_i$, *for any vector $\boldsymbol{x}\in\mathbb{Z}^n$.*

It is convenient to consider the integer $n \times m$-matrix

$$\Lambda = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$$

whose columns are the vectors $\boldsymbol{a}_i$ written in the standard basis of $\mathbb{Z}^n$.
Then the n linear forms $a_{j1}v_1 + \cdots + a_{jm}v_m$ corresponding to the rows of Λ vanish in $H^*(V;\mathbb{Z})$.

Theorem

There is the following isomorphism of complex vector bundles:

$$\mathcal{T}V \oplus \underline{\mathbb{C}}^{m-n} \cong \rho_1 \oplus \cdots \oplus \rho_m,$$

where $\mathcal{T}V$ is the tangent bundle, $\underline{\mathbb{C}}^{m-n}$ is the trivial $(m-n)$-plane bundle, and ρ_i is the line bundle corresponding to V_i, with $c_1(\rho_i) = v_i$.
In particular, the total Chern class of V is given by

$$c(V) = (1 + v_1) \cdots (1 + v_m).$$

Example

The complex projective space $\mathbb{C}P^n$ is the toric manifold, whose corresponding polytope is an n-simplex $P = \Delta^n$, and

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

The cohomology ring $H^*(\mathbb{C}P^n)$ is given by

$$\mathbb{Z}[v_1, \ldots, v_{n+1}]/(v_1 \cdots v_{n+1}, v_1 - v_{n+1}, \ldots, v_n - v_{n+1}) \cong \mathbb{Z}[v]/(v^{n+1}),$$

where v is any of the v_i, and

$$\mathcal{T}\mathbb{C}P^n \oplus \underline{\mathbb{C}} \cong \bar{\eta} \oplus \cdots \oplus \bar{\eta} \qquad (n+1 \text{ summands}),$$

where η is the tautological (Hopf) line bundle over $\mathbb{C}P^n$, and $\bar{\eta}$ is its conjugate, or the line bundle corresponding to a hyperplane $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$.

Example

Given $n_1, n_2 > 0$ and a sequence of integers $(i_1, \ldots, i_{n_2})$, define

$$V = \mathbb{C}P(\eta^{\otimes i_1} \oplus \cdots \oplus \eta^{\otimes i_{n_2}} \oplus \underline{\mathbb{C}}),$$

where $\eta^{\otimes i}$ denotes the ith tensor power of η over $\mathbb{C}P^{n_1}$ when $i \geqslant 0$ and the ith tensor power of $\bar{\eta}$ otherwise. Then V is the total space of a bundle over $\mathbb{C}P^{n_1}$ with fibre $\mathbb{C}P^{n_2}$. It is a projective toric manifold with

$$\varLambda = \begin{pmatrix} \overbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{matrix}}^{n_1} & \begin{matrix} -1 \\ \vdots \\ -1 \end{matrix} & & \text{\Large 0} & \\ & \text{\Large 0} & \begin{matrix} i_1 \\ \vdots \\ i_{n_2} \end{matrix} & \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{matrix}}_{n_2} & \begin{matrix} -1 \\ \vdots \\ -1 \end{matrix} \end{pmatrix}$$

The polytope P here is combinatorially equivalent to a product $\Delta^{n_1} \times \Delta^{n_2}$.

Example (continued)

The cohomology of V is

$$H^*(V) \cong \mathbb{Z}[v_1, \ldots, v_{n_1+1}, v_{n_1+2}, \ldots, v_{n_1+n_2+2}]/\mathcal{I},$$

where $\mathcal{I}$ is generated by the elements

$$v_1 \cdots v_{n_1+1},\; v_{n_1+2} \cdots v_{n_1+n_2+2},\; v_1 - v_{n_1+1}, \ldots, v_{n_1} - v_{n_1+1},$$
$$i_1 v_{n_1+1} + v_{n_1+2} - v_{n_1+n_2+2}, \ldots, i_{n_2} v_{n_1+1} + v_{n_1+n_2+1} - v_{n_1+n_2+2}.$$

In other words,

$$H^*(V) \cong \mathbb{Z}[u, v]/\bigl(u^{n_1+1}, v(v - i_1 u)\cdots(v - i_{n_2} u)\bigr),$$

where $u = v_1 = \cdots = v_{n_1+1}$ and $v = v_{n_1+n_2+2}$.

