On the rational formality of toric spaces and polyhedral products

based on joint works with Nigel Ray

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1. Preliminaries

$\mathcal{K}$ a simplicial complex on $[m] = \{1, \dots, m\}$.
$I = \{i_1, \dots, i_k\} \in \mathcal{K}$ a simplex. Always assume $\varnothing \in \mathcal{K}$.

- $\mathrm{CAT}(\mathcal{K})$: category of $\mathcal{K}$ (simplices $I \in \mathcal{K}$ and inclusions $I \subset J$);
- CDGA: commutative differential graded algebras over $\mathbb{Q}$;
- TOP: pointed topological spaces.

Given a sequence $\boldsymbol{C} = (C_1, \ldots, C_m)$ of cdga's, define the diagram

$$\mathcal{D}^{\mathcal{K}}(\boldsymbol{C})\colon \mathrm{CAT}(\mathcal{K})^{op} \to \mathrm{CDGA}, \qquad I \mapsto \bigotimes_{i \in I} C_i,$$

by mapping a morphism $I \subset J$ to the surjection $\bigotimes_{i\in J} C_i \to \bigotimes_{i\in I} C_i$ sending each C_i with $i \notin I$ to 1.

Proposition

Let $C_i = \mathbb{Q}[v]$, the polynomial algebra on one generator of degree 2. Then

$$\lim \mathcal{D}^{\mathcal{K}}(\boldsymbol{C}) = \mathbb{Q}[v_1, \ldots, v_m] / \big(v_{j_1} \cdots v_{j_k} \colon \{j_1, \ldots, j_k\} \notin \mathcal{K}\big),$$

the face ring (the Stanley–Reisner ring) of $\mathcal{K}$, denoted by $\mathbb{Q}[\mathcal{K}]$.

Example

Let $\mathcal{K} = \bullet\ \bullet$ (two points). Then $\mathbb{Q}[\mathcal{K}] = \mathbb{Q}[v_1, v_2]/(v_1 v_2)$.

Let $(\boldsymbol{X}, \boldsymbol{A}) = \{(X_1, A_1), \ldots, (X_m, A_m)\}$ be a collection of m pairs of spaces, $A_i \subset X_i$. For each subset $I = \{i_1, \ldots, i_k\} \subset [m]$, set

$$(\boldsymbol{X}, \boldsymbol{A})^I = \Big\{(x_1, \ldots, x_m) \in \prod_{j=1}^{m} X_j \colon\ x_j \in A_j \quad \text{for } j \notin I\Big\}.$$

Define the diagram

$$\begin{aligned} \mathcal{D}_{\mathcal{K}}(\boldsymbol{X}, \boldsymbol{A}) \colon \textsc{cat}(\mathcal{K}) &\longrightarrow \textsc{top}, \\ I &\longmapsto (\boldsymbol{X}, \boldsymbol{A})^I, \end{aligned}$$

which maps the morphism $I \subset J$ of $\textsc{cat}(\mathcal{K})$ to the inclusion of spaces $(\boldsymbol{X}, \boldsymbol{A})^I \subset (\boldsymbol{X}, \boldsymbol{A})^J$.

The polyhedral product of $(\boldsymbol{X}, \boldsymbol{A})$ corresponding to $\mathcal{K}$ is given by

$$(\boldsymbol{X}, \boldsymbol{A})^{\mathcal{K}} = \operatorname{colim} \mathcal{D}_{\mathcal{K}}(\boldsymbol{X}, \boldsymbol{A}) = \operatorname*{colim}_{I \in \mathcal{K}} (\boldsymbol{X}, \boldsymbol{A})^I.$$

$$(\boldsymbol{X}, \boldsymbol{A})^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (\boldsymbol{X}, \boldsymbol{A})^{I} = \bigcup_{I \in \mathcal{K}} \Bigl(\prod_{i \in I} X_i \times \prod_{i \notin I} A_i \Bigr).$$

Notation: $(\boldsymbol{X}, pt)^{\mathcal{K}} = \boldsymbol{X}^{\mathcal{K}}$.
If $X_i = X$ and $A_i = A$ for $i = 1, \ldots, m$, then $(\boldsymbol{X}, \boldsymbol{A})^{\mathcal{K}} = (X, A)^{\mathcal{K}}$.

