Bott Towers and Equivariant Cobordism

based on joint works with Victor Buchstaber and Nigel Ray

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1. Bounded flag manifolds and Bott towers

A bounded flag in $\mathbb{C}^{n+1}$ is

$$\mathcal{U} = \{U_1 \subset U_2 \subset \cdots \subset U_{n+1} = \mathbb{C}^{n+1}, \quad \dim U_i = i\}$$

such that $U_k \supset \mathbb{C}^{k-1} = \langle \boldsymbol{e}_1, \ldots, \boldsymbol{e}_{k-1} \rangle$, $k = 2, \ldots, n$.

Denote by BF_n the set of all bounded flags in $\mathbb{C}^{n+1}$.

Theorem

BF_n is a smooth compact toric variety under the action of the torus $(\mathbb{C}^\times)^n$

$$(\mathbb{C}^\times)^n \times BF_n \to BF_n$$

$$(t_1, \ldots, t_n) \cdot (w_1, \ldots, w_n, w_{n+1}) = (t_1 w_1, \ldots, t_n w_n, w_{n+1}),$$

BF_n bounded flag manifold.

ξ_n tautological line bundle over BF_n, whose fibre over $\mathcal{U}$ is $U_1 \cong \mathbb{C}$.

Proposition

$BF_n = \mathbb{C}P(\underline{\mathbb{C}} \oplus \xi_{n-1})$, where ξ_{n-1} is over BF_{n-1}.

Proof.

Consider

$$BF_n \to BF_{n-1}$$
$$\mathcal{U} \mapsto \mathcal{U}' = \mathcal{U}/\mathbb{C}^1 \quad \text{in } \mathbb{C}^{2,\ldots,n+1} = \mathbb{C}^n,$$

where $\mathcal{U}' = \{U_1' \subset U_2' \subset \cdots \subset U_{n-1}'\}$, $U_k' = U_{k+1}/\mathbb{C}^1$.
To recover $\mathcal{U}$ from $\mathcal{U}'$ one has to choose a line U_1 in $U_2 = \mathbb{C}^1 \oplus U_1'$. □

Get a sequence of fibre bundles with fibre $\mathbb{C}P^1$

$$BF_n \to BF_{n-1} \to \cdots \to BF_1 = \mathbb{C}P^1 \to pt$$

a Bott tower structure on BF_n.

A Bott tower is a sequence of fibre bundles

$$B_n \to B_{n-1} \to \cdots \to B_1 = \mathbb{C}P^1 \to pt$$

in which $B_k = \mathbb{C}P(\underline{\mathbb{C}} \oplus \eta_{k-1})$ for a line bundle η_{k-1} over B_{k-1}.

Theorem

$$H^*(B_n) \cong H^*(B_{n-1})[u_n]\big/\big(u_n^2 = c_1(\eta_{n-1})u_n\big),$$

where $u_n = c_1(\xi_n)$ and ξ_n is the tautological line bundle over B_n.

Example

When $\eta_k = \xi_k$ for each k, we get $B_n = BF_n$, the bounded flag manifold with the 'intrinsic' structure of a Bott tower. We have

$$H^*(BF_n) \cong H^*(BF_{n-1})[u_n]/(u_n^2 = u_{n-1}u_n).$$

2. Representing complex bordism classes

As a complex manifold BF_n, represents a $2n$-dimensional class in the complex bordism ring

$$\Omega^U = \{\text{stably complex manifolds}\}/\text{complex bordism relation}$$

Theorem (Milnor, Novikov'1960)

$$\Omega^U \cong \mathbb{Z}[a_1, a_2, \ldots], \quad \dim a_i = 2i.$$

A stably complex manifold M^{2n} can be taken as a representative of a_n iff

$$s_n[M^{2n}] = \begin{cases} \pm 1, & n \neq p^k - 1, \\ \pm p, & n = p^k - 1. \end{cases}$$

Here s_n is the characteristic class corresponding to the symmetric polynomial $x_1^n + \cdots + x_n^n$, where $c_n(\mathcal{T} M^{2n}) = (1 + x_1) \cdots (1 + x_n)$.

E.g., $s_n[\mathbb{C}P^n] = n + 1$, so $[\mathbb{C}P^1] = a_1$, $\mathbb{C}P^2 = [a_2]$, but $\mathbb{C}P^3 \neq [a_3]$.