The total Chern class is

$$c(V) = (1+u)^{n_1+1}(1 + v - i_1 u)\cdots(1 + v - i_{n_2} u)(1+v).$$

3. Toric representatives in unitary bordism classes

The classical family of generators for Ω^U is formed by the Milnor hypersufaces $H(n_1, n_2)$. Each $H(n_1, n_2)$ is a hyperplane section of the Segre embedding $\mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2} \to \mathbb{C}P^{(n_1+1)(n_2+1)-1}$ and may be given explicitly by the equation

$$z_0 w_0 + \cdots + z_{n_1} w_{n_1} = 0$$

in the homogeneous coordinates $[z_0 : \cdots : z_{n_1}] \in \mathbb{C}P^{n_1}$ and $[w_0 : \cdots : w_{n_2}] \in \mathbb{C}P^{n_2}$, assuming that $n_1 \leqslant n_2$.

Also, $H(n_1, n_2)$ can be identified with the projectivisation $\mathbb{C}P(\zeta)$ of a certain n_2-plane bundle over $\mathbb{C}P^{n_1}$. The bundle ζ is not a sum of line bundles when $n_1 > 1$, so $H(n_1, n_2)$ is *not* a toric manifold in this case.

Buchstaber and Ray introduced a family $B(n_1, n_2)$ of *toric* generators of Ω^U. Each $B(n_1, n_2)$ is the projectivisation of a sum of n_2 line bundles over the bounded flag manifold BF_{n_1}. Then $B(n_1, n_2)$ is a toric manifold, because BF_{n_1} is toric and the projectivisation of a sum of line bundles over a toric manifold is toric.

We have $H(0, n_2) = B(0, n_2) = \mathbb{C}P^{n_2-1}$, so $s_{n_2-1}[H(0, n_2)] = s_{n_2-1}[B(0, n_2)] = n_2$. Furthermore,

$$s_{n_1+n_2-1}[H(n_1, n_2)] = s_{n_1+n_2-1}[B(n_1, n_2)] = -\binom{n_1+n_2}{n_1} \quad \text{for } n_1 > 1.$$

The fact that each of the families $\{[H(n_1, n_2)]\}$ and $\{[B(n_1, n_2)]\}$ generates the unitary bordism ring Ω^U follows from the well-known identity

$$\gcd\left\{\binom{n}{i},\ 0 < i < n\right\} = \begin{cases} 1 & \text{if} \quad n \neq p^s \quad \text{for any prime } p, \\ p & \text{if} \quad n = p^s \quad \text{for a prime } p \text{ and } s > 0. \end{cases}$$

We proceed to describing another family of toric generators for Ω^U.

Given two positive integers n_1, n_2, define

$$L(n_1, n_2) = \mathbb{C}P(\eta \oplus \underline{\mathbb{C}}^{n_2}),$$

where η is the tautological line bundle over $\mathbb{C}P^{n_1}$.
It is a projective toric manifold with

$$\Lambda = \begin{pmatrix} \overbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{matrix}}^{n_1} & \begin{matrix} -1 \\ \vdots \\ -1 \end{matrix} & & \text{\huge 0} & \\ & & 1 & \begin{matrix} 1 & 0 & 0 \end{matrix} & -1 \\ \text{\huge 0} & & 0 & \begin{matrix} 0 & \ddots & 0 \end{matrix} & \vdots \\ & & 0 & \underbrace{\begin{matrix} 0 & 0 & 1 \end{matrix}}_{n_2} & -1 \end{pmatrix}$$

The cohomology ring is given by

$$H^*\big(L(n_1,n_2)\big) \cong \mathbb{Z}[u,v]/\big(u^{n_1+1}, v^{n_2+1} - uv^{n_2}\big)$$

with $u^{n_1}v^{n_2}\langle L(n_1,n_2)\rangle = 1$.