- $\mathcal{Z}_{\mathcal{K}} = (D^2, S^1)^{\mathcal{K}}$ the moment-angle complex.
  It has an action of the torus T^m.
  Let $\mathcal{K} = \bullet\ \bullet$. Then $(D^2, S^1)^{\mathcal{K}} = D^2 \times S^1 \cup S^1 \times D^2 \cong S^3$.
- $DJ(\mathcal{K}) = (\mathbb{C}P^{\infty}, pt)^{\mathcal{K}}$ the Davis–Januszkiewicz space.

Proposition

There exists a homotopy fibration

$$\begin{array}{ccccc}
\mathcal{Z}_{\mathcal{K}} & \longrightarrow & DJ(\mathcal{K}) & \longrightarrow & (\mathbb{C}P^{\infty})^m \\
\| & & \| & & \| \\
(D^2, S^1)^{\mathcal{K}} & & (\mathbb{C}P^{\infty}, pt)^{\mathcal{K}} & & (\mathbb{C}P^{\infty}, \mathbb{C}P^{\infty})^{\mathcal{K}}
\end{array}$$

It is often convenient to replace $\lim$ and colim by the homotopy invariant functors holim and $\operatorname{hocolim}$.

Proposition

(a) *The diagram $\mathcal{D}^{\mathcal{K}}(\boldsymbol{C})$ is Reedy fibrant. Therefore, there is a weak equivalence* $\lim \mathcal{D}^{\mathcal{K}}(\boldsymbol{C}) \xrightarrow{\simeq} \operatorname{holim} \mathcal{D}^{\mathcal{K}}(\boldsymbol{C})$.

(b) *The diagram $\mathcal{D}_{\mathcal{K}}(\boldsymbol{X}, \boldsymbol{A})$ is Reedy cofibrant whenever each $A_i \to X_i$ is a cofibration (e.g. when (X_i, A_i) is a cellular pair). Under this condition, there is a weak equivalence* $\operatorname{hocolim} \mathcal{D}_{\mathcal{K}}(\boldsymbol{X}, \boldsymbol{A}) \xrightarrow{\simeq} (\boldsymbol{X}, \boldsymbol{A})^{\mathcal{K}}$.

Proof.

(a) A $\mathrm{CAT}^{op}(\mathcal{K})$-diagram $\mathcal{C}$ is Reedy fibrant when the canonical map $\mathcal{C}(I) \to \lim \mathcal{C}|_{\mathrm{CAT}^{op}(\partial\Delta(I))}$ is a fibration for each $I \in \mathcal{K}$. In our case,

$$\mathcal{D}^{\mathcal{K}}(\boldsymbol{C})(I) = \bigotimes_{i\in I} C_i, \qquad \lim \mathcal{D}^{\mathcal{K}}(\boldsymbol{C})|_{\mathrm{CAT}^{op}(\partial\Delta(I))} = \bigotimes_{i\in I} C_i/\mathcal{I},$$

where $\mathcal{I}$ is the ideal generated by all products $\prod_{i\in I} c_i$ with $c_i \in C_i^+$. Hence the fibrance condition is satisfied. □

Proof.

(b) A $\text{CAT}(\mathcal{K})$-diagram $\mathcal{D}$ in TOP is Reedy cofibrant whenever each map $\operatorname{colim} \mathcal{D}|_{\text{CAT}(\partial\Delta(I))} \to \mathcal{D}(I)$ is a cofibration. In our case,

$$\operatorname{colim} \mathcal{D}_{\mathcal{K}}(\boldsymbol{X},\boldsymbol{A})|_{\text{CAT}(\partial\Delta(I))} = (\boldsymbol{X},\boldsymbol{A})^{\partial\Delta(I)} \times \boldsymbol{A}^{[m]\setminus I}, \; \mathcal{D}_{\mathcal{K}}(\boldsymbol{X},\boldsymbol{A})(I) = (\boldsymbol{X},\boldsymbol{A})^{I},$$

so the Reedy cofibrance condition is satisfied. □

2. Formality of polyhedral products

A space X is (rationally) formal if the singular cochain algebra $C^*(X;\mathbb{Q})$ is weakly equivalent to its cohomology $H^*(X;\mathbb{Q})$ (viewed as a dga with zero differential). That is, X is formal whenever there is a zig-zag of quasi-isomorphisms

$$(C^*(X;\mathbb{Q}),d) \longleftarrow \cdots \longrightarrow (H^*(X;\mathbb{Q}),0).$$

Over $\mathbb{Q}$ or $\mathbb{R}$ one can choose a commutative model for $C^*(X)$. When X is a manifold, this is provided by the de Rham differential forms $\Omega^*(X)$.