Given $i \leqslant j$, consider $\mathbb{C}^{i+1} \subset \mathbb{C}^{j+1}$ and define the Milnor hypersurface

$$H_{ij} = \big\{(\ell, W)\colon \ \ \ell \text{ is a line in } \mathbb{C}^{i+1},\ W \text{ is a hyperplane in } \mathbb{C}^{j+1},\ \ell \subset W\big\}.$$

It is given by the equation $z_0 w_0 + \cdots + z_i w_i = 0$ in $\mathbb{C}P^i \times \mathbb{C}P^j \subset \mathbb{C}P^{ij+i+j}$.

E.g., $H_{22} = Fl(\mathbb{C}^3)$, complete flags in $\mathbb{C}^3$.

Proposition

$$s_{i+j-1}[H_{ij}] = \binom{i+j}{i}.$$

Therefore, $\{[H_{ij}],\ 0 \leqslant i \leqslant j\}$ generate the complex bordism ring Ω^U.

However, H_{ij} is not a toric manifold when $i \geqslant 2$.

Theorem (Buchstaber–Ray)

The complex bordism ring Ω^U has a generator set consisting of toric manifolds.

Proof.

Consider the manifolds

$$B_{ij} = \big\{(\mathcal{U}, W)\colon\ \ \mathcal{U} \text{ is a bounded flag in } \mathbb{C}^{i+1}, \\ W \text{ is a hyperplane in } \mathbb{C}^{j+1},\ U_1 \subset W\big\}.$$

$$\begin{array}{ccc} B_{ij} & \to & H_{ij} \\ \downarrow & & \downarrow \\ BF_i & \to & \mathbb{C}P^i \end{array} \qquad\qquad \begin{array}{ccc} (\mathcal{U}, W) & \mapsto & (U_1, W) \\ \downarrow & & \downarrow \\ \mathcal{U} & \mapsto & U_1 \end{array}$$

$B_{ij} = \mathbb{C}P(\text{sum of line bundles})$, so it is toric. □

Quasitoric manifolds generalise toric manifolds topologically. A quasitoric manifold M^{2n} has an action of a torus T^n with quotient a simple polytope P. Quasitoric manifolds have canonical T^n-invariant stably complex structures, but are not complex or almost complex in general.

Theorem (Buchstaber–P–Ray)

In dimensions > 2, every complex bordism class contains a quasitoric manifold.

It remains open whether ring generators a_i of the complex cobordism ring Ω^U can be represented by toric or quasitoric manifolds. A partial result on this problem has been recently obtained by A. Wilfong.

3. The universal toric genus

Given a T^k-manifold, one has a universal transformations between the three version of equivariant cobordism:

$$\underset{\text{geometric}}{U^*_{T^k}(X)} \quad \to \quad \underset{\text{homotopic}}{MU^*_{T^k}(X)} \quad \to \quad \underset{\text{Borel}}{U^*(ET^k \times_{T^k} X)}$$

For $X = pt$ one gets a homomorphism of Ω_U-modules

$$\Phi \colon \Omega_{U:T^k} \to U^*(BT^k) = \Omega_U[[u_1, \ldots, u_k]]$$

called the universal toric genus.

It assigns to the equivariant cobordism class of a T^k-manifold M the 'cobordism class' of the map $ET^k \times_{T^k} M \to BT^k$.

We have

$$\Phi(M) \;=\; [M] + \sum_{\omega\colon\, |\omega|>0} g_\omega(M)\, u^\omega,$$

in $U^*(BT^k) = \Omega_U[[u_1,\dots,u_k]]$, where $[M] \in \Omega_U$, $u^\omega = u_1^{\omega_1}\cdots u_k^{\omega_k}$.

What are the coefficients $g_\omega(M)$?

The bounded flag manifold BF_n is the quotient of

$$(S^3)^n = \{(z_1, \dots, z_{2n}) \in \mathbb{C}^{2n} \colon |z_k|^2 + |z_{k+n}|^2 = 1,\ 1 \leqslant k \leqslant n\}$$

by the T^n-action given by

$$(z_1, \dots, z_{2n}) \mapsto (t_1 z_1,\ t_1^{-1} t_2 z_2,\ \dots,\ t_{n-1}^{-1} t_n z_n,\ t_1 z_{n+1},\ t_2 z_{n+2},\ \dots,\ t_n z_{2n})$$

This gives the stable splitting

$$\mathcal{T}(BF_n) \oplus \underline{\mathbb{C}}^n \cong \bar{\xi}_1 \oplus \xi_1 \bar{\xi}_2 \oplus \dots \oplus \xi_{n-1} \bar{\xi}_n \oplus \bar{\xi}_1 \oplus \bar{\xi}_2 \oplus \dots \oplus \bar{\xi}_n$$

where ξ_k is the tautological line bundle over BF_k pulled back to BF_n.