There is an isomorphism of complex bundles

$$\mathcal{T}L(n_1,n_2) \oplus \underline{\mathbb{C}}^2 \cong \underbrace{p^*\bar\eta \oplus \cdots \oplus p^*\bar\eta}_{n_1+1} \oplus (\bar\gamma \otimes p^*\eta) \oplus \underbrace{\bar\gamma \oplus \cdots \oplus \bar\gamma}_{n_2},$$

where γ is the tautological line bundle over $L(n_1,n_2) = \mathbb{C}P(\eta \oplus \underline{\mathbb{C}}^{n_2})$.

The total Chern class is

$$c\big(L(n_1,n_2)\big) = (1+u)^{n_1+1}(1+v-u)(1+v)^{n_2}$$

with $u = c_1(p^*\bar\eta)$ and $v = c_1(\bar\gamma)$.

We also set $L(n_1,0) = \mathbb{C}P^{n_1}$ and $L(0,n_2) = \mathbb{C}P^{n_2}$.

Lemma

For $n_2 > 0$, we have

$$s_{n_1+n_2}\left[L(n_1,n_2)\right] = \binom{n_1+n_2}{0} - \binom{n_1+n_2}{1} + \cdots + (-1)^{n_1}\binom{n_1+n_2}{n_1} + n_2.$$

Theorem (Lu-P.)

The bordism classes $[L(n_1,n_2)] \in \Omega^U_{2(n_1+n_2)}$ generate the ring Ω^U.

Proof. $s_{n_1+n_2}\left[L(n_1,n_2) - 2L(n_1-1,n_2+1) + L(n_1-2,n_2+2)\right]$

$$= (-1)^{n_1-1}\binom{n_1+n_2}{n_1-1} + (-1)^{n_1}\binom{n_1+n_2}{n_1} - 2(-1)^{n_1-1}\binom{n_1+n_2}{n_1-1} = (-1)^{n_1}\binom{n_1+n_2+1}{n_1}$$

It follows that any unitary bordism class can be represented by a disjoint union of products of projective toric manifolds. Products of toric manifolds are toric, but disjoint unions are not, as toric manifolds are connected.

A disjoint union may be replaced by a connected sum, representing the same bordism class. However, connected sum is not an algebraic operation, and a connected sum of two algebraic varieties is rarely algebraic.

Connected representatives in all bordism classes can be constructed by appealing to quasitoric manifolds [Davis and Januszkiewicz], which provide a topological generalisation of projective toric manifolds.

A quasitoric manifold is a smooth $2n$-dimensional closed manifold M with a locally standard action of a (compact) torus T^n whose quotient M/T^n is a simple polytope. An omniorientation of a quasitoric manifold provides it with an intrinsic stably complex structure.

One can form equivariant connected sum of quasitoric manifolds, but the resulting invariant stably complex structure does not represent the cobordism sum of the two original manifolds. A more intricate connected sum construction is needed, as described in [Buchstaber, P. and Ray].

The conclusion, which can be derived from the above construction and any of the toric generating sets $\{B(n_1, n_2)\}$ or $\{L(n_1, n_2)\}$ for Ω^U, is as follows:

Theorem (Buchstaber-P.-Ray)

In dimensions > 2, every unitary bordism class contains a quasitoric manifold, necessarily connected, whose stably complex structure is induced by an omniorientation, and is therefore compatible with the torus action.

4. Special unitary bordism

A stably complex manifold $(M, c_{\mathcal{T}})$ is special unitary (an SU-manifold) if $c_1(M) = 0$. Bordism classes of SU-manifolds form the special unitary bordism ring $\varOmega^{SU}$.

The ring structure of $\varOmega^{SU}$ is more subtle than that of $\varOmega^{U}$. Novikov described $\varOmega^{SU} \otimes \mathbb{Z}[\frac{1}{2}]$ (it is a polynomial ring). The 2-torsion was described by Conner and Floyd. We shall need the following facts.