For arbitrary X, one uses Sullivan's algebra of piecewise polynomial differential forms $A_{PL}(X)$, which is a commutative dga weakly equivalent to $C^*(X;\mathbb{Q})$.

Theorem

If each space X_i in $\boldsymbol{X} = (X_1, \ldots, X_m)$ is formal, then the polyhedral product $\boldsymbol{X}^{\mathcal{K}}$ is also formal.

Proof.

By the properties of $A_{PL}(X)$, there is a canonical quasi-isomorphism

$$A_{PL}(\boldsymbol{X}^{\mathcal{K}}) = A_{PL} \operatorname{colim}_I \boldsymbol{X}^I \xrightarrow{\simeq} \lim_I A_{PL}(\boldsymbol{X}^I).$$

Since each X_i is formal, there is a zigzag of quasi-isomorphisms $A_{PL}(X_i) \leftarrow \cdots \rightarrow H^*(X_i)$. Applying the previous Proposition for the case $C_i = A_{PL}(X_i)$ and $C_i = H^*(X_i)$ we obtain that both the corresponding diagrams $\mathcal{D}^{\mathcal{K}}(\boldsymbol{C})$ are fibrant, so their limits are weakly equivalent:

$$\lim_I A_{PL}(\boldsymbol{X}^I) \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} \lim_I H^*(\boldsymbol{X}^I)$$

(we also use the fact that $H^*(\boldsymbol{X}^I) \cong \bigotimes_{i \in I} H^*(X_i)$ with $\mathbb{Q}$-coefficients). The proof is finished by appealing to the isomorphism

$$\lim_I H^*(\boldsymbol{X}^I) \cong H^*(\boldsymbol{X}^{\mathcal{K}}).$$

□

Corollary

The Davis–Januszkiewicz space $DJ(\mathcal{K}) = (\mathbb{C}P^\infty, pt)^{\mathcal{K}}$ *is formal for any* $\mathcal{K}$.

The result cannot be extended to polyhedral products of the form $(\boldsymbol{X}, \boldsymbol{A})^{\mathcal{K}}$. Although $\lim_I A_{PL}((\boldsymbol{X}, \boldsymbol{A})^I)$ is still a model for $A_{PL}(\boldsymbol{X}, \boldsymbol{A})^{\mathcal{K}}$, the $\mathrm{CAT}(\mathcal{K})^{op}$-diagram $I \mapsto H^*((\boldsymbol{X}, \boldsymbol{A})^I)$ is *not* fibrant in general, and therefore its limit is neither isomorphic to $\lim_I A_{PL}((\boldsymbol{X}, \boldsymbol{A})^I)$, nor to $H^*((\boldsymbol{X}, \boldsymbol{A})^{\mathcal{K}})$.

Indeed, the moment-angle complex $\mathcal{Z}_{\mathcal{K}} = (D^2, S^1)^{\mathcal{K}}$ is not formal in general, as it may have nontrivial Massey products in cohomology [Baskakov].

3. Formality of quasitoric manifolds

A quasitoric manifold $M = M(P, \Lambda)$ is determined by

- a simple n-polytope P, and
- a characteristic map $\Lambda\colon \mathbb{Z}^m \to \mathbb{Z}^n$.

$\mathcal{K} = \mathcal{K}_P$ the dual triangulation of sphere S^{n-1}.

M can be identified with the quotient $\mathcal{Z}_\mathcal{K}/K(\Lambda)$,
where $\mathcal{Z}_\mathcal{K} = (D^2, S^1)^\mathcal{K}$ is the moment-angle manifold corresponding to $\mathcal{K}$,
and $K(\Lambda) = \mathrm{Ker}(\Lambda\colon T^m \to T^n)$ is a freely acting $(m-n)$-torus.