E.g., for $n = 1$ we obtain the standard isomorphism $\mathcal{T}\mathbb{C}P^1 \oplus \underline{\mathbb{C}} \cong \bar{\xi} \oplus \bar{\xi}$, where $\xi = \xi_1$ is the tautological line bundle.

Now we twist the torus action on $(S^3)^n$ as follows:

$$(z_1,\ldots,z_{2n}) \mapsto (t_1 z_1, t_1^{-1} t_2 z_2, \ldots, t_{n-1}^{-1} t_n z_n, t_1^{-1} z_{n+1}, t_2^{-1} z_{n+2}, \ldots, t_n^{-1} z_{2n}).$$

This gives the splitting

$$\mathcal{T}(BF_n) \oplus \underline{\mathbb{R}}^{2n} \cong \bar{\xi}_1 \oplus \xi_1 \bar{\xi}_2 \oplus \cdots \oplus \xi_{n-1} \bar{\xi}_n \oplus \xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_n,$$

and the corresponding complex bordism class is zero in Ω^U_{2n}, as an iterated sphere bundle.

We denote by $\beta_n \in \Omega_{2n}(\mathbb{C}P^\infty)$ the bordism class of $\widetilde{BF}_n \xrightarrow{\xi_n} \mathbb{C}P^\infty$.

Theorem (Ray)

The bordism classes $\{\beta_n \colon n \geqslant 0\}$ form a basis of the free Ω_U-module $U_*(\mathbb{C}P^\infty)$ which is dual to the basis $\{u^k \colon k \geqslant 0\}$ of the Ω_U-module $U^*(\mathbb{C}P^\infty) = \Omega_U[[u]]$.

Similarly, define $\beta_\omega \in \Omega_{2|\omega|}(BT^k)$ the bordism class of $\widetilde{BF}_\omega = \widetilde{BF}_{\omega_1} \times \cdots \times \widetilde{BF}_{\omega_n} \to BT^k$.

Given a T^k-manifold M, define the bundle

$$G_\omega(M) = (S^3)^\omega \times_{T^\omega} M \longrightarrow \widetilde{BF}_\omega = (S^3)^\omega / T^\omega.$$

Theorem (Buchstaber-P-Ray)

The manifold $G_\omega(M)$ represents the coefficient $g_\omega(M)$ in the expansion of the universal toric genus.

4. Rigidity and fibre multiplicativity

A genus is a homomorphism $\varphi\colon \Omega_U \to R$ where R is a commutative ring with unit (usually $\mathbb{Z}$).

By the Hirzebruch correspondence, a genus φ is determined by a series

$$f(x) = x + \cdots \in R \otimes \mathbb{Q}[[x]].$$

Namely,

$$\varphi(M) = \Big\langle \prod_{i=1}^{n} \frac{x_i}{f(x_i)}, [M] \Big\rangle,$$

where $c(\mathcal{T}M) = (1 + x_1) \cdots (1 + x_n)$.

Given a genus $\varphi\colon \Omega_U \to R$, define its equivariant extension

$$\varphi^T\colon \Omega_{U:T^k} \xrightarrow{\Phi} \Omega_U[[u_1,\dots,u_k]] \xrightarrow{h_\varphi} R\otimes\mathbb{Q}[[x_1,\dots,x_k]]$$

mapping $[M]\mapsto\varphi(M)$ and $u_i\mapsto f(x_i)$.

Definition

A genus φ is rigid on M if $\varphi^T=\varphi$ (a constant).

A genus φ is fibre multiplicative with respect to M if

$$\varphi(N)=\varphi(M)\varphi(B)$$

for any fibre bundle $N\to M\to B$ with structure group G of positive rank.

Theorem (Buchstaber-P-Ray)

A genus φ is rigid on M iff it is fibre multiplicative with respect to M.

Proof.

Use the expansion $\Phi(M)=[M]+\cdots$ with coefficients represented by $G_\omega(M)$, a bundle over null-bordant base $\widetilde{BF}_\omega$. □

References

- Victor M. Buchstaber and Taras E. Panov. *Toric Topology*. A book project; arXiv:1210.2368.
- Victor M. Buchstaber, Taras E. Panov and Nigel Ray. *Toric genera*. Internat. Math. Res. Notices 2010, no. 16, 3207–3262.
- Victor M. Buchstaber, Taras E. Panov and Nigel Ray. *Spaces of polytopes and cobordism of quasitoric manifolds*. Moscow Math. J. **7** (2007), no. 2, 219–242.