Theorem

- *The kernel of the forgetful map $\varOmega^{SU} \to \varOmega^{U}$ consists of torsion.*
- *Every torsion element in $\varOmega^{SU}$ has order 2.*
- *$\varOmega^{SU} \otimes \mathbb{Z}[\frac{1}{2}]$ is a polynomial algebra on generators in every even degree > 2:*

$$\varOmega^{SU} \otimes \mathbb{Z}[\tfrac{1}{2}] \cong \mathbb{Z}[\tfrac{1}{2}][y_i \colon i > 1], \quad \deg y_i = 2i.$$

Let $\partial\colon \Omega^U_{2n} \to \Omega^U_{2n-2}$ be the homomorphism sending a bordism class $[M^{2n}]$ to the bordism class $[V^{2n-2}]$ of a submanifold $V^{2n-2} \subset M$ dual to $c_1(M)$. It satisfies

$$\partial(a \cdot b) = a \cdot \partial b + \partial a \cdot b - [\mathbb{C}P^1] \cdot \partial a \cdot \partial b.$$

Let $\mathcal{W}_{2n}$ be the subgroup of Ω^U_{2n} consisting of bordism classes $[M^{2n}]$ such that every Chern number of M^{2n} of which c_1^2 is a factor vanishes. The restriction of the boundary homomorphism $\partial\colon \mathcal{W}_{2n} \to \mathcal{W}_{2n-2}$ is defined.

The direct sum $\mathcal{W} = \bigoplus_{i\geqslant 0} \mathcal{W}_{2i}$ is *not* a subring of Ω^U: one has $[\mathbb{C}P^1] \in \mathcal{W}_2$, but $c_1^2[\mathbb{C}P^1 \times \mathbb{C}P^1] = 8 \neq 0$, so $[\mathbb{C}P^1] \times [\mathbb{C}P^1] \notin \mathcal{W}_4$.

$\mathcal{W}$ is a commutative ring with respect to the twisted product

$$a * b = a \cdot b + 2[V^4] \cdot \partial a \cdot \partial b,$$

where $\cdot$ denotes the product in Ω^U and V^4 is a stably complex manifold with $c_1^2[V^4] = -1$, e.g. $V^4 = \mathbb{C}P^1 \times \mathbb{C}P^1 - \mathbb{C}P^2$.

Set

$$m_i = \begin{cases} 1 & \text{if} \quad i+1 \neq p^s \quad \text{for any prime } p, \\ p & \text{if} \quad i+1 = p^s \quad \text{for some prime } p \text{ and integer } s > 0, \end{cases}$$

so that $[M^{2i}] \in \Omega^U_{2i}$ represents a polynomial generator iff $s_i[M^{2i}] = \pm m_i$.

Theorem

$\mathcal{W}$ is a polynomial ring on generators in every even degree except 4:

$$\mathcal{W} \cong \mathbb{Z}[x_1, x_i \colon i > 2], \quad x_1 = [\mathbb{C}P^1], \quad \deg x_i = 2i,$$

with $s_i[x_i] = m_i m_{i-1}$ and the boundary operator $\partial \colon \mathcal{W} \to \mathcal{W}$, $\partial^2 = 0$, given by $\partial x_1 = 2$, $\partial x_{2i} = x_{2i-1}$, and satisfying the identity $\partial(a * b) = a * \partial b + \partial a * b - x_1 * \partial a * \partial b$.

Theorem

There is an exact sequence of groups

$$0 \longrightarrow \Omega^{SU}_{2n-1} \xrightarrow{\theta} \Omega^{SU}_{2n} \xrightarrow{\alpha} \mathcal{W}_{2n} \xrightarrow{\beta} \Omega^{SU}_{2n-2} \xrightarrow{\theta} \Omega^{SU}_{2n-1} \longrightarrow 0,$$

where θ is the multiplication by the generator $\theta \in \Omega^{SU}_1 \cong \mathbb{Z}_2$, α is the forgetful homomorphism, and $\alpha\beta = -\partial$.

We have

$$\mathcal{W} \otimes \mathbb{Z}[\tfrac{1}{2}] \cong \mathbb{Z}[\tfrac{1}{2}][x_1, x_{2k-1}, 2x_{2k} - x_1 x_{2k-1} \colon k > 1],$$

where $x_1^2 = x_1 * x_1$ is a ∂-cycle, and each x_{2k-1}, $2x_{2k} - x_1 x_{2k-1}$ is a ∂-cycle.