Results below are equally applicable to toric manifolds M (nonsingular compact toric varieties), in which case $\mathcal{K}$ is the underlying complex of the corresponding complete regular simplicial fan.

We consider the elements

$$t_i = \lambda_{i1} v_1 + \dots + \lambda_{im} v_m, \quad 1 \leqslant i \leqslant n,$$

in the face ring $\mathbb{Q}[\mathcal{K}] = \mathbb{Q}[v_1, \dots, v_m]/\mathcal{I}_\mathcal{K}$ corresponding to the rows of $\varLambda = (\lambda_{ij})$.

Lemma

For a toric or quasitoric manifold M, the algebra $A_{PL}(M)$ is weakly equivalent to the commutative dg-algebra

$$\bigl(\Lambda[x_1, \dots, x_n] \otimes \mathbb{Q}[\mathcal{K}], d\bigr), \quad \textit{with} \quad dx_i = t_i,\ dv_i = 0.$$

Proof.

We consider a $\mathrm{CAT}^{op}(\mathcal{K})$-diagram whose value on $I \subset J$ is the quotient map

$$\big(\Lambda[x_1,\ldots,x_n]\otimes\mathbb{Q}[v_i\colon i\in J],d\big)\to\big(\Lambda[x_1,\ldots,x_n]\otimes\mathbb{Q}[v_i\colon i\in I],d\big),$$

where $dx_i = t_i$ and $dx_i = 0$.
There are quasi-isomorphisms

$$\big(\Lambda[x_1,\ldots,x_n]\otimes\mathbb{Q}[v_i\colon i\in I],d\big)\xrightarrow{\simeq}A_{PL}\big((D^2,S^1)^I/K(\Lambda)\big)$$

which are compatible with the maps corresponding to inclusions of simplices $I \subset J$ and therefore provide a weak equivalence of Reedy fibrant diagrams in CDGA. Their limits are therefore quasi-isomorphic, and we obtain the required zigzag

$$A_{PL}(M)=A_{PL}\big((D^2,S^1)^{\mathcal{K}}/K(\Lambda)\big)\xrightarrow{\simeq}\lim_I A_{PL}\big((D^2,S^1)^I/K(\Lambda)\big)$$
$$\xleftarrow{\simeq}\lim_I\big(\Lambda[x_1,\ldots,x_n]\otimes\mathbb{Q}[v_i\colon i\in I],d\big)=\big(\Lambda[x_1,\ldots,x_n]\otimes\mathbb{Q}[\mathcal{K}],d\big).\quad\square$$

Theorem

Every toric or quasitoric manifold is formal.

Proof.

We use the model of the previous lemma and utilise the fact that $\mathbb{Q}[\mathcal{K}]$ is Cohen–Macaulay, i.e. $\mathbb{Q}[\mathcal{K}]$ is free as a module over $\mathbb{Q}[t_1, \ldots, t_n]$.

Hence $\otimes_{\mathbb{Q}[t_1,\ldots,t_n]}\mathbb{Q}[\mathcal{K}]$ is a right exact functor.

Applying it to the quasi-isomorphism $(\Lambda[u_1, \ldots, u_n] \otimes \mathbb{Q}[t_1, \ldots, t_n], d) \to \mathbb{Q}$ yields a quasi-isomorphism

$$(\Lambda[u_1, \ldots, u_n] \otimes \mathbb{Q}[\mathcal{K}], d) \xrightarrow{\simeq} \mathbb{Q}[\mathcal{K}]/(t_1, \ldots, t_n),$$

which is given by the projection onto the second factor.

Now $\mathbb{Q}[\mathcal{K}]/(t_1, \ldots, t_n) \cong H^*(M)$ by the theorem of Davis and Januszkiewicz, so the result follows from the previous lemma. □

Similar arguments apply to torus manifolds M with $H^{odd}(M;\mathbb{Z}) = 0$.
In this case, $\mathbb{Q}[\mathcal{K}]$ is replaced by the face ring $\mathbb{Q}[\mathcal{S}]$ of the corresponing simplicial poset $\mathcal{S}$.

Note also that the formality of projective toric manifolds follows immediately from the fact that they are Kähler.

References

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