Theorem

There exist elements $y_i \in \Omega^{SU}_{2i}$, $i > 1$, such that $s_2(y_2) = -48$ and

$$s_i(y_i) = \begin{cases} m_i m_{i-1} & \text{if } i \text{ is odd}, \\ 2m_i m_{i-1} & \text{if } i \text{ is even and } i > 2. \end{cases}$$

These elements are mapped as follows under the forgetful homomorphism $\alpha \colon \Omega^{SU} \to \mathcal{W}$:

$$y_2 \mapsto 2x_1^2, \quad y_{2k-1} \mapsto x_{2k-1}, \quad y_{2k} \mapsto 2x_{2k} - x_1 x_{2k-1}, \quad k > 1.$$

In particular, $\Omega^{SU} \otimes \mathbb{Z}[\tfrac{1}{2}]$ embeds into $\mathcal{W} \otimes \mathbb{Z}[\tfrac{1}{2}]$ as the polynomial subring generated by x_1^2, x_{2k-1} and $2x_{2k} - x_1 x_{2k-1}$.

5. Toric generators of the SU-bordism ring

Proposition

An omnioriented quasitoric manifold M has $c_1(M) = 0$ if and only if there exists a linear function $\varphi\colon \mathbb{Z}^n \to \mathbb{Z}$ such that $\varphi(\lambda_i) = 1$ for $i = 1, \dots, m$. Here the λ_i are the columns of characteristic matrix.
In particular, if some n vectors of $\lambda_1, \dots, \lambda_m$ form the standard basis $\mathbf{e}_1, \dots, \mathbf{e}_n$, then M is SU iff the column sums of Λ are all equal to 1.

Corollary

A toric manifold V cannot be SU.

Proof. If $\varphi(\lambda_i) = 1$ for all i, then the vectors λ_i lie in the positive halfspace of φ, so they cannot span a complete fan.

Theorem (Buchstaber-P.-Ray)

A quasitoric SU-manifold M^{2n} represents 0 in Ω^U_{2n} whenever $n < 5$.

Example

Assume that $n_1 = 2k_1$ is positive even and $n_2 = 2k_2 + 1$ is positive odd, and consider the manifold $L(n_1, n_2) = \mathbb{C}P(\eta \oplus \underline{\mathbb{C}}^{n_2})$. We change its stably complex structure to the following:

$$
\begin{aligned}
&\mathcal{T}L(n_1, n_2) \oplus \underline{\mathbb{R}}^4 \\
&\cong \underbrace{p^*\bar\eta \oplus p^*\eta \oplus \cdots \oplus p^*\bar\eta \oplus p^*\eta}_{2k_1} \oplus p^*\bar\eta \oplus (\bar\gamma \otimes p^*\eta) \oplus \underbrace{\bar\gamma \oplus \gamma \oplus \cdots \oplus \bar\gamma \oplus \gamma}_{2k_2} \oplus \gamma
\end{aligned}
$$

and denote the resulting stably complex manifold by $\widetilde{L}(n_1, n_2)$. It has

$$c\bigl(\widetilde{L}(n_1, n_2)\bigr) = (1 - u^2)^{k_1}(1 + u)(1 + v - u)(1 - v^2)^{k_2}(1 - v),$$

so $\widetilde{L}(n_1, n_2)$ is an SU-manifold of dimension $2(n_1 + n_2) = 4(k_1 + k_2) + 2$.

Example (continued)

$\widetilde{L}(n_1, n_2)$ is an omnioriented quasitoric manifold over $\Delta^{n_1} \times \Delta^{n_2}$ corresponding to the matrix

$$\Lambda = \begin{pmatrix} \overbrace{\begin{matrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{matrix}}^{n_1=2k_1} & \begin{matrix} 1 \\ -1 \\ \vdots \\ 1 \\ -1 \\ 1 \end{matrix} & \text{\Large 0} \\ \text{\Large 0} & & \underbrace{\begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{matrix}}_{n_2=2k_2+1} \begin{matrix} 1 \\ -1 \\ \vdots \\ 1 \end{matrix} \end{pmatrix}$$

The columns sum of this matrix are 1 by inspection.

Example

Given positive even $n_1 = 2k_1$ and odd $n_2 = 2k_2 + 1$, consider the omnioriented quasitoric manifold $\widetilde{N}(n_1, n_2)$ over $\Delta^1 \times \Delta^{n_1} \times \Delta^{n_2}$ with

$$
\Lambda = \left(\begin{array}{cccccccccccccc}
1 & 1 & & & & & & & & & & & & \\
 & & 1 & 0 & 0 & \cdots & 0 & 1 & & & & & & \\
 & & 0 & 1 & 0 & \cdots & 0 & -1 & & & & & & \\
0 & & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & & 0 & & & \\
 & & 0 & 0 & 0 & 1 & 0 & 1 & & & & & & \\
 & & \multicolumn{5}{c}{\underbrace{\begin{matrix}0 & 0 & 0 & 0 & 1\end{matrix}}_{n_1=2k_1}} & -1 & & & & & & \\
 & -1 & & & & & & 0 & 1 & 0 & 0 & \cdots & 0 & 1 \\
 & 1 & & & & & & 0 & 0 & 1 & 0 & \cdots & 0 & -1 \\
 & 0 & & & & & & 1 & 0 & 0 & 1 & \cdots & 0 & 1 \\
 & & & & 0 & & & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 & & & & & & & & \multicolumn{5}{c}{\underbrace{\begin{matrix}0 & 0 & 0 & 0 & 1\end{matrix}}_{n_2=2k_2+1}} & 1
\end{array}\right)
$$

The column sums are 1 by inspection, so $\widetilde{N}(n_1, n_2)$ is a quasitoric SU-manifold of dimension $2(1 + n_1 + n_2) = 4(k_1 + k_2) + 4$.

Lemma

- *For $k > 1$, there is a linear combination y_{2k+1} of SU-bordism classes $[\widetilde{L}(n_1, n_2)]$ with $n_1 + n_2 = 2k + 1$ such that $s_{2k+1}(y_{2k+1}) = m_{2k+1}m_{2k}$.*
- *For $k > 2$, there is a linear combination y_{2k} of SU-bordism classes $[\widetilde{N}(n_1, n_2)]$ with $n_1 + n_2 + 1 = 2k$ such that $s_{2k}(y_{2k}) = 2m_{2k}m_{2k-1}$.*

Proof. Calculating the characteristic numbers, we get

$$s_{n_1+n_2}\left[\widetilde{L}(n_1, n_2)\right] = -\tbinom{n_1+n_2}{1} + \tbinom{n_1+n_2}{2} + \cdots - \tbinom{n_1+n_2}{n_1-1} + \tbinom{n_1+n_2}{n_1}.$$

$$s_{n_1+n_2+1}\left[\widetilde{N}(n_1, n_2)\right] = 2\left(-n_1 - \tbinom{2k}{1} + \tbinom{2k}{2} + \cdots - \tbinom{2k}{n_1-1} + \tbinom{2k}{n_1}\right).$$

It follows that

$$s_{n_1+n_2}\left[\widetilde{L}(n_1, n_2) - \widetilde{L}(n_1 - 2, n_2 + 2)\right] = \tbinom{n_1+n_2}{n_1} - \tbinom{n_1+n_2}{n_1-1}.$$

Now the first statement of the lemma follows from

$$\gcd\left\{\tbinom{2k+1}{2i} - \tbinom{2k+1}{2i-1},\ 0 < i \leqslant k\right\} = m_{2k+1}m_{2k}.$$

The second statement is proved using similar (but more involved) identities.

The main result is as follows:

Theorem (Lu-P.)

There exist quasitoric SU-manifolds M^{2i}, $i \geqslant 5$, with $s_i(M^{2i}) = m_i m_{i-1}$ if i is odd and $s_i(M^{2i}) = 2m_i m_{i-1}$ if i is even. These quasitoric manifolds represent polynomial generators of $\Omega^{SU} \otimes \mathbb{Z}[\frac{1}{2}]$.

References

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